Chapter 4

Basic results on morphisms

4.1 Graphs

We saw that we can form products of quasiprojective varieties, and in particular $X \times X$, if X is quasiprojective.

Lemma 4.1.1. *If X is a quasiprojective variety, then the diagonal $\Delta_X = \{(x, x) \mid x \in X\} \subset X \times X$ is closed.*

Proof. When $X = \mathbb{P}^n$, this was a done a previous exercise. In general, embed $X \subset \mathbb{P}^n$ as a locally closed set, then

$$\Delta_X = \Delta_{\mathbb{P}^n} \cap X \times X$$

must be closed. □

If the topology of $X \times X$ were the product topology (although it isn't), this would be equivalent to the Hausdorff axiom. So we would really view this as an analogue. Given a morphism $f : X \to Y$, the graph is

$$\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\}$$

Lemma 4.1.2. *$\Gamma_f \subset X \times Y$ is closed.*

Proof. Consider the map $F : X \times Y \to Y \times Y$ given by $F(x, y) = (x, f(y))$. This is regular, and therefore continuous. The lemma follows because $\Gamma_f = F^{-1}\Delta_Y$. □

Corollary 4.1.3. *If $f_1, f_2 : X \to Y$ are two regular maps which agree on a nonempty open set, then they agree everywhere.*

Proof. Let $U \subseteq X$ be the subset in question, then

$$\Gamma_{f_1} = \overline{\Gamma_{f_1|_U}} = \overline{\Gamma_{f_2|_U}} = \Gamma_{f_2}$$

□

4.2 Completeness of projective varieties

Let $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n_k$ denote the natural projection. When $k = \mathbb{C}$, $\mathbb{C}^{n+1} - \{0\}$ has a classical topology, and so $\mathbb{P}^n_{\mathbb{C}} = \mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*$ can be given the quotient topology, which means that set is open or closed if and only if its preimage is open or closed. Denote this topological space $\mathbb{P}^n_{\mathbb{C},cl}$.

Lemma 4.2.1. *$\mathbb{P}^n_{\mathbb{C},cl}$ is compact Hausdorff.*

Proof. This follows from the fact that unit sphere $S^{2n-1} \subset \mathbb{C}^{n+1} - \{0\}$, is compact Hausdorff, and $\mathbb{P}^n_{\mathbb{C},cl} = \pi(S^{2n-1})$. □

Corollary 4.2.2. *If $f : \mathbb{P}^n_{\mathbb{C},cl} \to Y$ is a continuous map to another Hausdorff space, then f is closed, i.e. it takes closed sets to closed sets.*

Returning to pure algebraic geometry, we can ask if there is an analogous property. In fact, the Zariski topology for $\mathbb{P}^n$ is compact (by a previous exercise) but this is a rather weak statement. It's more useful to look for an analogue of the corollary.

Theorem 4.2.3. *A regular map $f : \mathbb{P}^n_k \to Y$ to another quasiprojective variety is closed.*

In fact, we will deduce this from a stronger result.

Theorem 4.2.4. *For any quasiprojective variety Y, the projection $p : \mathbb{P}^n_k \times Y \to Y$ is closed.*

Proof. Let $X \subset \mathbb{P}^n_k \times Y$ be a closed set, we have to prove that $p(X)$ is closed or equivalently that the complement is open. If Y_i is an affine open cover, then $Y - p(X) = \bigcup(Y_i - p(X))$. So it is enough to prove that each $Y_i - p(X)$ is open. This allows us to reduce to the case where $Y = Y_i$ is affine, and to the case where $Y = \mathbb{A}^m$ Let $x_0, \ldots, x_n$ denote homogeneous coordinates on $\mathbb{P}^n$ and $y_1, \ldots, y_m$ coordinates on $\mathbb{A}^m$. Then there exists a finite set of homogenous polynomials $f_i(x, y) \in R = k[x_0, \ldots, y_1, \ldots]$ which are homogenous in the x's such that X is the zero set of these polynomials. If $y = b \in \mathbb{A}^m$, then the preimage $p^{-1}(b) = V_{\mathbb{P}}(I_b)$ where $I_b = (f_1(x, b), \ldots)$. We have

$$\begin{aligned} Y - p(X) &= \{b \in Y \mid p^{-1}(b) = \emptyset\} \\ &= \{b \in Y \mid \sqrt{I_b} \supseteq (x_0, \ldots, x_n)\} \\ &= \bigcup_N \{b \in Y \mid I_b \supseteq (x_0, \ldots, x_n)^N\}, \end{aligned}$$

with the help of the Nullstellensatz. So it suffices to prove that

$$Y_N = \{b \in Y \mid I_b \supsetneq (x_0, \ldots, x_n)^N\}$$

is closed. To make this more explicit, suppose that f_i has degree d_i in the x's. Let $S_N \subset k[x_0, \ldots x_n]$ denote the subspace of homogenous polynomials of degree N. We have a linear map

$$\bigoplus S_{N-d_i} \xrightarrow{M_N(b)} S_N$$

where $M_N(b) = \sum f_i(x, b)$. After choosing bases for S_N, $M_N(b)$ can be represented by a matrix which is polynomial in b. Therefore

$$Y_N = \{b \in Y \mid \operatorname{rank} M(b) < \dim S_N\}$$

is the zero set of the $(\operatorname{rank} S_N)^2$ minors. So it is closed. □

The next corollary is really the same result restated explicitly in terms of solving equations.

Corollary 4.2.5 (The fundamental theorem of elimination theory)**.** *Given finitely many polynomials $f_i(x, y)$ which are homogenous in the x's, there exist finitely many polynomials $g_j(y)$ depending on the f's, so that the system*

$$f_1(x, b) = 0, f_2(x, b) = 0, \ldots$$

has a nonzero solution if and only if

$$g_1(b) = 0, g_2(b) = 0, \ldots$$

We can deduce theorem 4.2.3 in a slight stronger form.

Corollary 4.2.6. *A regular map $f : X \to Y$ from a projective variety to a quasiprojective variety is closed.*

Proof. Embedd $X \subset \mathbb{P}^n$ as a closed set. For any closed subset $Z \subset X \subset \mathbb{P}^n$, $\Gamma_{f|Z}$ is closed. Therefore $f(Z) = p(\Gamma_{f|Z})$ is closed. □

Corollary 4.2.7. *Any regular function on a projective variety is constant.*

Proof. Suppose that X is projective, and $f : X \to \mathbb{A}^1$ is regular. Then $f(X)$ is closed, so it is either finite or all of $\mathbb{A}^1$. If it is a finite, then it has to be single point, otherwise X would not be irreducible. If $f(X) = \mathbb{A}^1$, the image of the composite $X \to \mathbb{A}^1 \subset \mathbb{P}^1$ would not be closed, so this case is impossible.

□

4.3 Dimension of fibres

Dimensions came up in passing earlier. Now we want to study it more seriously now. Given a noetherian topological space X, the dimension $\dim X \in \mathbb{N} \cup \{\infty\}$, is $\sup n$ for chains of irreducible closed sets

$$X = X_0 \supsetneqq X_1 \supsetneqq \ldots X_n$$

We say that X has pure dimension n if all its irreducible components have dimension n. One sees immediately that when X is affine, $\dim X$ is the same thing as $\sup n$ for all chains of prime ideals

$$p_0 \subsetneqq p_1 \subsetneqq \ldots p_n \subset \mathcal{O}(X)$$

and this is precisely the Krull dimension $\dim \mathcal{O}(X)$. From commutative algebra, we have the following useful fact.

Theorem 4.3.1 (Krull's principal ideal theorem)**.** *If R is a noetherian ring, and p is a minimal prime of a proper ideal $I = (f_1, \dots, f_r)$ then $\dim R/p \geq R - r$. If $r = 1$ and f_1 is not a zero divisor, $\dim R/p = \dim R - 1$.*

Corollary 4.3.2. *If $X \subset \mathbb{A}^n$ is an irreducible component of $V(f)$, with $f \neq 0$, then it has pure dimension $n - 1$.*

Suppose that X is an affine variety. By Noether's normalization theorem, we can find an integral extension

$$k[x_1, \dots, x_n] \subset \mathcal{O}(X)$$

or geometric terms a finite map

$$X \to \mathbb{A}^n_k.$$

This implies that

Theorem 4.3.3. *When X is an affine variety, $\dim X$ is finite, and coincides with transcendence degree of the function field $K(X)$ (= fraction field of $\mathcal{O}(X)$).*

We can define the function field of any quasiprojective variety X, as $K(X) = K(U)$, for some (or any) nonempty affine open U. The above theorem applies for general X as well. This means that as far as dimensions are concerned, we can just work with affine varieties.

Corollary 4.3.4. *If $f : X \to Y$ is a dominant regular map, $\dim Y \leq \dim X$.*

Proof. We can replace X and Y be affine opens so that $f(X) \subset Y$. The map is still dominant, so that $\mathcal{O}(Y) \to \mathcal{O}(X)$ is injective. Therefore we have an inclusion of fields $K(Y) \subset K(X)$. This implies the inequality. □

Theorem 4.3.5. *Suppose that $f : X \to Y$ is a dominant regular map of varieties, and let $r = \dim X - \dim Y$.*

(a) For any $y \in f(X)$, $\dim f^{-1}(y) \geq r$.

(b) There is exists an nonempty open subset $U \subset f(X)$ such that for $y \in U$, $\dim f^{-1}(y)$ has pure dimension r.

One should understand the theorem as saying that most fibres have expected dimension, and fibres have at least that.

Proof. We first reduce to the case where X and Y are affine. We apply Noether's normalization theorem to obtain a finite map $\pi : Y \to \mathbb{A}^n$. Let $\pi(y) = (a_1, \dots, a_n)$. Then any irreducible component of $f^{-1}(y)$ is the zero set of minimal prime of $((\pi \circ f)^* x_1 - a_1, \dots, (\pi \circ f)^* x_n - a_n)$. So (a) follows from Krull's theorem.

By proposition 2.3.10 $f(X)$ contains an open subset of $U \subset f(X)$. The proof of that proposition in fact gives much more, namely $U = D(b)$ for some $b \in \mathcal{O}(Y)$, and also that we have an integral extension

$$\mathcal{O}(Y)[1/b] \subset \mathcal{O}(Y)[1/b][x_1, \dots, x_m] \subset \mathcal{O}(X)[1/b]$$

In geometric terms, we can factor the restriction of f through

$$f^{-1}(U) \to \mathbb{A}^m \times U \to U$$

where the first map is finite, and the second is the projection. In particular, this implies that $m = r$. We also can see that any irreducible component of $f^{-1}(y)$ is finite over $\mathbb{A}^r \times \{y\} \cong \mathbb{A}^r$. So it must also have dimension r. This proves (b). $\square$

Corollary 4.3.6. $\dim X \times Y = \dim X + \dim Y$.

4.4 Examples/Exercises

A hypersurface $X \subset \mathbb{P}^n$ (of degree d) is given by $X = V_{\mathbb{P}}(f)$, where f is a nonzero (degree d) polynomial. We won't insist that X is irreducible. Theorem 4.3.1 shows that X is pure of dimension $n-1$. The converse is also true[1]. Since $X = V_{\mathbb{P}}(\lambda f)$ for any $\lambda \neq 0$, the set of projective hypersurfaces degree d is parameterized by the projective space $\mathbb{P}S_d$ of homogeneous degree d polynomials, which has dimension $\binom{n+d}{n} - 1$. When $d = 1$, we get another copy of $\mathbb{P}^n$, called the dual projective space $\mathbb{P}S_1 = \check{\mathbb{P}}^n$. Of course, $\mathbb{P}^n \cong \check{\mathbb{P}}^n$, but the isomorphism isn't natural.

Example 4.4.1. *Let $F = F(1, n-1; n) = \{(p, H) \in \mathbb{P}^n \times \check{\mathbb{P}}^n \mid p \in H\}$. This and generalizations considered later are called flag varieties. In coordinates $([x_0, \ldots, y_0], [y_0, \ldots, y_n]) \in F$, if and only if $\sum x_i y_i = 0$, so it is a variety. One can also see that the projections π_i to $\mathbb{P}^n$ and $\check{\mathbb{P}}^n$ are regular and surjective. Given $p = [a_0, \ldots, a_n]$, $\pi_1^{-1}(p)$ is defined by $\sum a_i y_i = 0$. So each fibre has dimension $n-1$. Therefore $\dim F = n + (n-1) = 2n - 1$ by theorem 4.3.5.*

Example 4.4.2. *The last example generalizes to an incidence variety $I = \{(p, X) \in \mathbb{P}^n \times \mathbb{P}S_d \mid p \in X\}$. We have $\pi_2^{-1}(X) = X$. Therefore $\dim F = \binom{n+d}{n} - 1 + (n-1)$.*

A point $p \in V_{\mathbb{P}}(f)$ is called singular if all the derivatives $\partial f / \partial x_i$ vanish at p. A hypersurface is called nonsingular if has no singular points.

Example 4.4.3. *It's easy to see that*

$$Z = \{(p, X) \in \mathbb{P}^n \times \mathbb{P}S_d \mid p \text{ a singular point of } X\}$$

is closed in I from above. Thus theorem 4.2.4 shows that the set of singular hypersurfaces $Sing = \pi_2(Z)$ is closed in $\mathbb{P}S_d$

Let us study the set $Sing$ in more detail. Let us suppose, for simplicity, that $char\, k$ does not divide d (this is not essential). Then the Fermat hypersurface $x_0^d + x_1^d + \ldots x_n^d = 0$ is nonsingular. This implies that the complement of $Sing$ is nonempty. Therefore the codimension $codim\, Sing := \dim \mathbb{P}S_d - \dim Sing > 0$. We claim in fact that

[1] To see this, use the fact that $k[x_0, \ldots x_n]$ is a UFD, and that any height one prime in a UFD is principal.

Lemma 4.4.4. *codim Sing* $= 1$.

Proof. Let $N = \binom{n+d}{n} - 1 = \dim S_d$. We calculate $\dim Z$ using the projection $\pi_1 : Z \to \mathbb{P}^n$. The fibre $\pi_1^{-1}(p)$ is the set of degree d hypersurfaces containing p as a singular point. To make things more explicit, let us work with $p = [1, 0, \ldots, 0]$. For the end result, the choice won't matter. Let $f \in k[x_1, \ldots x_n]$ be dehomogenized equation of such a hypersurface. We can write $f = \sum f_i$ as a sum of homogeneous polynomials. The condition $f^H(p) = 0$ is equivalent to $f(0) = 0$, or that the constant term $f_0 = 0$. The vanishing of the derivatives of f^H, translates into the vanishing of the derivatives of f, and this implies that $f_1 = 0$. This means that fibre is given by $n + 1$ linearly independent linear equations. So it is isomorphic to $\mathbb{P}^{N-n-1}$. It follows that $\dim Z = n + N - n - 1 = N - 1$. We want to show that $\dim \pi_2(Z) = N - 1$. To do this, it is enough to show that some fibre is zero dimensional. But this is clear. $X = V_{\mathbb{P}}(x_1^d + \ldots x_n^d) \subset \mathbb{P}^n$ is a hypersurface with exactly one singularity at $p = [1, 0 \ldots, 0]$. Therefore $\pi_2^{-1}(X) = p$. $\square$

Exercise 4.4.5.

1. *Earlier, in example 1.3.5, we saw that*

$$E_1 = \{(A, v) \in \mathbb{A}^{n^2} \times \mathbb{A}^n \mid v \text{ is an eigenvector of } A\}$$

was closed. Use the projection to $p_1 : E_1 \to \mathbb{A}^{n^2}$ *to compute* $\dim E_1$.

2. *Calculate the dimension of the fibres of* $p_2 : E_1 \to \mathbb{A}^n$. *If it looks too messy, just do it for* $n = 2$.

3. *Show that the set*

$$E = \{(A, B, v) \in \mathbb{A}^{n^2} \times \mathbb{A}^{n^2} \times k^n \mid v \text{ is an eigenvector for } A\&B\}$$

is closed and conical in each variable in the sense that $(A, B, v) \in E$ *implies* $(\alpha A, \beta B, \gamma v) \in E$ *for scalars* $\alpha, \beta, \gamma \in k$. *Conclude that the image of* E *in* $\mathbb{P}^{n^2-1} \times \mathbb{P}^{n^2-1} \times \mathbb{P}^{n-1}$ *is closed.*

4. *Use the previous exercise to show that the* $(A, B) \in \mathbb{A}^{n^2} \times \mathbb{A}^{n^2}$ *such that* A *and* B *admit a simultaneous eigenvector is a Zariski closed set. Show by example, that it's a proper subset when* $n = 2$.

5. *Show that in characteristic* $p > 0$, $x^{p-1}y + y^{p-1}z + z^{p-1}x = 0$ *is nonsingular hypersurface in* $\mathbb{P}^2$.