2. D-MODULES AND RIEMANN-HILBERT

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The classical Riemann-Hilbert, or Hilbert's 21st, problem asks whether every representation of the punctured complex plane comes from a system of differential equations with regular singular points. We describe a huge generalization of this, which plays a key role in the later story.

1. D-modules

1.1. **Weyl Algebra.** Fix a positive integer n. The nth Weyl algebra D_n over $\mathbb{C}$ is the ring of differential operators with complex polynomial coefficients in n variables. More formally, D_n can be defined as the *noncommutative* $\mathbb{C}$-algebra generated by symbols $x_1, \ldots x_n$, $\partial_1 = \frac{\partial}{\partial x_1}, \ldots \partial_n = \frac{\partial}{\partial x_n}$ subject to relations

$$x_i x_j = x_j x_i$$
$$\partial_i \partial_j = \partial_j \partial_i$$
$$\partial_i x_j = x_j \partial_i, \text{ if } i \neq j$$
$$\partial_i x_i = x_i \partial_i + 1$$

The last two relations stem from the Leibnitz rule $\partial_i(x_j f) = \partial_i(x_j) f + x_j \partial_i f$. These relations can be expressed more succinctly, using commutators as

$$[x_i, x_j] = [\partial_i, \partial_j] = 0$$
$$[\partial_i, x_j] = \delta_{ij}$$

There is a sense in which D_n is almost commutative that I want to explain. From the defining relations, it follows that any $P \in D_n$ can be expanded uniquely as

$$P = \sum \alpha_{I,J} x^I \partial^J$$

where $I, J \in \mathbb{N}^n$, $x^I = x_1^{I_1} \ldots x_n^{I_n}$ etc. The maximum value of $J_1 + \ldots J_n$ occurring in this sum is the *order* of P. We write $F_k D_n$ for the space of operators of order at most k. It is easy to see that $F_k F_m \subseteq F_{k+m}$. Thus the associated graded

$$Gr(D_n) = \bigoplus_k F_k / F_{k-1}$$

inherits a graded algebra structure.

Lemma 1.2. *Given $P, Q \in D_n$, we have $order([P, Q]) < order(P) + order(Q)$*

Sketch. It's enough to check this when P, Q are monomials, i.e. expressions of the form $x^I \partial^J$. In this case, it is a straight forward consequence of induction and the defining relations. □

Corollary 1.3. *$Gr(D_n)$ is commutative.*

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Slightly more work yields:

Theorem 1.4. *$Gr(D_n)$ is isomorphic to the polynomial ring $\mathbb{C}[x_1, \ldots x_n, \xi_1, \ldots \xi_n] = R_{2n}$*

I want to sketch a slightly nonstandard proof of this. First "quantize" D_n to obtain a ring H_n which has an additional variable q subject to the relations that q commutes with x_i and ∂_j and

$$[\partial_i, x_j] = q\delta_{ij}$$

The remaining relations are the same as for D_n: the x's and ∂'s commute among themselves. I will call H_n the Heisenberg algebra, since it nothing but the universal enveloping algebra of the Heisenberg Lie algebra. We see from the relations that $D_n = H_n/(q-1)$ and the "classical limit" $H_n/(q)$ is the polynomial ring R_{2n}, where $(q - \lambda)$ is the two sided ideal generated by this element.

Now form the Rees algebra

$$Rees = \bigoplus t^k F_k \subset \mathbb{C}[t] \otimes D_n$$

with t a central element. The theorem will follow from the next result which is straightforward.

Lemma 1.5.
(1) *$Gr(D_n) \cong Rees/(t)$*
(2) *The map $Rees \to H_n$ sending $x_i \mapsto x_i$, $t\partial_j \mapsto \partial_j$ and $t \mapsto q$ is an isomorphism.*

In more geometric terms, we have an identification between $Gr(D_n)$ and the ring of polynomial functions on the cotangent bundle $T^*\mathbb{C}^n$.

1.6. **D_n-modules.** The notion of a D_n-module gives an abstract way to think about systems of linear partial differential equations in n-variables. Since the ring D_n is noncommutative, we have to be careful about distinguishing between left-modules and right D_n-modules. I will often be lazy, and refer to a left D_n-module simply as a D_n-module. Here are some examples.

Example 1.7. *D_n is automatically both a left and right D_n-module.*

Example 1.8. *Let $R_n = \mathbb{C}[x_1, \ldots x_n]$ be the polynomial ring. This is a left D_n-module where x_i acts by multiplication, and ∂_i by $\frac{\partial}{\partial x_i}$.*

Example 1.9. *Given operators $P_1 \ldots P_N \in D_n$, the left (resp. right) ideal $\sum D_n P_i$ (resp. $\sum P_i D_n$) are left (resp. right) D_n-modules. Likewise for the quotients $D_n/\sum D_n P_i$ (resp. $D_n/\sum P_i D_n$). Note that $D_n/\sum D_n \partial_i = R_n = \mathbb{C}[x_1, \ldots, x_n]$.*

Example 1.10. *Given a nonzero polynomial, $R_n[f^{-1}] = \mathbb{C}[x_1, \ldots x_n, f^{-1}]$ is a D_n-module, where the derivatives act by differentiation of rational functions.*

Example 1.11. *Let F be any space of complex valued functions on $\mathbb{C}^n$ which is an algebra over the polynomial ring and and closed under differentiation, then it becomes a left D_n-module as above. In particular, this applies to holomorphic, C^∞ and C^∞ functions with compact support.*

Example 1.12. *The space of distributions is the topological dual of the space of C^∞ functions with compact support or test functions. This has a right module structure defined a follows. Given a distribution δ, a test function ϕ and $P \in D_n$, let $\delta P(\phi) = \delta(P\phi)$.*

The first four examples above are finitely generated. (The last example requires some thought. In the special case $f = x_1$, we see immediately that x_1^{-N} can be obtained by repeated differentiation.)

Fix a space of functions F as in example 1.11. Given a left D_n-module M, define the space of solutions by

$$Sol(M) = Hom_{D_n}(M, F)$$

To justify this terminology consider the example 1.9 above. We see immediately that there is an exact sequence

$$0 \to Sol(M) \to F \stackrel{\sum P_i}{\longrightarrow} F^N$$

Therefore $Sol(M)$ is the space of solutions of the system $P_i(f) = 0$.

There is a symmetry between right and left modules that I will refer to as "flipping".

Lemma 1.13. *There is an involution $P \mapsto P^*$ of D_n determined by $x_i^* = x_i$ and $\partial_i^* = -\partial_i$. Given a right D_n-module M, the operation $P \cdot m = mP^*$ makes M into a left module, which I denote by $Flip^{R\to L}(M)$. This gives an equivalence between the categories of (finitely generated) left and right modules. The inverse operation will denote by $Flip^{L\to R}$.*

Suppose that M is a finitely generated D_n-module. We define *good* filtration on M to be a filtration F_pM such that

(1) The filtration $F_pM = 0$ for $p \ll 0$ and $\cup F_pM = M$.
(2) Each F_pM is a finitely generated R_n-submodule.
(3) $F_pD_n \cdot F_qM \subseteq F_{p+q}M$.

Lemma 1.14. *Every finitely generated module possess a good filtration.*

Proof. Write it as a quotient of some free module D_n^N and take the image of $(F_pD_n)^N$ □

The filtration is *not* unique, however it does lead to some well defined invariants. Given a module with good filtration, the associate graded

$$Gr(M) = \bigoplus F_pM/F_{p-1}M$$

is a finitely generated $Gr(D_n)$-module. The annihilator of $Gr(M)$ gives an ideal in $Gr(D_n) \cong R_{2n}$. The zero set of this ideal defines a subvariety $Ch(M,F) \subset \mathbb{C}^{2n}$ called the *characteristic variety* or singular support. Since $Gr(M)$ is graded with respect to the natural grading on $Gr(D_n)$, we see that, the annihilator is homogeneous. Therefore

Lemma 1.15. *$Ch(M,F)$ is invariant under the action of $t \in \mathbb{C}^*$ by $(x_i, \xi_j) \mapsto (x_i, t\xi_j)$.*

We can view this another way. Consider the Rees module $Rees(M,F) = \oplus F_pM$. This is a finitely generated module over H_n such that $Gr(M) = H_n/(q) \otimes Rees(M,F)$. So in some sense $Ch(M,F)$ is the classical limit of M as $q \to 0$.

Theorem 1.16. *$Ch(M,F)$ is independent of the filtration. Thus we can, and will, drop F from the notation.*

Example 1.17. *In the previous examples, we see that*

(1) *The annihilator of* $Gr(D_n)$ *is* 0*, so that* $Ch(D_n) = \mathbb{C}^{2n}$.
(2) *Taking* $R_n = D_n/\sum D_n\partial_i$*, yields* $Gr(R_n) = \mathbb{C}[x_1, \dots x_n]$*. Its annihilator is the ideal* $(\xi_1, \dots \xi_n)$*. Therefore* $Ch(R_n) = \mathbb{C}^n \times 0$
(3) *Let* $M = R_1[x^{-1}]$*, where we* $x = x_1$*. Then* $1, x^{-1}$ *generate* M*. Let*

$$F_kM = F_k \cdot 1 + F_k \cdot x^{-1} = \mathbb{C}[x]x^{-k-1}$$

This gives a good filtration. A simple computation shows that

$$Gr(M) \cong \mathbb{C}[x,\xi]/(\xi) \oplus \mathbb{C}[x,\xi]/(x)$$

So $Ch(M) = V(x\xi)$ *is a union of the axes.*

Theorem 1.18 (Bernstein's inequality)**.** *For any nonzero finitely generated* D_n*-module, we have* $\dim Ch(M) \geq n$.

There are a number of ways to prove this. Perhaps the most conceptual, though not the easiest, way is to deduce it from the involutivity of the annihilator [Ga]. This means that this ideal is closed under the Poisson bracket induced from the symplectic structure of $\mathbb{C}^{2n} = T^*\mathbb{C}^n$. This implies that the tangent space of any smooth point $p \in Ch(M)$ satisfies $T_p^\perp \subseteq T_p$, and the inequality follows. Note that the $\mathbb{C}^*$-action of lemma 1.15 is precisely the natural action on the fibers of the cotangent bundle.

We say that finitely generated D_n-module M is *holonomic* if $\dim Ch(M) = n$ or if $M = 0$. Thanks to Bernstein's inequality, this is equivalent to $\dim Ch(M) \leq n$. For example R_n and $R_1[x^{-1}]$ are holonomic, but D_n isn't.

Proposition 1.19. *The class of holonomic modules is closed under submodules, quotients and extensions. Therefore the full subcategory of holonomic modules is Abelian.*

Proof. One checks that $Ch(M_2) = Ch(M_1) \cup Ch(M_3)$ for any exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$. □

From the symplectic viewpoint, holonomic modules are precisely the ones with Lagrangian characteristic varieties. There is also a homological characterization of such modules.

Theorem 1.20. *A finitely generated* D_n*-module is holonomic if and only if* $Ext^i(M, D_n) = 0$ *for* $i \neq n$*. If* M *is holonomic, then the module* $Ext^n(M, D_n)$ *is a finitely generated holonomic right* D_n*-module. The contravariant functor* $M \mapsto Flip^{R\to L}Ext^n(M, D_n)$ *is an involution on the category of holonomic modules.*

Corollary 1.21. *Holonomic modules are artinian (which means the descending chain condition holds).*

Sketch. Any descending chain in M gets flipped around to an ascending chain in $N = Ext^n(M, D_n)$. D_n is known to be right (and left) noetherian, so the same goes for N. □

It will follow that holonomic modules can built up from simple holonomic modules.

1.22. **Inverse and direct image.** Suppose that $X = \mathbb{C}^n$ with coordinates x_i and $Y = \mathbb{C}^m$ with coordinates y_j. Consider a map $F : X \to Y$ given by

$$F(x_1, \ldots x_n) = (F_1(x_1, \ldots x_n), \ldots)$$

where the F_i are polynomials. Let $\mathcal{O}_X = \mathbb{C}[x_1, \ldots x_n]$, and $\mathcal{O}_Y = \mathbb{C}[y_1, \ldots y_n]$, and let D_X and D_Y denote the corresponding Weyl algebras. (To avoid confusion, I will label the derivatives ∂_{x_i} etc.) Then F determines an algebra homomorphism

$$\mathcal{O}_Y \to \mathcal{O}_X$$

$$f \mapsto f \circ F$$

Given a left D_Y-module M, we define a left D_X module F^*M, called the *inverse image*, as follows. First define

$$F^*M = \mathcal{O}_X \otimes_{\mathcal{O}_Y} M$$

as an $\mathcal{O}_X$-module. We now define an action of the derivatives by the chain rule

$$\partial_{x_i}(f \otimes m) = \frac{\partial f}{\partial x_i} \otimes m + \sum_j f \frac{\partial F_j}{\partial x_i} \otimes \partial_{y_j} m$$

Lemma 1.23. *This formula determines a D_X-module structure on F^*M*

Example 1.24. *Let $X = \mathbb{C}^n$ with coordinates $x_1, \ldots x_n$, $Y = \mathbb{C}^{n-1}$ with coordinates $x_1, \ldots x_{n-1}$. Let $p(x_1, \ldots x_n) = (x_1, \ldots x_{n-1})$. Then*

$$p^*M = \mathbb{C}[x_n] \otimes_{\mathbb{C}} M$$

Given $f(x_n) \otimes m$, x_n and ∂_n acts in the usual way through the first factor, and remaining generators of D_X act through the second.

There is a second description that is useful. Define

$$D_{X \to Y} = F^*D_Y = \mathcal{O}_X \otimes_{\mathcal{O}_Y} D_Y$$

This has the structure of a left D_X-module as above, as well as a right D_Y-module structure, where D_Y acts by right multiplication on itself in the above formula. These two actions commute, so they determine a so called bimodule structure. If we flip both of these actions, we get left D_Y right D_X bimodule

$$D_{Y \leftarrow X} = Flip_{D_X}^{L \to R} Flip_{D_Y}^{R \to L}(D_{X \to Y}).$$

Lemma 1.25. $f\, F^*M = D_{X \to Y} \otimes_{D_Y} M$

Proof.

$$F^*M = \mathcal{O}_X \otimes \mathcal{O}_Y M = (\mathcal{O}_X \otimes_{\mathcal{O}_Y} D_Y) \otimes_{D_Y} M = D_{X \to Y} \otimes_{D_Y} M$$

□

Given a left D_X-module N, the direct image

$$F_*N = D_{Y \leftarrow X} \otimes_{D_X} N$$

is a left D_Y-module. This operation is sometimes denoted with an integral sign to suggest the analogy with integration along the fibers.

Example 1.26. *Let $X = \mathbb{C}^n$ with coordinates $x_1 \dots x_n$, $Y = \mathbb{C}^{n+1}$ with coordinates $x_1, \dots x_{n+1}$ and suppose $i(x_1, \dots) = (x_1, \dots x_n, 0)$. We have $i_*M = M^{\mathbb{N}}$, a countable direct sum. Here $x_1, \partial_1, x_2, \dots \partial_n$ acts as componentwise using the given D_X-module structure, x_{n+1} acts by 0, ∂_{n+1} acts as the shift operator*

$$(m_1, m_2, \dots) \mapsto (0, m_1, m_2, \dots)$$

Thus it is more suggestive to write

$$i_*M = \bigoplus_j \partial_{n+1}^j M$$

These operations are compatible with composition, as one would hope.

Theorem 1.27. *If $F : X \to Y$ and $G : Y \to Z$ are polynomial maps of affine spaces, then for any D_Z-module M and D_X-module M, we have*

1. $(G \circ F)^*M \cong F^*G^*M$
2. $(G \circ F)_*N \cong G_*F_*N$.

Theorem 1.28. *These operations preserve finite generation and holonomicity.*

Since any map can be factored as an embedding followed by a projection, it suffices by the theorem 1.27 to check these two cases. This theorem provides many additional examples of holonomic modules.

1.29. **Differential operators on affine varieties.** Let X be a nonsingular affine variety over $\mathbb{C}$. This is a complex manifold which can be described as the set of solutions to a system of polynomial equations in some $\mathbb{C}^n$. We write $\mathcal{O}(X)$ for ring regular (= polynomial) functions on X. This is a finitely generated commutative algebra. A differential operator of order $\leq k$ on X is a $\mathbb{C}$-linear endomorphism T of $\mathcal{O}(X)$ such that

$$[\dots[[T, f_0], f_1]\dots f_k] = 0$$

for all $f_i \in \mathcal{O}(X)$. Let $\mathrm{Diff}_k(X)$ denote the space of these operators. We define

$$D_X = \bigcup \mathrm{Diff}_k(X)$$

Lemma 1.30. *D_X becomes a ring under composition such that $Diff_k Diff_m \subset Diff_{k+m}$.*

We note the following characterization (c.f. [K2, lemma 1.7]) which sometimes useful.

Proposition 1.31. *D_X is a quotient of the univeral enveloping algebra of the Lie algebra of vector fields $Der_{\mathbb{C}}(\mathcal{O}(X))$ by the relations $[\xi, f] = \xi(f)$ for all $\xi \in Der_{\mathbb{C}}(\mathcal{O}(X))$ and $f \in \mathcal{O}(X)$.*

When $X = \mathbb{C}^n$, $(D_X, \mathrm{Diff}_\bullet) = (D_n, F_\bullet)$. Everything that we have done so far generalizes to the setting of affine varieties. For example

Theorem 1.32. *The associated graded with respect to $Diff_\bullet$ is isomorphic to the ring of regular functions on the cotangent bundle T^*X.*

We can define left/right D-modules as before. All of the previous examples generalize. We give a new example.

Example 1.33. *Let $f \in R_n$, and let X be complement of the zero set of f in $\mathbb{C}^n$. This is an affine variety with coordinate ring $R = R_n[\frac{1}{f}]$. Let $A = \sum A_i dx_i$ be an $r \times r$ matrix of 1-forms with coefficients in R satisfying the integrability condition $[A_i, A_j] = 0$. Then $M = R^r$ carries a left D_X-module structure with*

$$\partial_i v = \frac{\partial v}{\partial x_i} + A_i v;\ v \in M$$

Note that this construction is equivalent to defining an integrable connection on M. There are nontrivial examples only when X is non-simply connected, and in particular none unless $f \neq 1$.

The "flipping" operation for affine varieties is more subtle than before. Let ω_X denote the canonical module or equivalently the module of algebraic n-forms, where $n = \dim X$. This has right D_X module structure dual to left module structure on $\mathcal{O}(X)$. Heuristically, this can be undertood by the equation

$$\int_X (\alpha P) f = \int_X \alpha(Pf)$$

where α is an n-form, P a differential operator, and f a function and X is replaced by a compact manifold. A rigorous definition can be given via the Lie derivative L. Given a vector field ξ and an element of $\alpha \in \omega_X$,

$$L_\xi \omega(\xi_1, \ldots \xi_n) = \xi(\omega(\xi_1, \ldots \xi_n)) + \sum \omega(\xi_1, \ldots [\xi, \xi_i], \ldots \xi_n)$$

Then $\omega \cdot \xi = -L_\xi \alpha$ can be shown to extend to a right action of the whole ring D_X with the help of proposition 1.31. Under this action, the difference $(\alpha P)f - \alpha(Pf)$ can be shown to be exact, and so above integral formula would follow.

Lemma 1.34. *If M is a left D_X-module, then $Flip^{L\to R}(M) = \omega_X \otimes_{\mathcal{O}(X)} M$ carries a natural right D_X-module structure. This operation induces an equivalence between the categories of left and right D_X-modules; its inverse is $Flip^{R\to L}(N) = \omega_X^{-1} \otimes N$.*

Note that $\omega_{\mathbb{C}^n} \cong R_n$, which was why we could ignore it.

The notions of characteristic variety and holonomocity can be defined as before. The characteristic variety of example 1.33 is X embedded in T^*X as the zero section. Therefore it is holonomic.

Given a morphism of affine varieties $F : X \to Y$, we can define bimodules $D_{X\to Y}$, $D_{Y\leftarrow X}$, and inverse and direct images as before.

1.35. **Non-affine varieties.** Now we want to generalize to the case where X is a nonsingular non-affine variety, for example projective space $\mathbb{P}^n$. First, recall that in its modern formulation a variety consists of a space X with a Zariski topology and a sheaf of commutative rings $\mathcal{O}_X$, such that for any open set $\mathcal{O}_X(U)$ is the space of regular functions [Ha]. By definition, X possesses an open covering by affine varieties. Our first task is to extend D_X to this world:

Lemma 1.36. *There exists a unique sheaf of noncommutative rings D_X on X such that for any affine open U, $D_X(U)$ is the ring of differential operators on U.*

We can define a filtration by subsheaves $F_p D_X \subset D_X$ as above. The previous result globalizes easily to:

Theorem 1.37. *The associated graded is isomorphic to $\pi_* \mathcal{O}_{T^*X}$ where $\pi : T^*X \to X$ is the cotangent bundle.*

A left or right D_X-module is sheaf of left or right modules over D_X. For example, $\mathcal{O}_X$ (resp. ω_X) has a natural left (resp. right) D_X-module structure. D_X has both. We have an analogue of lemma 1.34 in this setting, so we can always switch from right to left.

We will be primarily interested in the modules which are coherent (i.e. locally finitely generated) over D_X. The notion of good filtration for a D_X-module M can be extended to this setting. The associated graded $Gr(M)$ defines a sheaf over the cotangent bundle, and the characteristic variety $Ch(M)$ is its support. This depends only on M as before and is $\mathbb{C}^*$-invariant. We have Bernstein's inequality $\dim Ch(M) \geq \dim X$, and M is holonomic if equality holds. We again have:

Proposition 1.38. *The full subcategory of holomonic modules is an artinian Abelian category.*

Given a morphism of varieties $F : X \to Y$, we define

$$D_{X\to Y} = \mathcal{O}_X \otimes_{F^{-1}\mathcal{O}_Y} F^{-1}D_Y$$

where F^{-1} is the inverse image in the category of sheaves [Ha]. This is a left D_X right $F^{-1}D_Y$ bimodule. We define a right D_X left $F^{-1}D_Y$ bimodule by flipping both actions:

$$D_{Y\leftarrow X} = Flip^{L\to R}_{D_X} Flip^{R\to L}_{F^{-1}D_Y}(D_{X\to Y})$$

As an $\mathcal{O}_X$-module, it this is isomorphic to $\omega_X \otimes D_{X\to Y} \otimes F^*\omega_Y^{-1}$.

Given a left D_Y-module M, we can define the naive[1] inverse image as the D_X-module:

$$F_n^*M = D_{X\to Y} \otimes_{F^{-1}D_Y} F^{-1}M$$

For a left D_X-module N, the naive direct image as the D_Y-module

$$F_*^n N = F_*(D_{Y\leftarrow X} \otimes_{D_X} N)$$

where F_* on the right is the sheaf theoretic direct image.

The above definitions proceed in complete analogy with the affine case. The bad news is that the naive direct image is somewhat pathological. For example, the composition rule (theorem 1.27) may fail. The solution is to work in the setting of derived categories. Let $D^b(D_X)$ denote the bounded derived category of quasicoherent left D_X-modules. We can define the inverse image

$$F^* : D^b(D_Y) \to D^b(D_X)$$

by

$$F^*M^\bullet = D_{X\to Y} \otimes^{\mathbb{L}}_{F^{-1}D_Y} F^{-1}M^\bullet$$

and the direct image $F_* : D^b(D_X) \to D^b(D_Y)$ by

$$F_*N^\bullet = \mathbb{R}F_*(D_{Y\leftarrow X} \otimes^{\mathbb{L}}_{D_X} N^\bullet)$$

This is sometimes denote by an integral symbol. These behave well under composition. At the end of the day, we can compose these operations with $\mathcal{H}^i$ to get actual D-modules. There are cases where this can be made explicit. For F a closed immersion, F_* coincides with F_+ described earlier. For F is a smooth and projective or relative dimension n, we have

$$D_{Y\leftarrow X} \cong (D_X \to \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} D_X \to \Omega^2_{X/Y} \otimes_{\mathcal{O}_X} D_X \ldots)[n]$$

[1]This is nonstandard terminology.

The right side is the relative de Rham complex associated to D_X. From this, we obtain that

$$F_*N = \mathbb{R}F_*(N \to \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} N \to \ldots)[n]$$

is the complex associated to the relative de Rham cohomology of N.

1.39. **Connections.** Let E be a vector bundle on a nonsingular variety X, i.e. a locally free $\mathcal{O}_X$-module. A D_X-module structure on E is the same thing as an integrable connection on E, which is given locally as in example 1.33. Globally, this is a $\mathbb{C}$-linear map from the tangent sheaf

$$\nabla : \mathcal{T}_X \to End(E)$$

such that $\nabla(v)$ is a derivation and such that ∇ preserves Lie bracket

$$\nabla([e_1, e_2]) = [\nabla(e_1), \nabla(e_2)]$$

Equivalently, it is given by a $\mathbb{C}$-linear map

$$\nabla : E \to \Omega^1_X \otimes E$$

satisfying the Leibnitz rule and the having curvature $\nabla^2 = 0$.

From, the local description, it is easy to see that the characteristic variety of an integrable connection is the zero section of T^*X. Thus it is holonomic. Conversely,

Proposition 1.40. *M is a vector bundle with integrable connection if and only if its characteristic variety is the zero section of T^*X*

Corollary 1.41. *If M is a holonomic module, there exists an open dense set $U \subseteq X$ such that $M|_U$ is an integrable connection.*

Proof. We can assume that the support of M is X, otherwise the statement is trivially true. Then the map $Ch(M) \to X$ is generically finite, and therefore finite over some open $U \subset X$. Since $Ch(M) \cap T^*U$ is $\mathbb{C}^*$ invariant, it must be the zero section. □

Given a morphism $F : X \to Y$ and an integrable connection (E, ∇) on Y. The pullback of the associated D_Y-module coincides with the pullback F^*E in the category of O-modules with its induced connection. If (E', ∇') is an integrable connection on X, then the pushforward of the associated D_X-module does not come from a connection in general.

We finally discuss the notion of *regular singularities* which is a growth condition at infinity. The classical condition is the following.

Example 1.42. *Let A be an $r \times r$ matrix of rational 1-forms on $\mathbb{P}^1$. Let U be the complement of the poles in $\mathbb{P}^1$ of the entries of A, and let $j : U \to \mathbb{P}^1$ the inclusion. Then we can define a D_U-module structure on $M = \mathcal{O}_U^r$ by*

$$\partial v = \frac{dv}{dx} + Av$$

M is holomonic. The D_X-module j_*M has regular singularities if and if the differential equation $\partial v = 0$ has regular singularities in the classical sense; this is the case if the poles of A are simple.

In general, we have the following extension due to Deligne. A vector bundle (E, ∇) with a connection on a smooth variety X has regular singularities if there exists a nonsingular compactification $\bar{X}$, with $D = \bar{X} - X$ a divisor with normal crossings, such that (E, ∇) extends to a vector bundle with a map

$$\bar{\nabla} : \bar{E} \to \Omega^1_X(\log D) \otimes \bar{E}$$

The notion of regular singularities can be extended to arbitrary holonomic D_X-modules. If M is a simple holonomic module with support Z, then $M|_Z$ is generically given by an integrable connection as above. Say that M has regular singularities if this connection is regular. In general, M has regular singularities if each of its simple subquotients have regular singularities. This notion behaves well with respect to the operations defined earlier. See [B, Bo, K2] for details.

1.43. **Riemann-Hilbert correspondence.** In the 19th century Riemann completely analyzed the hypergeometric equation in terms of its monodromy. Hilbert, in his 21st problem, proposed that a similar analysis should be carried out for more general differential equations. Here I want to explain a very nice interpretation and solution in D-module language due to Kashiwara-Kawai and Mebkhout.

Fix a smooth variety X over $\mathbb{C}$. We can treat X as a complex manifold, and we denote this by X^{an}. Most algebraic objects give rise to corresponding analytic ones, usually marked by "an". In particular, $D_{X^{\mathrm{an}}}$-module is the sheaf of holomorphic differential operators. Any D_X-module gives rise to a $D_{X^{\mathrm{an}}}$-module.

Let $\Omega^p_{X^{\mathrm{an}}}$ denote the sheaf of holomorphic p-forms on X^{an}. Recall that we have a complex, $\Omega^\bullet_{X^{\mathrm{an}}}$, called the de Rham complex, which is quasi-isomorphic to the constant sheaf $\mathbb{C}_{X^{\mathrm{an}}}$. We can modify this to allow coefficients in any $D_{X^{\mathrm{an}}}$-module M:

$$DR(M)^\bullet = \Omega^\bullet_{X^{\mathrm{an}}} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} M[\dim X]$$

(The symbol $[n]$ mean shift the complex n places to the left). The differential is given in local coordinates by

$$d(dx_{i_1} \wedge \ldots \wedge dx_{i_p} \otimes m) = \sum_j dx_j \wedge dx_{i_1} \wedge \ldots dx_{i_p} \otimes \partial_j m$$

We can define a complex

$$\ldots D_X \otimes_{\mathcal{O}_X} \wedge^2 \mathcal{T}_X \to D_X \otimes_{\mathcal{O}_X} \mathcal{T}_X \to D_X$$

with differentials dual to $DR(D_X)^\bullet$ under the identification

$$Hom_{\text{right-}D_X\text{-mod}} Hom(\Omega^p_X \otimes_{\mathcal{O}_X} D_X, D_X) \cong D_X \otimes_{\mathcal{O}_X} \wedge^p T_X$$

The complex $D_{X^{\mathrm{an}}} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \wedge^\bullet \mathcal{T}_{X^{\mathrm{an}}}$ gives a locally free resolution of $\mathcal{O}_{X^{\mathrm{an}}}$. This comes down to the fact that it becomes a Koszul complex after taking the associated graded with respect to F. Therefore

$$DR(M)^\bullet \cong Hom(D_{X^{\mathrm{an}}} \otimes_{\mathcal{O}_{X^{\mathrm{an}}}} \wedge^\bullet \mathcal{T}_{X^{\mathrm{an}}}, M) \cong \mathbb{R}\mathcal{H}om(\mathcal{O}_{X^{\mathrm{an}}}, M)$$

We can extend the definition of DR to the derived category $D^b(D_X)$ by using the last formula.

We can now give classical version of the Riemann-Hilbert correspondence.

Proposition 1.44. *If E is a holomorphic vector bundle with an integrable connection ∇, $DR(E)[-\dim X]$ is a locally constant sheaf of finite dimensional $\mathbb{C}$-vector spaces. The functor $E \mapsto DR(E)[-\dim X]$ induces an equivalence of categories between these categories.*

Sketch.

$$DR(E)[-\dim X] = E \xrightarrow{\nabla} \Omega^1_{X^{\rm an}} \otimes E \xrightarrow{\nabla} \ldots$$

gives a resolution of $ker\nabla$, which is locally constant. Conversely, given a locally constant sheaf L, $\mathcal{O}_{X^{\rm an}} \otimes L$ can be equipped with an integrable connection such L is the kernel. □

By imposing regularity assumptions, Deligne [De] was able to make this correspondence algebraic.

Theorem 1.45 (Deligne)**.** *There is an equivalence of categories between the category of locally constant sheaves $\mathbb{C}$-vector spaces on X^{an} and algebraic vector bundles equipped with regular integrable connections.*

The point is that regularity ensures that the holomorphic data extends to a compactification, where GAGA applies. For general D-modules, we impose holonomicity as well. $DR(M)$ will no longer be a locally constant sheaf in general, but rather a complex with constructible cohomology. So it is natural to formulate the general result as an equivalence of derived categories. On one side, we have $D^b_{rh}(D_X) \subset D^b(D_X)$ the subcategory of the derived category of left modules with regular holonomic cohomology. On the other side, we have the constructible derived category.

Theorem 1.46 (Kashiwara, Mebkhout)**.** *The de Rham functor DR induces an equivalence of categories between $D^b_{rh}(D_X)$ and $D^b_c(X, \mathbb{C})$. Moreover the inverse and direct images constructions are compatible under this correspondence.*

There is one more aspect of this, which is worth noting. The duality $M \mapsto Flip^{R\to L} Ext^n(M, D_n)$ constructed earlier can be generalized naturally to this setting to

$$M \mapsto Flip^{R\to L} \mathbb{R}\mathcal{H}om(M, D_X).$$

This corresponds to the Verdier dual. The Riemann-Hilbert correspondence is sometimes phrased using the sheaf of "solutions"

$$Sol(M) = \mathbb{R}\mathcal{H}om(M, \mathcal{O}_{X^{\rm an}})$$

This formulation is equivalent, because Sol corresponds to DR of the dual module.

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