0. INTRODUCTION

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Given a family of complex algebraic varieties $f : X \to Y$, we can associate the following topological data:

(1) the homeomorphism types of the fibres, and in particular, their rational cohomology $H^i(X_y, \mathbb{Q})$
(2) the monodromy which describes the way that cohomology classes get moved as you travel between topologically similar fibres.

How do we compute the singular cohomology of X in terms of this? The classical answer, at least since the 1950's, is by the Leray spectral sequence.

$$H^p(Y, R^q f_* \mathbb{Q}) \Rightarrow H^{p+q}(X, \mathbb{Q})$$

But analyzing this is easier said than done. However, an entirely new technique evolved in the early 1980's which turns out to be often easier. This involves the notion of a "perverse sheaf" which as the originators [BBD] point out is neither a sheaf nor perverse, but more about that later. Although this is a purely topological theory, in practice calculations are facilitated by the remarkable decomposition theorem of [BBD] which relies on the underlying algebraic geometry. By general nonsense we have $H^i(X, \mathbb{Q}) = H^i(Y, \mathbb{R} f_* \mathbb{Q})$, where $\mathbb{R} f_* \mathbb{Q}$ is the derived image in the derived category of Y. The decomposition theorem, says that this splits up into a sum of simple pieces which are essentially the intersection cohomology complexes of Goresky-MacPherson attached to local systems supported on subvarieties of Y. Computing the cohomology of X reduces to identifying these pieces. The original proof of the theorem made use of the fact, that since the map is algebraic, it can be reduced mod p and then the statement can be deduced from a strong form of the Weil conjectures. Prior to this, Gabber did important related work which was unfortunately never published. I've made a slightly expanded translation of Deligne's unpublished write up of Gabber's purity theorem available here.

There is a second proof of the decomposition theorem due to Saito [S] which doesn't involve a characteristic p methods, but instead uses Hodge theory. Saito does more than just prove this theorem; he sets up a general framework in which to do Hodge theory with coefficients on singular spaces. Unfortunately, this is very complicated and not widely understood. The main goal of the seminar is to go through an outline of the proof.

References

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