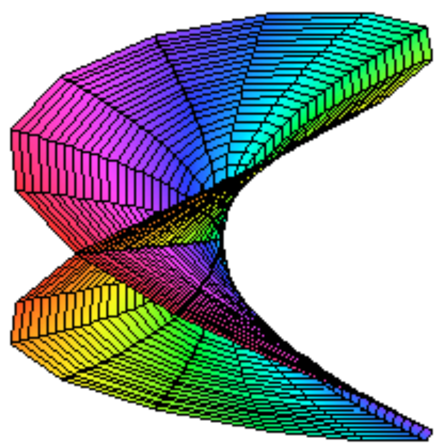
Visualizing Elliptic Curves

Donu Arapura

In this essay, I will explain how to visualize a Riemann surface (aka complex curve) with our 3-d eyes. As a warm up, we start with a standard example $y^2 = x$. When x,y are treated as real variables, this is simply a parabola opening sideways. Things get more interesting when the variables are allowed to take complex values. Since the graph would live in 2 complex or 4 real dimensions, it is impossible visualize completely. Nevertheless, we can get a sense of it projecting to 3 real dimensions, by setting x_1 and x_2 to be real and imaginary parts of x, and x_3 to be the real part of y. (We will use x_i as coordinates of R^3.) In polar coordinates, this gives

$$x_3 = Re(\sqrt{re^{i\theta}}) = \sqrt{r}\cos(\theta/2)$$

We can now plot the graph, using colour to encode the imaginary part of y.

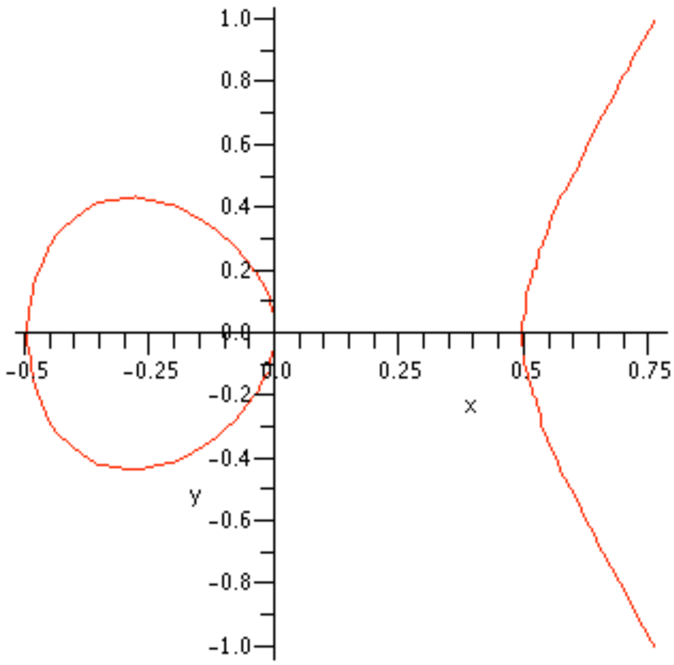


Here θ runs from 0 to 4π. Note that even though the surface appears to cross itself. We can see that it doesn't since the colours of sheets are different where they meet.

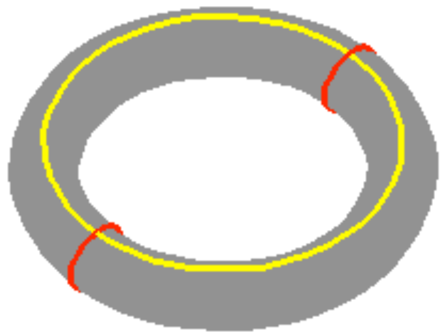
Now we turn to a more subtle example of the elliptic curve

$$y^2 = 4x^3 - x = x(2x-1)(2x+1)$$

(It's not an ellipse of course. The reason for this name comes from its connection to elliptic integrals and functions, where an integral is called elliptic if the integrand contains a square root of cubic polynomial such as the one above. A classical reference -- in spite of the name -- is Whittaker and Watson's "A course in modern analysis". Modern references are the books by Husemullor, Knapp, Silverman... on Elliptic Curves.) When x and y are treated as real variables, the graph of the above equation looks like:



After adding a point at infinity to the right, we get two circles topologically. Now let us treat the variables x and y are treated as complex. (In fact, historically signifcant progress in the study of elliptic integrals was made only after the introduction of complex analysis in the 19th century.) Now the above equation defines a complex elliptic curve which topologically is just a torus (after adding a point at infinity). The two real circles are marked in red here and below. We will also keep track of transevrse circle in yellow. It may be helpful to think of this as a "purely imaginary" curve, even though this isn't quite accurate.



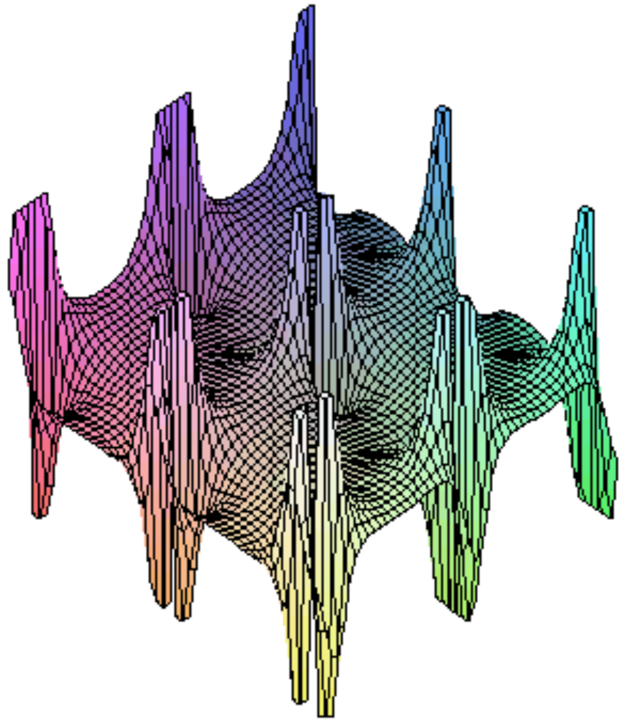
The standard way to see this is to use the parameterization

$$x = \wp(t), y = \wp'(t)$$

where $t \in C$ and

$$\wp(t) = \frac{1}{t^2} + \sum_{(m,n)\in Z^2-(0,0)} \left[\frac{1}{(t - m\alpha_1 - n\alpha_2)^2} - \frac{1}{(m\alpha_1 + n\alpha_2)^2}\right]$$

is the Weierstrass $\wp$-function (for an appropriate choice of parameters given below).



This is an elliptic function on the complex plane. Such functions are doubly periodic which means that the graph repeats itself in both the horizontal and vertical directions. More precisely, we can tile the plane into equally sized parallelograms (called period parallelograms), such that the piece of the graph over each tile is the same. In the case of *!18;* and its derivative, we can arrange that singularities occur exactly at the corners of these parallelograms. Using standard formulas [Whittaker-Watson p. 444], the corners of the central period parallelogram can be taken to be 0, α_1,α_2 and $\alpha_1 + \alpha_2$ where

$$\alpha_1 = 2\int_{1/2}^{\infty} \frac{dx}{\sqrt{4x^3 - x}} = 3.7081\ldots$$

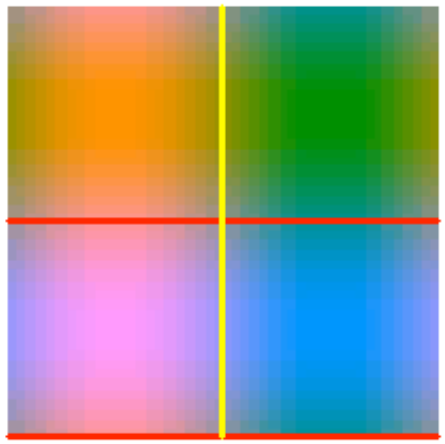
$$\alpha_2 = 2i\int_{-\infty}^{-1/2} \frac{dx}{\sqrt{4x^3 - x}} = (3.7081\ldots)i$$

Since these parameterizing functions are doubly periodic, the elliptic curve can be identified with a period parallelogram (in fact a square in this case) with the sides glued together i.e. a torus. Note that the corners of the parallelogram get identified to the same point "at infinity" on the torus.

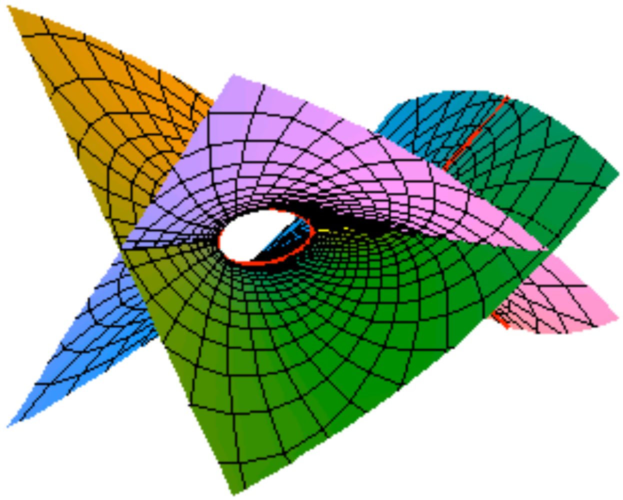
Even though, we now know the elliptic curve abstractly, we really want understand the way it is embedded into C^2. As above, we can get a sense of it projecting to 3 real dimensions. The graph of

$$x_1 = Re(\wp(t)), x_2 = Im(\wp(t)), x_3 = Re(\wp'(t))$$

represents a projection of the ellptic curve down to R^3. For the domain of *t*, we use a period parallegram. We've colour coded it, and will use the same colour scheme on all the remaining graphs. Also we've marked certain horizontal and vertical lines in red and yellow. The last couple of lines are bisectors. These lines clearly map to the above curves on the torus. Furthermore, these red lines really do map to the real curves because $\wp$ and its derivative can be shown to take real values along them [Whittaker-Watson p. 444].



With this preparation, we can now plot the graph. To get a better sense of it, we can generate an animation where it rotates about the x_3-axis:

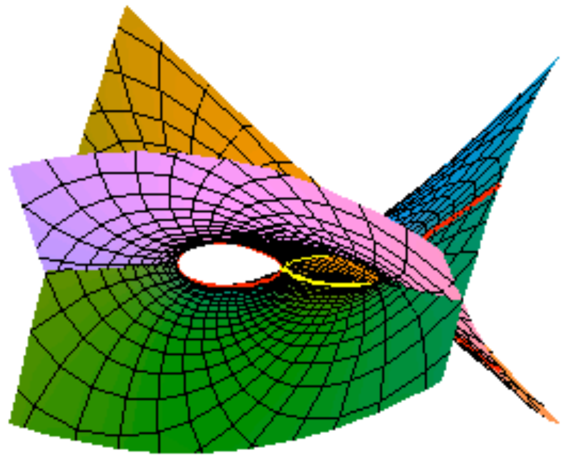


Click on picture to start animation.

The colours of the domain and the graph match. The red curves are the real part of the graph considered at the beginning. Note the local topology near the branch points (- $\frac{1}{2}$,0,0) ... is same as in the first example $y^2 = x$. In particular, the singularities in the graph are artifacts of the projection. The elliptic curve has none: it's just a torus as we saw. For example, the yellow curve which is really a loop on the torus gets crushed down to a line in the projection. To get more insight we can use a different projection. Or better yet, use a family of orthogonal projections:

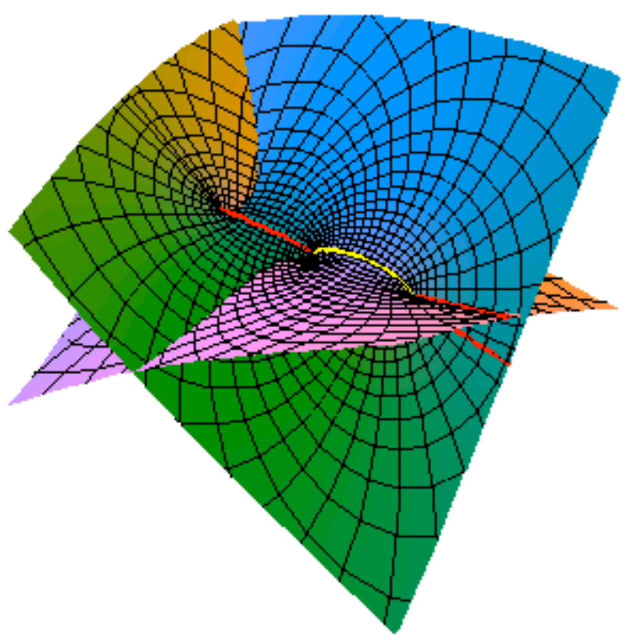
$$x_1 = Re(\wp(t)), x_2 = Im(\wp(t)), x_3 = (\cos\theta)Re(\wp'(t)) + (\sin\theta)Im(\wp'(t))$$

The original projection corresponds to $\theta = 0$. As θ oscillates between 0 and $\pi/2$, we we see the yellow curve open up into a loop and collapse, while the red curves do the opposite.



Click on picture to start animation.

For the final graph, we give a top down view with $\theta = \pi/5$.



Notes:

The plots were done in Maple. Since the functions involved are singular, I used the "view" option to truncate values. For example the plot of the real part of the $\wp$-function
was given by

```
P := (x,y) -> WeierstrassP(x + y*I,1,0);
plot3d(Re(P(x,y)), x= 0..8, y=0..8, view=-4..4, grid=[50,50]);
```

For the last few graphs, I construction a function "Gr" from angles to graphs.

```
PP := (x,y) -> WeierstrassPPrime(x + y*I,1,0);
IR := (theta, z) -> cos(theta)*Re(z) + sin(theta)*Im(z);
Gr1 := theta -> plot3d([Re(P(x,y)), Im(P(x,y)), .3*IR(theta,PP(x,y))] , x=0..3.74,y=0..3.74,
color=[sin(2*Pi*x/3.74), 0.5, sin(2*Pi*y/3.74)], view=[-1..1,-1..1, -1..1], grid=[35,35]);
L1 := theta -> spacecurve([Re(P(x,1.854)), Im(P(x,1.854)), .3*IR(theta,PP(x,1.854))] ,
x=0.0..3.74, color=red, thickness=2);
L2 := theta -> spacecurve([Re(P(x,0)), Im(P(x,0)), .3*IR(theta, PP(x,0))] , x=1..2.7, color=red,
thickness=2);
L3 := theta->spacecurve([Re(P(1.854,y)), Im(P(1.854,y)), .3*IR(theta,PP(1.854,y))] ,
y=0.0..3.74, color=yellow, thickness=2);
Gr := theta-> display({Gr1(theta) , L1(theta),L2(theta),L3(theta)});
```

Note that some reason (I expect involving numerical errors) I had to go a bit over the period to 3.74 to get the curves to close. Also the compression factor .3 in front of IR(...) was thrown in for aesthetic reasons. The function Gr used to generated frames which were assembled into an animation later.

This document was produced by using TeXht to convert latex to html, and then hand editing the html file.