PERVERSE SHEAVES AND t-STRUCTURES

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The Riemann-Hilbert correspondence, which gives an equivalence $D^b_{rh}(X) \cong D^b_c(X, \mathbb{C})$, raises the question what does the image of the category of regular holonomic modules look like on the right? That is what we will try to answer here; it will involve a new class of objects called perverse sheaves. In order to make the theory work over arbitrary fields, it is important to have an intrinsic characterization of these objects, which we give. Also, aside from D-module theory, another source of examples came from intersection cohomology.

1. Perverse sheaves

Let us with the simplest case of the disk Δ. We assume that our D-modules M are singular only at the origin, or in other words that the restriction to Δ^* is given by the trivial bundle V with a connection ∇. The complex $DR(M)$ is

$$M \xrightarrow{\partial} M$$

shifted so that the first M starts in degree -1. Then the Poincaré lemma gives an exact sequence

$$0 \to \ker \nabla \to M|_{\Delta^*} \xrightarrow{\partial} M_{\Delta^*}$$

thus $DR(M)|_{\Delta^*} \cong \ker \nabla$ up to shift. The dual $V^* = Hom(V, \mathcal{O})$ carries a natural connection

$$\langle \nabla^* v^*, v \rangle = d\langle v^*, v\rangle - \langle v^*, \nabla v\rangle$$

This operation is compatible with duality for local systems. With respect to a local dual basis, the connection matrix is given by $-A^T$. Which suggests that this operation can be extended to D-modules by sending a module with presentation matrix $P = (p_{ij})$ to the module with presentation matrix $P^* = (p^*_{ji})$. To say this more invariantly, we define

$$M^* = Ext^1_{D_\Delta}(M, D_\Delta)$$

A provisional definition of perverse sheaf is that it is a complex of sheaves of $\mathbb{C}$-vector spaces on Δ quasi-isomorphic to $DR(M)$ for a regular holonomic D-module M. From this, we can infer the following: The collection of perverse sheaves should form an artinian abelian category because the M's form one. To obtain the actual definition, we note that $K = DR(M)$ and dually $DR(M^*)$ have cohomology in exactly in degrees 0 and -1. The dual $DR(M^*)$ can be understood directly in terms of K. It is the Verdier dual $DK = \mathbb{R}\mathcal{H}om(K, \mathbb{C}[-1])$. So now arrive at the definition:

A *perverse sheaf* on Δ is a bounded complex of sheaves K such that

(P1) The cohomology sheaves $H^i(K)$ are zero unless $i = 0, -1$. H^0 supported at 0, and H^{-1} gives a local system on Δ^*.

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(P2) The same conditions holds for DK.

Let's look at some basic examples of perverse sheaves corresponding to the simple D-modules.

1. $\mathbb{C}_\Delta[-1]$ is perverse. We can realize this as $DR(\mathcal{O}_\Delta)$.
2. $j_*L[-1]$ is perverse, where L is the rank one local system on Δ^* with monodromy given by $a \in \mathbb{C}^*$. To see this, choose a logarithm $r = \frac{1}{2\pi i}\log a$, and set $M = \mathcal{O}_\Delta[z^{-1}]$ with $\partial \cdot 1 = \frac{r}{z}$. Then $DR(M)$ consists of $\langle z^{-r}\rangle$ in degree -1. Writing $z = \exp(-2\pi i t)$, we can see that $z^{-r} \mapsto az^{-r}$ under $t \mapsto t+1$.
3. The sky scraper sheaf $\mathbb{C}_0$ is perverse. It is given by $DR(\mathcal{O}[z^{-1}]/\mathcal{O}) = \mathbb{C}_0$.

Let us know turn the general case.

Theorem 1.1. *Let X be a smooth complex algebraic variety. An object $\mathcal{F} \in D^b_c(X,\mathbb{C})$ lies in the image of DR if and only if for every subvariety $\iota : S \to X$*

1. $\mathcal{H}^i\iota^*\mathcal{F} = 0$ *for* $i > -\dim S$ *and*
2. $\mathcal{H}^i\iota^!\mathcal{F} = 0$ *for* $i < -\dim S$.

Proof. In one direction suppose that $\mathcal{F} = DR(M)$ for M regular holonomic. Let $j : T \subset S$ be a smooth open set such that M is a connection. Then we see that $\iota^!\mathcal{F}_T = DR((\iota \circ j)^! M)$ is concentrated in degree $-\dim S$. Thus $\mathcal{H}^i\iota^!\mathcal{F}$ is supported on $S - T$ when $i < -\dim S$. By induction, we see that the restriction of $\mathcal{H}^i\iota^!\mathcal{F}$ to $S - T$ also vanishes. Thus (2) holds. For (1), we apply (2) to the dual $D\mathcal{F} = DR(D(M))$.

□

An object of $D^b_c(X)$ is a *perverse sheaf* if the above conditions hold. It follows from the theorem that the full subcategory of these is abelian. We will give a more direct argument later on.

2. Intersection homology

Basic examples of perverse sheaves come from intersection homology. Goresky and MacPherson modified the definition of homology, to obtain a theory with good properties on singular spaces. The idea was to place restrictions on the chains that met the singular set using a function called perversity. Note that their original approach was entirely geometric in the spirit of Lefschetz. One thing that was not obvious from this point of view was the topological invariance of the theory. This was solved later using sheaf theoretic methods.

Here is the set up. Fix an n dimensional pseudomanifold $X \supset X_{n-2} \supset X_{n-3}\ldots$, which means that

1. $X_i - X_{i+1}$ is a topological i-manifold.
2. $X - X_{n-2}$ is dense.
3. X is locally trivial along $X_i - X_{i+1}$ in the appropriate sense.

A perversity is a function from strata $\{X_i - X_{i-1}\}$ (or equivalently the labels $\{n-2, n-3\ldots\}$) to $\mathbb{Z}$. The case of interest for us is when X and the strata complex algebraic varieties and $p(c) = (c-2)/2$ is the so called middle perversity.

Fix a field k. Suppose that X is triangulated in such a way that $X_\bullet$ are subcomplexes. Let C^{-i} be the sheaf associated to the presheaf

$$U \mapsto \{k\text{-valued } i\text{-chains on } U\}$$

where we use possibly infinite, but locally finite, simplicial chains on a triangulation refining the initial one. This is becomes a complex of fine sheaves which realizes the dualizing complex D. Given a chain ξ, let $|\xi|$ denotes its support. Given a perversity p, let $IC_p^{-i}(U) \subset IC^{-i}(U)$ denote the chains ξ satisfying

$$\dim(|\xi| \cap X_{n-c}) \leq i - c + p(c)$$

$$\dim(|\partial\xi| \cap X_{n-c}) \leq i - 1 - c + p(c)$$

To help parse this, note that this looks like a transversality statement when $p(c)$ is omitted, so $p(c)$ measures the deviation from that. Intersection homology $IH_*^p(X,k) = H^{-*}(\Gamma(X, IC_p^*))$ by definition. By taking $p(c) \gg 0$, we see that this includes usual homology. Among other things, Goresky-MacPherson showed that when X is compact, Poincaré duality holds

$$IH_i^p(X,k)^* \cong IH_{n-i}^p(X,k)$$

This follows from Verdier duality once one observes that IC_p is self dual with respect to Verdier duality. Although this was not their original argument. After reindexing IC_p gives a basic example of a perverse sheaf.

3. The étale topology

Let X be a variety over a field k with characteristic p (possibly 0). Grothendieck realized that the Zariski topology is too coarse for many purposes, so he introduced the étale topology X_{et}. A map $Y \to X$ is étale if it is locally of the form $SpecA[x_1, \ldots, x_n]/(f_1, \ldots f_n) \to SpecA$ where the Jacobian $\det(\partial f_i/\partial x_j)$ is invertible. This is the analogue of a (finite) covering space. An "open set" of X_{et} is an étale map $U \to X$. A family of opens $\{U_i \to X\}$ is a covering if the images cover X. This is enough to make sheaf theory work. If $G \to X$ is an étale group scheme, then we can form the sheaf of sections on X_{et}. In particular, this remark applies to constant group schemes, leading to constant sheaves. If n is prime to p, then Grothendieck showed that $H^i(X_{et}, \mathbb{Z}/n)$ behaves like singular cohomology, and in fact, Artin showed that it coincides with it when $k = \mathbb{C}$. Most of the previous notions generalize. For example, a sheaf $\mathcal{F}$ on X_{et} is locally constant if $\mathcal{F}|_{U_i}$ is constant for some covering. These correspond to representations of the étale fundamental group $\pi_1^{et}(X)$.When X is normal,

$$\pi_1^{et}(X) = Gal(\bigcup\{k(X) \subset L \subset \overline{k(X)}\}/k(X))$$

where L varies over fields where the normalization of X in L is etale over X. A sheaf $\mathcal{F}$ is constructible if there is a Zariski locally closed partition for which the restrictions are locally constant. There exists analogues of $\mathbb{R}f_*, f^*$ etc. which preserve constructibility.

For various reasons, it is useful to have a theory which produces vector spaces over a field of characteristic 0. Just plugging in the field into $H^i(X_{et}, -)$ produces a bad theory, basically because the étale topology cannot "see" infinite sheeted covers. The trick is to approximate by finite coefficients and take a limit. Let ℓ be a prime different from p. Define

$$H^i(X_{et}, \mathbb{Q}_\ell) = \varprojlim_n H^i(X_{et}, \mathbb{Z}/\ell^n) \otimes \mathbb{Q}_\ell$$

To work with more general coefficients, define an ℓ-adic sheaf to be an inverse system $\ldots \mathcal{F}_n \to \mathcal{F}_{n-1} \ldots$ where each $\mathcal{F}_n$ is a constructible sheaf of $\mathbb{Z}/\ell^n$-modules, and the maps induce $\mathcal{F}_n \otimes \mathbb{Z}/\ell^{n-1} \cong \mathcal{F}_{n-1}$. There are a number of technicalities and subtleties that we will ignore. The upshot is that there is a version of the constructible derived category $D_c^n(X_{et}, \mathbb{Q}_\ell)$ in this setting, and it is stable under operations $\mathbb{R}f_*, f^*, \mathbb{D}$ etc. Thus we can define perverse sheaves of $\mathbb{Q}_\ell$ sheaves by imitating the above definition.

4. t-structures

Given $D = D^b(\mathcal{A})$, where $\mathcal{A}$ is abelian, set $D^{\geq n} = D^{\geq n}(\mathcal{A})$ (resp. $D^{\leq n} = D^{\leq n}(\mathcal{A})$) to be full subcategory of complexes such that $H^i(A) = 0$ unless $i \geq n$ (resp. $i \leq n$). This is the prototype of a t-structure. Then

TS1 If $A \in D^{\leq 0}$ and $B \in D^{\geq 1}, Hom(A, B) = 0$.
TS2 $D^{\leq 0} \subset D^{\leq 1}$ and $D^{\geq 0} \supset D^{\geq 1}$.
TS3 For any $A \in D$, there is a distinguished triangle

$$X \to A \to Y \to$$

with $X \in D^{\leq 0}$ and $Y \in D^{\geq 1}$.

To verify TS3, we use the truncation functors

$$X = \tau_{\leq 0} A = \dots A^{-1} \to \ker d^0 \to 0 \dots$$

$$Y = \tau_{\geq 1} A = A/\tau_{\leq 0}$$

For TS1, using triangles such as

$$\tau_{\leq -1} \to A \to H^0(A) \to$$

$$H^1(B) \to B \to \tau_{\geq 2} B \to$$

plus induction, we can assume that A and B are sheaves F and G translated to degree ≤ 0 and ≥ 1 respectively. Then

$$Hom(A, B) = Ext^i(F, G) = 0$$

since i will be negative.

A *t-structure* on a triangulated category D is a pair $(D^{\leq 0}, D^{\geq 0})$ satisfying TS1, TS2, TS3, where $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$. Although, we have only one example so far, we will shortly see that there are (non-obvious) perverse t-structures.

Proposition 4.1. *For any t-structure, the inclusion $D^{\leq n} \to D$ (resp $D^{\geq n} \to D$) admits a right (resp. left) adjoint $\tau_{\leq n}$ (resp. $\tau_{\geq n}$). Any object fits into a canonical distinguished triangle*

$$\tau_{\leq 0} A \to A \to \tau_{\geq 1} A \to$$

Proof. In outline, for each $A \in D$ choose a triangle as in TS3. Define $\tau_{\leq 0} A = X$. Observe that by TS1 and TS2

$$Hom(X', A) \cong Hom(X', \tau_{\leq 0} A) \tag{1}$$

for $X' \in D^{\leq 0}$. Thus given $A' \to A$, we get an induced morphism $\tau_{\leq 0} A' \to \tau_{\leq 0} A$, so this is a functor. Equation (1) shows this is the right adjoint to inclusion. The remaining cases are similar. □

Proposition 4.2. *Suppose that $a \leq b$. Then $\tau_{\leq a}\tau_{\leq b} \cong \tau_{\leq a}$, $\tau_{\geq b}\tau_{\geq a} \cong \tau_{\geq b}$, and $\tau_{\geq a}\tau_{\leq b} \cong \tau_{\leq b}\tau_{\geq a}$.*

Proof. The first two isomorphisms are routine, so we prove only the last. The map $\tau_{\le b}X \to \tau_{\ge a}X$, given as the composition $\tau_{\le b}X \to X \to \tau_{\ge a}X$, factors through $\tau_{\ge a}\tau_{\le b}X$. As $\tau_{\ge a}\tau_{\le b}X \in D^{\le b}$, we see that $\tau_{\ge a}\tau_{\le b}X \to \tau_{\ge a}X$ factors through $\tau_{\le b}\tau_{\ge a}X$. We have to show that this is an isomorphism.

Let Y fit into a distinguished triangle

$$\tau_{<a}X \to \tau_{\le b}X \to Y \to \tag{2}$$

we can use this along with

$$\tau_{\le b}X \to X \to \tau_{>b}X \to$$

to generate

$$\tau_{<a}X \to X \to \tau_{\ge a}X \to$$

and

$$Y \to \tau_{\ge a}X \to \tau_{>b}X \to \tag{3}$$

by T4. Since $\tau_{<a}X = \tau_{<a}\tau_{\le b}X$, (2) implies that $Y \cong \tau_{\ge a}\tau_{\le b}X$. And since $\tau_{>b}X = \tau_{>b}\tau_{\ge a}X$, we can conclude from (3) that $Y \cong \tau_{\le b}\tau_{\ge a}X$. □

The heart ("le coeur" in the original) of the t-structure is $D^{\le 0} \cap D^{\ge 0}$. For the standard t-structure on $D^b(\mathcal{A})$, we can identify the heart with $\mathcal{A}$ itself. Remarkably, the axioms lead to a similar structure in general.

Theorem 4.3 (Beilinson, Bernstein, Deligne)**.** *The heart is abelian.* $H^0 = \tau_{\le 0}\tau_{\ge 0}$ *is a cohomological functor from* D *to the heart, which means that it takes a triangle* $A \to B \to C \to$ *to a long exact sequence*

$$\dots H^{-1}C \to H^0(A) \to H^0(B) \to H^0(C) \to H^1(A)\dots$$

where $H^i(A) = H^0(A[i])$.

Proof. We prove the first statement that the heart $\mathcal{A} = D^{\le 0} \cap D^{\ge 0}$ is abelian. If $f : A \to B$ is a morphism in $\mathcal{A}$, we need to construct a kernel and cokernel. Extend this to a distinguished triangle

$$A \to B \to S \to$$

Then using

$$B \to S \to A[1] \to$$

we can see that $S \in D^{\le 0} \cap D^{\ge -1}$. It follows that $C = \tau_{\ge 0}S$ and $K = (\tau_{\le -1}S)[1]$ are in $\mathcal{A}$. We have a natural map $B = \tau_{\le 0}B \to C$ which we claim is the cokernel of f. To see this obverse that for any $X \in \mathcal{A}$ we have an exact sequence

$$Hom(A[-1], X) \to Hom(S, X) \to Hom(B, X) \to Hom(A, X)$$

$$Hom(A[-1], X) = 0$$

by the axioms. Also

$$Hom(S, X) = Hom(\tau_{\ge 0}S, X)$$

Thus

$$0 \to Hom(C, X) \to Hom(B, X) \to Hom(A, X)$$

is exact, and this proves the that C is the cokernel. The proof that $K \to A$, induced from $S[-1] \to A$, is the kernel of f is similar.

The final step is to show that the image $im(f) = \ker(B \to C)$ is isomorphic to the coimage $coim(f) = coker(K \to A)$. Using T4, we can use

$$\begin{tikzcd}
C \arrow[dd, squiggly] & & B \arrow[ll] \arrow[dl] \\
& S \arrow[ul] \arrow[dr, squiggly] & \\
K[1] \arrow[ur] \arrow[rr, squiggly] & & A \arrow[uu]
\end{tikzcd}$$

to build

$$\begin{tikzcd}
C \arrow[dd, squiggly] \arrow[dr, squiggly] & & B \arrow[ll] \\
& I \arrow[ur] \arrow[dl] & \\
K[1] \arrow[rr, squiggly] & & A \arrow[uu] \arrow[ul]
\end{tikzcd}$$

Using the upper triangle in the second diagram, we see that $I \cong im(f)$. The bottom triangle shows that $I \cong coim(f)$. $\square$

Remark 4.4. *This does* **not** *say that D is the derived category of its heart. This not always true.*

5. Perverse t-structure

Let X be a variety over a field k, and let $D^b_c(X)$ denote either $D^b_c(X_{an}, \mathbb{Q})$ if $k = \mathbb{C}$ or $D^b_c(X_{et}, \mathbb{Q}_\ell)$ To begin with we fix a stratification $\mathcal{S}$ on an algebraic variety X, consisting smooth locally closed subvarieties such that $\bar{S}$ is union of strata whenever $S \in \mathcal{S}$. For example, $\mathcal{S}$ may be a Whitney stratification when $k = \mathbb{C}$. Let $\iota_S : S \to X$ denote the inclusions of strata. Define ${}^pD^{\leq 0}_{\mathcal{S}}(X)$ (resp. ${}^pD^{\geq 0}_{\mathcal{S}}(X)$) to be the subcategory of complexes $K \subset D^b_c(X)$ such that $H^a(\iota_S^* K) = 0$ for $a > -\dim S$ (resp. $H^a(\iota^!_S K) = 0$ for $a < -\dim S$ for each $S \in \mathcal{S}$.

Theorem 5.1. *$({}^pD^{\leq 0}_{\mathcal{S}}(X), {}^pD^{\geq 0}_{\mathcal{S}}(X))$ forms a t-structure.*

The theorem is proved by induction on the number of strata. For $X = S$ consisting of a single stratum, we can see that

$$ {}^pD^{\leq 0}_{\mathcal{S}}(X) = D^{\leq -\dim X}_{\mathcal{S}}(X)$$
$$ {}^pD^{\geq 0}(X) = D^{\geq -\dim X}_{\mathcal{S}}(X)$$

where the right side has the usual meaning. So it is just the usual t-structure translated by $-\dim X$), and therefore a t-structure in its own right.

The inductive step is supplied by

Proposition 5.2. *Let $j : U \hookrightarrow X$ be open and $i : Z = X - U \hookrightarrow X$ be the complement. Given t-structures $D^{\leq 0, \geq 0}_U \subset D^+(U)$ and $D^{\leq 0, \geq 0}_Z \subset D^+(Z)$. Then*

$$D^{\leq 0} = \{K \in D^+(X) \mid j^*K \in D^{\leq 0}_U, i^*K \in D^{\leq 0}_Z\}$$
$$D^{\geq 0} = \{K \in D^+(X) \mid j^*K \in D^{\geq 0}_U, i^!K \in D^{\geq 0}_Z\}$$

determines a t-structure on $D^+(X)$.

Sketch. Suppose that $A \in A^{\leq 0}$ and $B \in D^{\geq 1}$, we have to show that $Hom(A, B) = 0$. Observe that there is an exact sequence

$$Hom(i_* i^* A, B) \to Hom(A, B) \to Hom(j_! j^* A, B)$$

The first term $Hom(i_* i^* A, B) = Hom(i^* A, i^* B) = 0$ and second $Hom(j_! j^* A, B) = Hom(j^* A, j^* B) = 0$ by adjointness. The second axiom $D^{\leq 0} \subset D^{\leq 1}$ etc. holds because it holds on U and Z. The final axiom is the only one that takes a bit of work. See [BBD]. □

To get rid of the dependence on $\mathcal{S}$, we take the direct limit ${}^p D^{\leq 0}(X) = \lim_{\mathcal{S}} {}^p D^{\leq 0}_{\mathcal{S}}(X)$ etc.. This again yields a t-structure. As a first corollary, we see that the heart of this t-structure $P(X) = {}^p D^{\leq 0}(X) \cap {}^p D^{\geq 0}(X)$ is therefore an abelian category called the category of *perverse sheaves*. As a second corollary, we obtain perverse truncation functors ${}^p\tau_{\leq 0} : D^b_c(X) \to {}^p D^{\leq 0}(X)$, ${}^p\tau_{\geq 0} : D^b_c(X) \to {}^p D^{\geq 0}(X)$, and a cohomological functor ${}^p H^0 = {}^p\tau_{\leq 0} {}^p\tau_{\geq 0}$ from $D^b_c(X) \to P(X)$. We define perverse variant of standard functors T by ${}^p T = {}^p H^0 T$. This applies in particular to j_* and $j_!$ for an open immersion $j : U \to X$. Given $\mathcal{F} \in P(U)$, there is a canonical map ${}^p j_! \mathcal{F} \to {}^p j_* \mathcal{F}$, we define

$$j_{!*}\mathcal{F} = im({}^p j_! \mathcal{F} \to {}^p j_* \mathcal{F}) \in P(X)$$

After one unravels the definition a bit, it can be shown that when this is applied to the constant sheaf, one recovers the intersection cohomology complex of Goresky-MacPherson.

References

[BBD] Beilinson, Bernstein, Deligne, *Faisceux Pervers* (1982)

[dM] M. de Cataldo, L. Migliorini, *The decomposition theorem, perverse sheaves, and the topology of algebraic maps,* BAMS 2009

[Go] Goresky, MacPherson, *Intersection homology II*, Inventiones (1983)

[KW] Kiehl, Weissauer, *Weil conjectures, perverse sheaves, and the ℓ-adic Fourier transform*, Springer (2001)