1. THE CONSTRUCTIBLE DERIVED CATEGORY

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Given a family of varieties, we want to be able to describe the cohomology in a suitably flexible way. We describe with the basic homological framework.

1. Derived categories

We need to say a few words about the derived category. However, I'm going to make a number of white lies for the sake of expedience. Fortunately, there are no lack of good references Given complexes $A^\bullet, B^\bullet$ in abelian category, say $\mathcal{A}$, a map between them $f : A^\bullet \to B^\bullet$ is called a *quasi-isomorphism* if it induces an isomorphism of the cohomologies. The point of the derived category is to allow us to work up to quasi-isomorphism. More formally, the bounded derived category $D^b(\mathcal{A})$ is obtained from the category of bounded complexes of $\mathcal{A}$ by first moding out homotopies and by formally inverting quasi-isomorphisms. The objects of $D^b(\mathcal{A})$ are still complexes, but the morphisms from $A \to B$ are now equivalence classes of diagrams

$$A \overset{\sim}{\leftarrow} C \to B$$

where the first arrow is a quasi-isomorphism. Another diagram, $A \overset{\sim}{\leftarrow} C' \to B$ is equivalent if it embeds into a homotopy commutative diagram

$$\begin{tikzcd}
 & C \arrow[dl, "\sim"'] \arrow[dr] & \\
A & C'' \arrow[l, "\sim"'] \arrow[u] \arrow[d] & B \\
 & C' \arrow[ul, "\sim"] \arrow[ur] &
\end{tikzcd}$$

One thing we should observe right away is $D^b(\mathcal{A})$ is generally *not* abelian, so exact sequences don't make sense here. Actually the problem shows up in the homotopy category $K^b(\mathcal{A})$: given a homotopy class of maps $f : A^\bullet \to B^\bullet$, the kernel and cokernel are not well defined. However, the isomorphism class of the mapping cone

$$cone(f)^n = A^{n+1} \oplus B^n, \quad d(a,b) = (-da, \pm db + f(a))$$

is well defined, and this is almost as good. We denote the shifted complex $A^{\bullet+1}$ by $A[1]$. A diagram

$$A \to B \to cone(f) \to A[1]$$

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or more suggestively

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl] \\ & cone(f) \arrow[ul, rightsquigarrow] & \end{tikzcd}$$

is called a distinguished triangle, and it should be thought of a substitute for a short exact sequence. The set of (diagrams isomorphic to) distinguished triangles enjoy the following properties T1-T4, whose proofs range from obvious to challenging. I'll state them a bit imprecisely.

- T1 Every morphism embeds into a distinguished triangle. For the identity, the third vertex is 0.
- T2 The set of distinguished triangles is stable under rotation and translation.
- T3 Any pair of compatible maps between two vertices of two distinguished

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & {} \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & {} \end{tikzcd}$$

can be extended to a map of distinguished triangles in the obvious sense.

This list is already sufficient for many arguments. But there is one more somewhat technical property called the octahedral axiom because of the way it's sometimes depicted. The previous axioms implies that the third vertex of a triangle extending $f : X \to Y$ is determined up to noncanonical[1] isomorphism. It will be convenient to denote this by Y/X below.

- T4 Given distinguished triangles

$$A \to B \to B/A \to$$

$$B \to C \to C/B \to$$

arranged in the upper cap of an octahedral diagram

$$\begin{tikzcd} C/B \arrow[dd, rightsquigarrow] \arrow[dr, rightsquigarrow] & & C \arrow[ll] \\ & B \arrow[ur] \arrow[dl] & \\ B/A \arrow[rr, rightsquigarrow] & & A \arrow[ul] \arrow[uu] \end{tikzcd}$$

[1]The noncanonicity is the source of some headaches and occasional errors

(the nondistinguished triangles commute), we can complete this to an octahedral diagram with lower cap

$$\begin{tikzcd}
C/B \arrow[dd, squiggly] & & C \arrow[ll] \arrow[dl] \\
& C/A \arrow[ul] \arrow[dr, squiggly] & \\
B/A \arrow[ur] \arrow[rr, squiggly] & & A \arrow[uu]
\end{tikzcd}$$

with

$$A \to C \to C/A \to$$
$$B/A \to C/A \to C/B \to$$

distinguished. The last triangle, whose existence is the real point, can be expressed more suggestively as

$$(C/A)/(B/A) \cong C/B$$

This can be abstracted as follows. A *triangulated category* is an additive category equipped with a endofunctor $A \mapsto A[1]$ called translation, and a set of diagrams, called distinguished triangles, satisfying T1-T4. The bounded homotopy and derived categories are both triangulated. Another useful variant is the bounded below derived category $D^+(\mathcal{A})$. In fact, it is technically convenient to redefine $D^b(\mathcal{A}) \subset D^+(\mathcal{A})$ as the subcategory of complexes with finitely many nonzero cohomology groups. When $\mathcal{A}$ has enough injectives, $D^+(\mathcal{A})$ can be identified with the subcategory of $K^+(\mathcal{A})$ of injective complexes. With this interpretation, we can define the right derived functor $\mathbb{R}F : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ associated to a right exact functor. Under suitable finiteness conditions, $\mathbb{R}F(D^b(\mathcal{A})) \subset D^b(\mathcal{B})$. This functor is compatible with the triangulated structure in a natural sense. There is a similar story for left derived functors denote by $\mathbb{L}\ldots$, or $\otimes^{\mathbb{L}}$ in the case of tensor product.

2. Constructible Sheaves

Fix a connected topological space X which is nice enough that the fundamental group has the expected properties. A sheaf of abelian groups $\mathcal{F}$ on X is *locally constant* or a *local system* if there exists an open cover $\{U_i\}$ so that $\mathcal{F}|_{U_i}$ is constant. If $\tilde{X} \to X$ is the universal cover and $\rho : \pi_1(X) \to Aut(A)$ is a representation, then the sheaf of sections of the bundle $(\tilde{X} \times A)/\pi_1(X) \to X$ is locally constant. In fact, all locally constant sheaves arise this way. More precisely:

Theorem 2.1. *There is an equivalence of categories between the category of $\pi_1(X)$-modules and locally constant sheaves of abelian groups.*

We want to relax this notion to allow "singularities". A sheaf $\mathcal{F}$ on X is called *constructible* if it has finitely generated stalks and if there is a partition $\{X_i\}$ of X into "nice" locally closed sets such that $\mathcal{F}|_{X_i}$ is locally constant. Of course, the definition won't be precise until we define what nice means. In the literature, there are several standard choices:

(1) When X is a complex algebraic variety with its classical topology, the $\{X_i\}$ can be chosen Zariski locally closed. This will be the default choice for us.

(2) When X is complex or real analytic manifold, X_i can be chosen to be locally closed with respect to the analytic locally closed topology.
(3) In the previous cases, we could require that the partition satisfy the Whitney conditions which says roughly that the topology of X is locally trivial along each X_i.
(4) ....

The basic example which shows why an algebraic geometer would care about this is as follows. Let $f : X \to Y$ be a map of complex algebraic varieties. The higher direct image $R^i f_*\mathbb{Z}$ is the sheaf associated to the presheaf $U \mapsto H^i(f^{-1}U, \mathbb{Z})$. We can also define higher direct images with compact support by taking derived functors of

$$f_!(\mathcal{F})(U) = \{s \in \mathcal{F}(f^{-1}U) \mid supp(s) \text{ is proper over } U\}$$

When f is topologically a fibre bundle, then these are locally constant. For example, Ehresmann showed that this is the case when f is smooth and proper. But this is not true in general. However, Thom-Whitney stratification theory shows that X is in a suitable sense a stratified fibre bundle over Y, and therefore that:

Theorem 2.2. *The sheaves $R^i f_*\mathbb{Z}$ and $R^i f_!\mathbb{Z}$ are constructible. More generally higher direct images (with compact support) of constructible sheaves are again constructible.*

The original version was probably first proved for étale cohomology by Artin and Grothendieck. We define the constructible derived category $D^b_c(X)$ to be the subcategory of the bounded derived category of sheaves $D^b(Sh(X))$ of abelian groups on X with constructible cohomology. As a corollary of the previous theorem, we get

Corollary 2.3. *$\mathbb{R}f_*$ and $\mathbb{R}f_!$ take $D^b_c(X) \to D^b_c(Y)$.*

Constructible sheaves are also stable under standard linear algebra operations such as $\mathcal{H}om$ and $\otimes$. There are derived versions of these operations, whose cohomologies are the sheaf Ext and Tor.

3. Verdier duality

Although, it is not essential, let us briefly switch from sheaves of abelian groups to sheaves of vector spaces over a field k. Then the statements get a bit simpler. Let $D^+(X, k)$ denote the derived category of the category of sheaves of k-vector spaces. We have a derived functor

$$\mathbb{R}\Gamma_c : D^+(X, k) \to D^+(k)$$

to the derived category of k-vector space. Set $H^i_c = \mathcal{H}^i \circ \mathbb{R}\Gamma_c$. Let us assume that the spaces are locally compact Hausdorff with finite cohomological dimension, i.e. $H^i(X, -) = 0$ for $i >$ a fixed constant. Manifolds, varietes and finite simplicial complexes all satisfy this. These spaces satisfy a very general form of Poincaré duality.

Theorem 3.1 (Verdier)**.** *Let X be as above, then there exists an object $\mathbb{D} \in D^+(X, k)$ called the dualizing complex such that*

$$Hom(H^{-i}_c(\mathcal{F}), k) \cong Ext^i_{D^+(X,k)}(\mathcal{F}, \mathbb{D})$$

When X is an oriented N-manifold, $\mathbb{D} \cong k[N]$.

Proof. We just give the construction. Details can be found in [I, V1]. Let $S^\bullet$ be a soft resolution of k. Define

$$\mathbb{D}^i(U) = \Gamma_c(U, S^i)^*$$

□

To see why this is a generalization of Poincaré, observe that when $i = -j$, $\mathcal{F} = k$ and $\mathbb{D} = k[N]$, the right sider $Ext^i_{D^+(X,k)}(\mathcal{F}, \mathbb{D}) \cong H^{N-j}(X, k)$.

Now let $D^+(X)$ denote the derived category of vector spaces or abelian groups. Let $\mathcal{H}om$ denote sheaf Hom. We can extend this to an operation on complexes by

$$\mathcal{H}om(A^\bullet, B^\bullet)^n = \prod_i \mathcal{H}om(A^i, B^{i+n})$$

$$d(f) = d_B f \pm f d_A$$

By identifying objects of $D^+(X, k)$ with injective complexes, and using the above formula, we get a well defined extension denoted by $\mathbb{R}\mathcal{H}om$ to this. We can now state the stronger form of Verdier duality.

Theorem 3.2 (Verdier)**.** *Given a continuous map $f : X \to Y$ of finite dimensional locally compact spaces. There exists a functor $f^! : D^+(Y) \to D^+(X)$ satisfying*

$$\mathbb{R}\mathcal{H}om(\mathbb{R}f_! A, B) \cong \mathbb{R}\mathcal{H}om(A, f^! B)$$

We can deduce the previous theorem by taking $Y = pt$ and $B = k$ and setting $\mathbb{D} = f^! k$. So

$$\mathbb{R}\mathcal{H}om(\mathbb{R}f_! A, k) \cong \mathbb{R}\mathcal{H}om(A, \mathbb{D})$$

Now take cohomology of both sides.

By applying the global section functor to both sides of the isomorphism in the theorem, we obtain

Corollary 3.3. *$f^!$ is right adjoint to $\mathbb{R}f_!$. In particular, it is uniquely determined.*

When f is a closed immersion, $f^!$ is defined at the sheaf level and it has an elementary description

$$f^! \mathcal{F}(U) = \{s \in \mathcal{F}(U) \mid supp(s) \subset X\}$$

Finally, we have

Theorem 3.4 (Verdier)**.** *When X is a complex algebraic variety $\mathbb{D} \in D^b_c(X)$, and the dualizing operation $D(\mathcal{F}) = \mathbb{R}\mathcal{H}om(\mathcal{F}, D)$ is an involution. More generally, given an algebraic map $f : X \to Y$ of complex algebraic varieties, $f^!(D^b_c(Y)) \subset D^b_c(X)$.*

There are further relations, such as $\mathbb{D}f^! = f^*\mathbb{D}$ and $\mathbb{D}\mathbb{R}f_* = \mathbb{R}f_*\mathbb{D}$.

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