Chapter 8

Grassmanians and Applications

8.1 The functor of points for projective space.

Recall $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[x_0, \dots, x_n]$, and for a field $\mathbb{P}^n_{\mathbb{Z}}(k)$ is just the set of one dimensional subspaces of k^{n+1}. For a more general ring, the story is more complicated.

Theorem 8.1.1. *Given a finitely presented module M over a commutative ring R, the following are equivalent*

(a) M_p is free (of rank d) for every $p \in \operatorname{Spec} R$.

(b) There exists an open cover $\{D(f_i)\}$ of $\operatorname{Spec} R$ such that $M[1/f_i]$ is free (of rank d)

(c) M is a projective, i.e. a direct summand of a free module of finite rank.

Proof. The equivalence of (a) and (b) is straightforward. For (a) $\Leftrightarrow$ (c), see Matsumura Comm. Alg. p 21. □

If these conditions hold, M is called locally free (of rank d) Free modules are locally free, but the converse need not hold (see exercises).

Lemma 8.1.2. $\mathbb{P}^n_{\mathbb{Z}}(R)$ *is the set of submodules of R^n, which are locally free of rank* 1.

Sketch. We recall that $\mathbb{P}^n$ is covered by affine schemes $U_i = D_+(x_i) \cong \mathbb{A}^n_{\mathbb{Z}}$. To give an element of $\mathbb{P}^n_{\mathbb{Z}}(R)$ amounts to giving a covering $D(r_i)$ of $\operatorname{Spec} R$ and a compatible family of morphisms $f_i : D(r_i) \to U_i$. f_i can be viewed as a vector in $R[1/r_i]^{n+1}$, and compatibility means that f_i and f_j both generate the same submodule of $R[1/r_i r_j]^{n+1}$. Then $M = \langle r_1 f, r_2 f_2 \dots \rangle \subset R^{n+1}$ is locally free of rank 1.

Conversely, given a rank one locally free $M \subset R^{n+1}$, we can find a finite set of $r_i \in R$ such that $M[1/r_i]$ is free and generated by a vector whose ith entry is invertible. Then we use these vectors to construct a compatible family of maps from $D(r_i) \to U_i$. □

We can generalize this.

Theorem 8.1.3. *There exists a scheme $\mathbb{G}_{\mathbb{Z}} = \mathbb{G}_{\mathbb{Z}}(d, n)$ over $\operatorname{Spec} \mathbb{Z}$, such that $\mathbb{G}_{\mathbb{Z}}(R)$ is the set of submodules of R^n, which are locally free of rank d.*

8.2 Grassmanians

Rather than proving theorem 8.2.4 in the generality that it's stated, we do it in classical setting of quasiprojective varieties over an algebraically closed field k. Given the vector space $V = k^n$, we define the Grassmanian

$$\mathbb{G}_k(d, n) = \{W \subset V \mid \dim W = d\}$$

When $d = 1$, this is just $\mathbb{P}_k^{n-1}$. In general, $\mathbb{G}_k(d, n)$ can be identified with the set of $d-1$ dimensional linear subspaces of $\mathbb{P}^{n-1}$, so many people prefer to call this $\mathbb{G}(d-1, n-1)$ or something similar.

Right now $\mathbb{G} = \mathbb{G}_k(d, n)$ is just a set. We can give it more structure by describing it in more explicit terms. To give $W \in \mathbb{G}$, it is enough to give an ordered basis $v_1, \ldots, v_d$. The $n \times d$ matrix $A = [v_1 \ldots v_d]$ lies in the set $R(d, n)$ of $n \times d$ matrices of rank d. $R(d, n)$ in open is the affine space of all matrices, so it's quasiprojective. The group $GL_n(k)$ acts on this by left multiplication, and $GL_d(k)$ by right multiplication. The following is straightforward.

Lemma 8.2.1.

(a) The left action on $R(d, n)$ is easily seen to be transitive.

(b) The right action is transitive on the fibres of the projection $R(d, n) \to \mathbb{G}$. Therefore $\mathbb{G}$ is the orbit space $R(d, n)/GL_d(k)$.

Given $M \in R(d, n)$, the Plücker vector $\mathrm{pl}(M)$ is the vector of $d \times d$ minors of M (in some chosen order). This is a nonzero vector of length $N = \binom{n}{d}$. We can express this in more coordinate free language by introducing exterior powers. Recall that the tensor algebra is the noncommutative graded algebra

$$T^*V = k \oplus V \oplus (V \otimes V) \oplus \ldots$$

The exterior algebra $\wedge^* V$ is a quotient[1] of this by the two sided ideal generated by $\{v \otimes v \mid v \in V\}$. The product in this algebra is denote by $\wedge$. It satisfies $v \wedge v = 0$, and consequently $u \wedge v = -v \wedge u$. This algebra has a natural grading,

[1] If you learned this in a differential geometry class, then you might have seen this defined as the *subspace of antisymmetric multilinear forms on V^**. This works over a field of characteristic 0, but not in general.

where the $\wedge^d V$ is generated by $v_{i_1} \wedge v_{i_2} \wedge \dots v_{i_d}$. In fact, if $e_1, \dots, e_n$ is the standard basis of V, products $e_{i_1} \wedge e_{i_2} \wedge \dots e_{i_d}$ with $i_1 < \dots i_d$ form a basis of $\wedge^d V$. From this point of view,

$$\text{pl}([v_1, \dots, v_d]) = v_1 \wedge \dots \wedge v_d$$

We can see this by expanding out this expression in the above basis. These constructions a functorial. In particular, given a matrix $A \in GL_n(k)$, it acts naturally on $\wedge^d V \cong V^N$. So that we have a homomorphism (of algebraic groups) $\rho_d : GL_d(k) \to GL_N(k)$.

Lemma 8.2.2.

(a) $\text{pl}(MB) = \det(B)\,\text{pl}(M)$*, for* $B \in GL_d(k)$

(b) $\text{pl}(AM) = \rho_d(A)\,\text{pl}(M)$*, for* $A \in GL_n(k)$

(c) pl *induces an injective map from* $\mathbb{G} \to \mathbb{P}^{N-1}$.

Proof. (a) and (b) are easy to see using the first and second definitions of pl respectively. So we focus on (c). First of all, by (a), pl gives a well defined map $\mathbb{G} \to \mathbb{P}^{N-1}$, which we will denote by $\overline{\text{pl}}$ (but eventually just pl). We just have to prove injectivity.

Suppose that $\overline{\text{pl}}(M) = \overline{\text{pl}}(M')$, then we have to show that $M' = MB$ for some $B \in GL_d(k)$. By (b), and transitivity of the $GL_n(k)$-action, we may take

$$A = [e_1 e_2 \dots e_d] = \begin{pmatrix} I \\ 0 \end{pmatrix}$$

All $d \times d$ minors of A except for the topmost are zero, and the same goes for A'. Write

$$A' = \begin{pmatrix} B \\ C \end{pmatrix}$$

where B is $d \times d$. Then B is invertible, so replacing A' by $A'B^{-1}$ allows us to take $B = I$. Since all but the topmost minors vanish, we can see that C must be 0. □

Theorem 8.2.3. *With previous notation, the image of* $\mathbb{G}$ *in* $\mathbb{P}^{N-1}$ *is closed.*

Proof. We identify $\mathbb{P}^{N-1}$ with the projective space associated to $\wedge^d V$ as above. The condition for an element $[w]$ to lie in $\mathbb{G}$ is that $w = v_1 \wedge \dots v_d$ for some linearly independent $v_i \in V$. One says that w is decomposable. Observe that if such v_i exist, then

$$w \wedge : V \to \wedge^{d+1} V$$

must have rank $n - d$.

Conversely, suppose that the rank of the map $w \wedge -$ is less than or equal to $n - d$, then we could find at a least d linearly independent elements v_i such that $w \wedge v_i = 0$. We extend these elements a basis, and express

$$w = \sum a_{i_1 \dots i_d} v_{i_1} \wedge \dots v_{i_d}$$

Expanding $w \wedge v_1 = 0$, we can see that it implies that $w = w_1 \wedge v_1$ for some $w_1 \in \wedge^{d-1} V$. The condition $w \wedge v_2 = w_1 \wedge v_1 \wedge v_2 = 0$ can be seen to implies that $w_1 = w_2 \wedge v_2$ for some w_2. Continuing in this way, we see that $w = v_1 \wedge \ldots v_d$. This proves that $[w] \in \mathbb{G}$ if and only if $\text{rank}(w \wedge -) \leq n - d$. This rank condition is given by the vanishing of $(n-d+1) \times (n-d+1)$ minors of a matrix representing $w \wedge -$. This shows that $\mathbb{G}$ is algebraic. □

With more work, one can get an explicit set of equations for $\mathbb{G}$ called Plücker equations. We will give a special case later on. But first, we find the dimension. As a first step observe that the map $\text{pl} : R(d,n) \to \mathbb{G}(d,n)$ is actually regular.

Theorem 8.2.4. $\mathbb{G}(d,n)$ *is irreducible and* $\dim \mathbb{G}(d,n) = d(n-d)$.

Proof. We first observe that the image of an irreducible space under a continuous map is irreducible. By lemma 8.2.1, there is a surjective map $GL_n(k) \to R(d,n)$, given by the $A \mapsto AM$, for some fixed $M \in R(d,n)$. Therefore $R(d,n)$ is irreducible. The map $\text{pl} : R(d,n) \to \mathbb{G}$ is surjective, consequently $\mathbb{G}$ is irreducible.

Let $\tilde{U} \subset R(d,n)$ be the subset of matrices such that the top $d \times d$ block is nonsingular. This is a $GL_d(k)$-invariant open set, and the image $\text{pl}(\tilde{U})$ is also open. This means that $\dim \mathbb{G} = \dim \text{pl}(\tilde{U})$. Consider the subset of $\tilde{U}$

$$U = \left\{ \begin{pmatrix} I \\ M \end{pmatrix} \mid M \in \mathbb{A}^{d(n-d)} \right\}$$

where the identity I is $d \times d$ and M is an arbitrary $d(n-d)$ matrix. This is isomorphic to $\mathbb{A}^{d(n-d)}$. We can see that $UB \cap U = \emptyset$ unless $B = I$ and that $\tilde{U}$ is a union of $GL_d(k)$-orbits. This implies that $\text{pl}|_U$ is injective and $\text{pl}(U) = \text{pl}(\tilde{U})$. Therefore theorem 4.3.5 shows that $\dim \text{pl}(U) = \dim U = d(n-d)$.

□

8.3 Lines in $\mathbb{P}^3$

Let's study the first interesting case of a Grassmanian which is not a projective space. We can identify $\mathbb{G} = \mathbb{G}(2,4)$ with the set of lines in $\mathbb{P}^3$, where to each line L, we associate the two dimension space $Cone(L) \subset k^4$. The Plücker embedding sends $\mathbb{G} \hookrightarrow \mathbb{P}(\wedge^2 k^4) \cong \mathbb{P}^5$, and $\dim \mathbb{G} = 4$. Therefore $\mathbb{G}$ is a hypersurface. If $w = \text{pl}([v_1, v_2]) = v_1 \wedge v_2$ then $w \wedge w = 0$. Writing

$$w = \sum x_{ij} e_i \wedge e_j,$$

we obtain

$$w \wedge w = (x_{12}x_{34} + x_{13}x_{24} + x_{14}x_{23}) e_1 \wedge e_2 \wedge e_3 \wedge e_4 = 0$$

The coefficients can be viewed as homogeneous coordinates of $\mathbb{P}^5$. Under the Plücker embedding they correspond to minors. In summary:

Theorem 8.3.1. $\mathbb{G}(2,4)$ *is a degree* 2 *hypersurface, or quadric, in* $\mathbb{P}^5$.

Let us try to look at some geometrically defined subvarieties. Fix a line $\ell_0 \subset \mathbb{P}^3$, say $x_0 = x_1 = 0$. Consider the subset, which is an example of a Schubert variety

$$S = \{\ell \in \mathbb{G} \mid \ell \cap \ell_0 \neq \emptyset\}$$

This can viewed as the subset of two dimensional subspaces which have nonzero intersection with the span of e_3 and e_4. A matrix M lies in $\mathrm{pl}^{-1}(S) \subset R(2,4)$ if a nonzero linear combination of its columns is of the form $(0,0,*,*)^T$. This means that $S = V_{\mathbb{P}}(x_{12}) \cap \mathbb{G}$ is a hypersurface in the Grassmanian. We can generalize this as follows:

Proposition 8.3.2. *For any curve $C \subset \mathbb{P}^3$, that is a subvariety of dimension* 1*, the set*

$$Ch_C = \{\ell \in \mathbb{G} \mid \ell \cap C \neq \emptyset\}$$

is a closed set of dimension 3*.*

Proof. We start by forming the flag variety $F = \{(p,\ell) \in \mathbb{P}^3 \times \mathbb{G} \mid p \in \ell\}$. This has two projections p_i to the factors. Then $Ch_C = p_1(p_2^{-1}(C))$, and this implies that it is closed. We can also use this to calculate the dimension. The fibre $p_2^{-1}(p)$ is the the set of lines through p. We will see in the exercises, that this is isomorphic to $\mathbb{P}^2$. In particular, $\dim p_2^{-1}(C) = \dim C + \dim(\text{general fibre}) = 3$. Now consider the restriction of the first projection $f : p_2^{-1}(C) \to S$. For $\ell \in S$, $f^{-1}(\ell) = C \cap \ell$. This is finite unless $C = \ell$; in particular, almost all fibres are finite. Therefore $\dim S = 3$. $\square$

Thus to any curve, we can associate a hypersurface $Ch_C \subset \mathbb{G}$. This can be shown to be given by vanishing of a single polynomial in the Plücker coordinates, i.e. $Ch_C = \mathbb{G} \cap V_{\mathbb{P}}(f)$. The coefficients of these polynomials are called the Chow coordinates of C. They completely determine the curve.

8.4 Exercises

Exercise 8.4.1.

1. *Show that an ideal I in a commutative ring R is free if and only if it is principal.*

2. *Let $X = V(y^2 - x(x-1)(x-2)) \subset \mathbb{A}^2_{\mathbb{C}}$, and let $R = \mathcal{O}(X)$. Show that the ideal $I = (x,y)$ is locally free but not free. For local freeness you can assume that the local rings are PID's if you need to.*

3. *Show that the group $GL_n(k)$ acts transitively on $\mathbb{G}_k(d,n)$.*

4. *Show that any point of $\mathbb{G}_k(d,n)$ has an open neighbourhood isomorphic to $\mathbb{A}^{d(n-d)}$. Conclude that it is regular. Hint: Use the proof of theorem 8.2.4 and the previous exercise.*

5. *Identify* $\mathbb{G} = G_k(2,4)$ *with the set of lines in* $\mathbb{P}^3_k$. *Show that the map* $\mathbb{P}^3 \times \mathbb{P}^3 - \Delta \to \mathbb{G}$ *sending* (p,q) *to the line through* p *and* q *is a morphism of quasiprojective varieties.*