Chapter 11

Counting Problems involving Symmetry

Group theory can be applied to counting problems invloving symmetry. Here are a few such problems.

Example 11.1. *How many dice can be constructed by labeling the face of a cube by the numbers* $1, \ldots 6$*?*

Example 11.2. *Suppose* 3 *identical decks of* 52 *cards are combined into a big deck. How many* 3*-card hands can be delt out of the big deck?*

Example 11.3. *How many ways can a necklace be constructed with* 2 *black and* 2 *white beads?*

To analyze the second problem, let's first keep track of the order in which cards are dealt. Let's suppose that the kinds of cards are labeled by numbers $1, \ldots 52$. Since we have three decks we can be confident about not running out of kinds. Thus the set of ordered hands can be identified with

$$H = \{(a, b, c) \mid a, b, c = 1, 2 \ldots 52\}$$

The cardinality of this set is 52^3. Of course, we want to disregard order. As first guess, we might think that we should divide this by the number of ways of permuting the cards to get $52^3/6$. Unfortunately, this is not an integer so it doesn't make sense. To explain the correct answer, we will introduce some more group theory.

Definition 11.4. *We say that a group* $(G, *, e)$ *acts on a set* X *if there is an operation* $\cdot : G \times X \to X$ *satisfying:*

1. $e \cdot x = x$.
2. $(g * h) \cdot x = g \cdot (h \cdot x)$

(We're are really defining a left action here, there is also a notion of right action, but we won't need that.)

Definition 11.5. *Given a group G acting on a set X, the orbit of $x \in X$ is the set $Gx = \{g \cdot z \mid g \in G\}$. The set of orbits is denoted by X/G.*

Definition 11.6. *An element x is called a fixed point of g is $g \cdot x = x$. Let $Fix(g) = \{x \in X \mid g \cdot x = x\}$ be the set of fixed points. The action is called fixed point free if no element $g \neq e$ has a fixed point.*

Theorem 11.7. *If finite group G has a fixed point free action on a finite set X, then $|X/G| = |X|/|G|$.*

Now, we can solve one the problems mentioned earlier. For problem 11.1, we first choose an initial labelling, or marking, of our blank cube. This allows us to talk about the first face, second face and so on. Let X be the set of labellings of the faces of this marked cube. There are 6 choices for the first face, 5 for the second..., therefore there are $6! = 720$ elements of X. $G = C$ is the symmetry group for the cube which has order 24. G acts by rotating the cube. Clearly there are no fixed points for any rotation (other than I). Therefore the number of labellings for a *blank* cube is

$$|X/G| = \frac{720}{24} = 30$$

This formula does not work for the examples. For example 11.2, S_3 acts on H by moving the positions of the cards. For example,

$$f = \begin{cases} 1 & \to & 2 \\ 2 & \to & 3 \\ 3 & \to & 1 \end{cases}$$

then $(a_1, a_2, a_3) \cdot f = (a_2, a_3, a_1)$. We want to treat two hands as the same if you can permute one to get the other, i.e. if they lie in the same orbit. Therefore our problem is to count the set of orbits H/S_3. If the first two cards are repeated, so $a_1 = a_2$, then (a_1, a_2, a_3) is a fixed point for (12).

Theorem 11.8 (Burnside's theorem)**.** *If G is a finite group acting on a finite set X, then*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|$$

This will be proved below. We note

Next consider problem 11.2. Let H_{ij} be the set of ordered triples (a_1, a_2, a_3) with $a_i = a_j$. This is the set of fix points of (ij). Since there only two free choices, we have

$$|H_{ij}| = 52^2$$

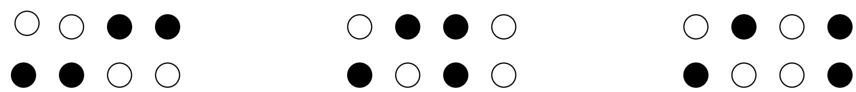
Let H_{123} be the set of triples where all the cards are the same. This is the set of fixed points for (123) and (132). Clearly

$$|H_{123}| = 52$$

Therefore

$$\begin{aligned} H/S_3 &= \frac{1}{6}[|H| + |H_{12}| + |H_{13}| + |H_{23}| + |H_{123}| + |H_{123}|] \\ &= \frac{1}{6}[52^3 + 3(52^2) + 2(52)] \\ &= 24804 \end{aligned}$$

Finally, consider problem 11.3. Let us first arrange the beads in sequence. There are 6 possibilities



and we let X be the set of these. The group in this problem is the symmetry group of the square $D_4 \subset S_4$ which permutes the positions of the beads. All of these are fixed by the identity. The rotations (1234) and (1432) have no fixed points. The element (12)(34) has two fixed points, namely the leftmost sequences in the above diagram. The remaining 4 elements also have two fixed points a piece (exercise). Therefore

$$|X/D_4| = \frac{1}{8}[6 + 5(2)] = 2$$

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Now we need to prove the above results. We first prove a strengthened version of theorem 10.7. Given a a group acting a set X, the stabilizer of x is

$$stab(x) = \{g \in G \mid g \cdot x = x\}$$

Theorem 11.9. *Let G be a finite group acting on a set X, then $|G| = |stab(x)||Gx|$*

Proof. For each $y \in xG$ let

$$T(y) = \{g \in G \mid g \cdot x = y\}$$

Choose $g_0 \in T(y)$, then the function $f(g) = g_0 * g$ maps $stab(x) \to T(y)$. This is a one to one correspondence since it has an inverse $f^{-1}(h) = g_0^{-1} * h$. This implies that $T(y)| = |stab(x)|$.

If $y \neq z$ then $T(y)$ and $T(z)$ must be disjoint, otherwise $y = g \cdot x = z$ for $g \in T(y) \cap T(z)$. Every g lies in some $T(y)$ namely $T(g \cdot z)$. Therefore $T(y)$ is a partition of G. By corollary 3.10,

$$|G| = \sum_{y \in xG} |T(y)| = |Gx||stab(x)|$$

□

Given a subgroup H of a group G, a (left) coset is a set of the form $g * H = \{g * h \mid h \in H\}$ for $g \in G$. The set of cosets is denoted by G/H. We can define a left action of G on G/H by

$$\gamma \cdot (g * H) = (\gamma * g) * H$$

Theorem 11.10 (Lagrange)**.** *Let G be a group of finite order. Then for any subgroup H, $|G| = |H||G/H|$.*

Proof. This is a transitive action which means that there is only one orbit, and the stabilizer $stab(H) = H$ (this will be expanded in the exercises). So this theorem follows from theorem 11.9. $\square$

We now prove Burnside's theorem.

Proof of theorem 11.8. Let

$$C = \{(x, g) \in X \times G \mid g \cdot x = x\}$$

Consider the map $p : C \to G$ given by $p(x, g) = g$. Then an element of $p^{-1}(g)$ is exactly a fixed point of g. Therefore proposition 3.11 applied to p yields

$$|C| = \sum_{g \in G} |p^{-1}(g)| = \sum_{g \in G} |Fix(g)| \tag{11.1}$$

Next consider the map $q : C \to X$ given by $q(x, g) = x$. Then $q^{-1}(x) = stab(x)$. Therefore proposition 3.11 applied to q yields

$$|C| = \sum_{x \in X} |p^{-1}(x)| = \sum_{x \in X} |stab(x)|$$

We group the the last sum into orbits

$$C = \sum_{x \in \text{ 1st orbit}} |stab(x)| + \sum_{x \in \text{ 2nd orbit}} |stab(x)| + \dots$$

For each orbit Gx_0 has $|G|/|stab(x_0)|$ elements by theorem 11.9. Furthermore, for any $x \in Gx_0$, we have $|stab(x)| = |stab(x_0)|$. Therefore

$$\sum_{x \in Gx_0} |stab(x)| = \sum_{x \in Gx_0} |stab(x_0)| = \frac{|G|}{|stab(x_0)|} |stab(x_0)| = |G|$$

Consequently

$$|C| = |G| \sum_{\text{orbits}} 1 = |G||X/G|$$

Combining this with equation 11.1 yields

$$|G||X/G| = \sum_{g \in G} |Fix(g)|$$

Dividing by $|G|$ yields the desired formula. $\square$

11.11 Exercises

1. How many necklaces can be constructed using 4 different colored beads? Explain.

2. How many necklaces can be formed using 2 black, one red and one yellow bead? Explain.

3. In how many ways can the faces of a tetrahedron be labelled by the numbers $1, 2, 3, 4$? Explain.

4. Suppose 2 identical decks of 52 cards are combined into a big deck. How many 3 card hands can be delt out of the big deck?

In the above calculations, certain numbers occured multiple times. This can be explained with the help of the following definition. Two elements $g_1, g_2 \in G$ are *conjugate* if $g_2 = h^{-1} * g_1 * h$ for some $h \in G$.

5. Prove that in S_3, (12) is conjugate to (13) and (23), and (123) is conjugate to (132).

6. Suppose that a finite group G acts on a set X. Prove that if g_1 and g_2 are conjugate, then the number of fixed points of g_1 and g_2 are the same. Therefore if we can choose elements $g_1, \ldots, g_n \in G$, such that every element of G is conjugate to exactly one g_i, then Burnside's can be rewritten as

$$|X/G| = \frac{1}{|G|}\sum_{1}^{n}(\#\text{ elements conjugate to } g_i)|Fix(g_i)|$$

   is a bit more efficient for computing.

7. Fill in the following missing details from the proof of Lagrange's theorem:

   a) Prove that G acts transitively on G/H

   b) Prove that $stab(H) = H$.

8. Prove that if G is a group with $|G|$ a prime, then G is cyclic (Hint: use Lagrange's theorem).