Chapter 1

Algebra of Rotations

One of our goals is to make precise the idea of symmetry, which is important in math and other parts of science. Something like a square has a lot of symmetry, but circle has even more. But what does this mean? One way of expressing this, is that we can rotate the circle to take any point to any other point. While for the square, a corner has to go to another corner. To analyze this further, let us start with line L, which we can think of as a circle with infinite radius. Mark a point called 0 on it. Then any other point on L is given by a unique real number: $+x$ if it lies to the right of 0 with distance x, and $-x$ if it lies to the left of 0 with distance x. In this way, we may identify L with the set of $\mathbb{R}$ of real numbers. "Rotation" in this case is operation which takes a point labelled by $x \in \mathbb{R}$, and shifts it to the right or left by a certain number of units, say y; in other words, it sends x to $x + y$, where y could be negative. Basic laws of algebra have a natural meaning in this context: Shifting by 0 does nothing because

$$x + 0 = x$$

Shifting by y and then by z can be achieved in one step because of the associative law

$$(x + y) + z = x + (y + z)$$

Shifting x by y is that same shifting y by x. In other words the commutative law

$$x + y = y + x$$

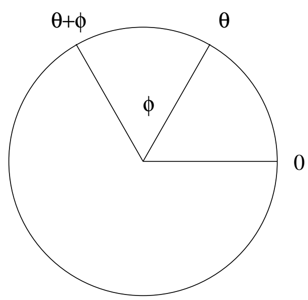
holds. Finally, we can always shift back to where we started because

$$\text{Given } x\text{, we can find } y \text{ with } x + y = 0 \quad \text{(existence of the inverse)}$$

Now we are ready to consider an honest circle C of radius 1 in the plane. Fix a point, called 0, on it. Then every point on C is determined by the distance along C measured from 0 in the counterclockwise direction: this is the same as the angle measured in radians. E very point on C is determined by exactly one number in

$$[0, 2\pi) = \{x \in \mathbb{R} \mid 0 \le x < 2\pi\}$$

and we will identify these sets. Now we define addition in C as follows: given $\theta, \phi \in C$, let $\theta \oplus \phi$ be given by rotating θ by the additional angle ϕ.



Here are a few simple examples

$$\pi/2 \oplus \pi/2 = \pi$$

$$\pi \oplus \pi = 0$$

In general, we can see that

$$\theta \oplus \phi = \begin{cases} \theta + \phi & \text{if } \theta + \phi < 2\pi \\ \theta + \phi - 2\pi & \text{if } \theta + \phi \geq 2\pi \end{cases}$$

And it will be convenient to adopt this last equation as the official definition.

At first it may seem like a strange operation, but notice that many familiar rules apply:

Lemma 1.1. *If $\theta \in C$, then $\theta \oplus 0 = \theta$.*

Proof. Since $\theta < 2\pi$, $\theta \oplus 0 = \theta + 0 = \theta$ □

Lemma 1.2. *If $\theta, \phi \in C$, then $\theta \oplus \phi = \phi \oplus \theta$.*

Proof. If we compare

$$\phi \oplus \theta = \begin{cases} \phi + \theta & \text{if } \phi + < 2\pi \\ \phi + \theta - 2\pi & \text{if } \phi + \theta \geq 2\pi \end{cases}$$

we see that it is identical to $\theta \oplus \phi$. □

Lemma 1.3. *Given $\theta \in C$, we have $\phi \in C$ such that $\theta \oplus \phi = 0$.*

Proof. We can take $\phi = \ominus\theta = 2\pi - \theta$. □

We omit the proof, but we also have the associative law

$$\theta \oplus (\phi \oplus \psi) = (\theta \oplus \phi) \oplus \psi$$

also holds.

So in summary, the set C with the operation $\oplus$ shares the same 4 laws as $\mathbb{R}$ with usual addition. We have a name for such a thing. It is called an *abelian group*, and it will be one of the key concepts in this class. C has additional symmetries called reflections or flips which takes a point on C to its mirror image. If C is given by the set of solutions to $x^2+y^2=1$ in the plane, then flips include the operations $V:(x,y)\mapsto(x,-y)$ and $H:(x,y)\mapsto(-x,y)$. In terms of the angles, V is the operation $\theta\mapsto\ominus\theta$ above. If we include all the flips along with the rotations, the algebra becomes more complicated: we get a *nonabelian group.* But we are jumping ahead of our story.

1.4 Exercises

1. Describe the set C_3 of all solutions to the equation $x\oplus x\oplus x=0$.

2. Prove that if $x,y\in C_3$ then $x\oplus y\in C_3$ (you can use all the previously stated properties including associativity).

3. If you forgot what complex numbers are, now is the time to remind yourself. Given $z=a+bi\in\mathbb{C}$, recall that $\bar{z}=a-bi$. Check that $z\bar{z}=a^2+b^2$, and also that $\bar{z}\bar{w}=\overline{zw}$ for $w=c+di$.

4. Let S be the set of complex numbers of the form $a+bi$, where $a^2+b^2=1$. With the help of the previous exercise, prove that if $z\in S$, then $z^{-1}\in S$, and that the product of any two numbers in S is also in S. (This gives another way to make the circle into an abelian group, although it will turn out to be the same in a sense we will make precise later.)

5. A rotation matrix is a matrix

$$R(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

where $\theta\in[0,2\pi)$. Show that the product of two rotation matrices satisfies $R(\theta\oplus\phi)=R(\theta)R(\phi)$ Hint: you may need to brush up on your linear algebra and trigonometry as well.