SUPPLEMENT FOR M341

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1. Max/Min

A closed interval is a set $[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$. This is bounded if a and b are finite The goal is to outline a proof that continuous functions on closed bounded intervals have maxima and minima. The argument here is a bit more direct than the one in the book [A] but it's essentially the same.

Lemma 1.1. *If $S \subset \mathbb{R}$ is bounded above, then $\sup S$ is the limit of some sequence $x_n \in S$.*

Proof. This will be left as an exercise. Hint: choose $x_n > \sup S - \frac{1}{n}$. $\square$

The next result says that closed bounded intervals are compact.

Lemma 1.2. *Any sequence in a closed bounded interval contains a subsequence which converges to an element of the interval.*

Proof. By the Bolzano-Weierstrass theorem, any sequence $x_n \in [a,b]$ contains a convergent subsequence. Say the limit is c. We have to show that $a \le c \le b$. But this follows from the the order limit theorem in the book [A, p 48]. To see this directly, suppose either $c > b$ or $c < a$. In the first case, set $\epsilon = (c-b)/2$. We see that an infinite number of points of x_n would have to lie within distance ϵ from c, and therefore outside $[a,b]$. This contradicts what we said initially. So $c \le b$. The argument that $c \ge a$ is similar. $\square$

Theorem 1.3. *A continuous function on a closed bounded interval must have a maximum and a minimum.*

Proof. Let f be continuous on $[a,b]$. Consider the set

$$f([a,b]) = \{y \in \mathbb{R} \mid \exists x \in [a,b] \text{ such that } y = f(x)\}$$

Suppose that this has no upper bound. Then there exists a sequence $y_n \in f([a,b])$ such that $y_n \to \infty$, which means that for each $r \in \mathbb{R}$, $y_n > r$ for all but finitely n. We can write $y = f(x_n)$ for some $x_n \in [a,b]$. The second lemma implies that after replacing it by a subsequence, we can asssume that x_n converges to some $c \in [a,b]$. Since f is continuous, this means that $y_n = f(x_n)$ converges to $f(c)$. But this contradicts the fact that $y_n \to \infty$. Therefore $f([a,b])$ is bounded above.

Set $M = \sup(f([a,b])$. If we can prove that $M \in f([a,b])$, then this will be the maximum value. By the first lemma, we have a sequence $y_n \in f([a,b])$ converging to M. Choose that $x_n \in [a,b]$ so that $y_n = f(x_n)$. By passing to a subsequence as above, we can assume that x_n converges to some value $c \in [a,b]$. Once again, we see that $y_n \to f(c)$. Therefore $f(c) = M$, which is what we wanted.

We have proved half of the theorem. To get the other half, observe that a maximum of $-f(x)$ corresponds to a minimum of $f(x)$. $\square$

2. Intermediate value theorem

We give a proof of the intermediate value theorem which is (paradoxically) both more elementary and more complicated than the one in the book [A]. It is based on the method of bisection.

Theorem 2.1. *If* $f : [a, b] \to \mathbb{R}$ *is continuous, and if* L *is a number such that either* $f(a) < L < f(b)$ *or* $f(a) > L > f(b)$ *holds, then there is a* $c \in [a, b]$ *such that* $f(c) = L$.

Proof. By replacing $f(x)$ by $\pm(f(x) - L)$, we can see that it is enough to prove the special case where $L = 0$ and $f(a) < 0 < f(b)$. Set $[a_1, b_1] = [a, b]$. Consider the midpoint $m_1 = (a + b)/2$. If $f(m_1) = 0$ we are done. If $f(m_1) < 0$, set $[a_2, b_2] = [m_1, b]$, otherwise set $[a_2, b_2] = [a, m_1]$. Choose the midpoint m_2 of $[a_2, b_2]$ and proceed as above, and continue. Then eventually we either arrive at point m_n with $f(m_n) = 0$, or we obtain an infinite nested sequence

$$[a_1, b_1] \supset [a_2, b_2] \supset \dots$$

such that $(b_n - a_n) = (b - a)/2^n$, $f(a_n) < 0$ and $f(b_n) > 0$ Since $a_m \in [a_n, b_n]$ for $m \geq n$, it follows that $|a_n - a_m| < \epsilon$ when $m \geq n$ and n is chosen so that $(b - a)/2^n < \epsilon$. Therefore a_n is Cauchy. So it converges to number we will call A. As f is continuous, $f(a_n) \to f(A)$. Since all $f(a_n) < 0$, we must have $f(A) \leq 0$. This is follows from the order limit theorem in the book [A, p 48].

By a similar argument, b_n converges to a number B, such that $f(B) \geq 0$. Notice that for any $\epsilon > 0$, we have

$$\begin{aligned} |B - A| &= |(B - b_n) + (b_n - a_n) + (a_n - A)| \\ &\leq |B - b_n| + |b_n - a_n| + |a_n - A| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

for some n. This implies that the distance between A and B can be made as small we like. This can only happen if $A = B$. Since $f(A) = f(B)$ is both greater than or equal to 0, and less than or equal to 0, we must have $f(A) = 0$.

References

[A] Abbott, Understanding Analysis

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