# NOTES ON THE HODGE CONJECTURE

DONU ARAPURA

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We work over $\mathbb{C}$ throughout. Unless stated otherwise, $X, Y$ can be assumed to be smooth projective varieties. Given a variety $X$, let $H^i(X)$ stand for singular cohomology with rational coefficients for the classical topology. Recall [GH, W] that we have a Hodge decomposition

$$H^i(X) \otimes \mathbb{C} = \bigoplus_{p+q=i} H^{pq}(X)$$

where $\bar{H}^{qp}(X) = H^{pq}(X) \cong H^q(X, \Omega_X^p)$ is the space of de Rham cohomology classes represented by forms of type $(p,q)$. It will be convenient to abstract things at this point. A (pure rational) Hodge structure $H$ is a rational vector space with a bigrading on $H \otimes \mathbb{C}$ satisfying $\bar{H}^{pq} = H^{qp}$. For arbitrary complexes varieties, Deligne [D] has shown that the cohomology carries a canonical mixed Hodge structure. We won't define this notion precisely except to note that this gives a nonobvious extension of pure Hodge structures.

Given a closed subvariety $Z \subset X$ of codimension $p$, we get a fundamental class $[Z] \in H^{2p}(X)$. There is a well known necessary condition for a class to be of this form.

**Lemma 0.1.** *$[Z]$ is a Hodge cycle i.e. it lies in $H^{2p}(X, \mathbb{Q}) \cap H^{pp}(X)$.*

*Proof.* Let $\tilde{Z} \to Z$ be a resolution of singularities. Then for any differential form $\alpha$ of type $(a,b)$ with $a + b = 2(\dim X - p)$,

$$\int_{\tilde{Z}} \alpha = 0$$

unless $a = b$. □

In the ICM in 1950 [Ho], Hodge suggested that the converse was true. More precisely: **that the space of Hodge cycles coincides with the space of algebraic cycles, i.e. is it spanned by fundamental class of subvarieties.** This has come to be known as the Hodge conjecture[1], which we sometimes abbreviate as (HC). This is arguably the deepest conjecture in complex algebraic geometry. It implies Grothendieck's standard conjectures over $\mathbb{C}$; the principal one being

(LSC) Lefschetz standard conjecture: The $\Lambda$ operator in Hodge theory, which gives a sort of inverse to the hard Lefschetz operator, preserves the space of algebraic cycles.

[1] At the risk of being pepandic, I want to point that Hodge didn't actually state this as a conjecture. His precise words are "It is clearly a matter of great importance to extend Lefschetz's condition for a 2-cycle to be algebraic. The problem is as follows..."

These conjectures ensure a good theory of homological motives. In fact, an equivalent formulation of HC in this language is that the Hodge realization of the category of homological motives is full and faithful. We point out an arithmetic consequence of HC

(AC) Absoluteness conjecture: Hodge cycles are absolute, which roughly means that they are invariant under the action of $Aut(\mathbb{C})$ arising from identifying $H^i(X, \mathbb{C})$ with $\ell$-adic cohomology tensored with $\mathbb{C}$.

Deligne has proved this for Abelian varieties [DMOS].

## 1. Caveats

These days the Hodge conjecture is formulated with rational coefficients, but the following question is perhaps closer to what Hodge had in mind[2]: *Given a class $\gamma \in H^{2p}(X, \mathbb{Z})$ whose image in complex cohomology lies in $H^{pp}(X)$, is it a $\mathbb{Z}$-linear combination of fundamental classes?* This does hold for $p = 1$ by the Lefschetz $(1, 1)$ theorem, but it is known to be false in general. Historically the first negative answer was due to Atiyah-Hirzebruch [AH]. They showed that there are extra conditions for the torsion class to be algebraic, which are in general nontrivial. Some further examples involving torsion cycles were found by Totaro [T] and Soulé-Voisin [SV]. If we restrict our attention to classes in the torsion free part, it is still false! Here is a simple counterexample. Let $X \subset \mathbb{P}^4$ be a general hypersurface of degree $d \geq 6$. The weak Lefschetz theorem gives $H^2(\mathbb{P}^4, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$. The generator is the class of a hyperplane $h$. By Poincaré duality $H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ is generated by a class $\gamma$ satisfying $\gamma \cdot h = 1$. This is a Hodge class, and in fact it is a rational multiple of $h^2$. We claim that it is not an integral algebraic cycle. Suppose that $\gamma = \sum n_i[C_i]$ for irreducible curves $C_i$. Since $H^4(X)$ is cyclic, we see that $\gamma$ is equal to one of them, say $\gamma = [C_1]$. Then $h \cdot C_1 = 1$ implies that $C_1 \subset X$ is a line. An easy dimension counting argument shows this is impossible for general $X$.

In a completely different direction, it is clear that the Hodge conjecture makes sense for compact Kähler manifolds. However counterexamples (with $\mathbb{Q}$-coefficients) have been constructed by Zucker [Z] and later by Voisin [V2].

Finally, we note that Hodge made a more ambitious proposal in [Ho]. But this, as Grothendieck [Gr2] observed, was overly optimistic even with rational coefficients. However, he suggested that the following modification ought to be true.

(GHC) Generalized Hodge conjecture: Any sub Hodge structure of $H^i(X)$ of level[3] $\leq i - 2p$ is supported on a subvariety of codim $\geq p$.

## 2. Easy Examples

**Proposition 2.1.** *The Hodge conjecture holds for varieties of dimension up to* 3.

*Proof.* For $H^2$ this this follows from the Lefschetz $(1, 1)$. We have, by the Hard Lefschetz theorem, an isomorphism $L : H^2(X) \to H^4(X)$ given by cupping by a hyperplane class. Since $L$ induces an isomorphism of Hodge cycles and preserves the space of algebraic cycles, the conjecture holds for $H^4$ as well. □

[2]The wording in [Ho] is somewhat ambiguous, but certainly in the introduction to [AH] they state explicitly that this is what he meant.

[3]The level is $\max\{|p - q| \mid H^{pq} \neq 0\}$.

The Hodge conjecture is trivially true for varieties where all the cohomology is spanned by algebraic cycles. Examples of such include projective spaces, Grassmanians, flag varieties, and toric varieties. Most of these examples can be checked using [F, 19.1.11]. We can combine these observations with the Künneth formula to get

**Corollary 2.2.** *If $X$ is a product of variety whose cohomology is algebraic with a variety of dimension $\leq 3$, then the Hodge conjecture holds for it.*

To set up the notation for the next proposition. Suppose that $X$ is a smooth projective variety. Let $Z \subset X$ be Zariski closed subset such that each component has codimension $p$, and let $\iota : \tilde{Z} \to X$ be a desingularization. Then by [D, 8.2.8], we have an exact sequence of mixed Hodge structures

$$H^{i-2p}(\tilde{Z})(-p) \to H^i(X) \to H^i(X - Z)$$

If $i = 2k$, then under the twist, rational $(k-p, k-p)$ cycles in $H^{i-2p}(\tilde{Z})$ become rational $(k, k)$ cycles in $H^{i-2p}(\tilde{Z})(-p)$.

**Proposition 2.3.** *If the Hodge conjecture holds for $\tilde{Z}$ and some nonsingular compactification of $U = X - Z$, then it holds for $X$.*

*Proof.* Suppose that $X_1$ is a nonsingular compactification of $U$ for which the Hodge conjecture holds. Let $X'$ be the closure of the image of $U$ in $X \times X_1$ under the diagonal embedding. Then let $X_2 \to X$ be a resolution of singularites which is an isomorphism over $U$. In this, we obtain a third nonsingular compactification $X_2$ mapping to $X$ and $X_1$.

We have an exact sequence of pure Hodge structures

$$H^{i-2p}(\tilde{Z})(-p) \to H^i(X) \to im[H^i(X) \to H^i(U)] \to 0$$

Since these Hodge structures are polarizable[4], it splits. Therefore we get an exact sequence of Hodge cycles

$$H^{2k-2p}(\tilde{Z})_{hodge} \to H^{2k}(X)_{hodge} \to im[H^{2k}(X) \to H^{2k}(U)]_{hodge} \to 0$$

By [D]

$$im[H^i(X) \to H^i(U)] = W_i H^i(U) = im[H^i(X_1) \to H^i(U)]$$

Therefore if $\alpha \in H^{2k}(X)$ is a Hodge cycle. Then its image in $H^{2k}(U)$ lifts to a Hodge cycle $\beta \in H^{2k}(Y)$. This is algebraic by assumption. Therefore, the pushforward of the preimage $\beta' \in H^{2k}(X)$ is algebraic. Since $\alpha - \beta'$ maps to zero in $H^{2k}(U)$, it is the image of a Hodge cycle $\gamma \in H^{2k-2p}(\tilde{Z})$ which is again algebraic. Therefore $\alpha = \beta + \iota_*(\gamma)$ is algebraic. □

**Exercises:**

(1) Show that the Hodge conjecture holds for $H^{2n-2}(X)$ where $\dim X = n$.
(2) If $f : X \to Y$ is a surjective map of smooth projective varieties, such that $X$ satisfies the Hodge conjecture then so does $Y$.
(3) Show that corollary 2.2 holds for $\mathbb{P}(E)$ where $E$ is a vector over a variety of dimension $\leq 3$.

[4] That is they possess bilinear forms satisfying the Hodge-Riemann relations. It is easy to see that short exact sequences of polarizable Hodge structures always split.

(4) Let $X$ be a smooth projective variety, and let $Z \subset X$ be a smooth closed subvariety. If the $X$ and $Z$ satisfy the Hodge conjecture, then so does the blow up $\pi : Bl_Z X \to X$.
(5) The Hodge conjecture is birationally invariant in dimensions up to 5. In other words, if $X$ and $X'$ are two birational smooth projective varieties with dimension $\leq 5$, then one of them satisfies the Hodge conjecture if and only if the other does.

## 3. Varieties with many rational curves

We look at some more substantial examples in this section in dimensions 4 and 5. A variety $X$ is uniruled if there is a rational curve passing through every general point of $X$. This is equivalent to the existence of a generically finite rational map $\mathbb{P}^1 \times Y \dashrightarrow X$, [Ko]. Some examples are:

(1) Smooth hypersurfaces in $\mathbb{P}^{n+1}$ of degree $d$ are uniruled if and only if $d \leq n+1$. When the inequality holds the family of rational curves can be constructed explicitly as a family of conics [CM] (or lines when $d < n+1$ [Lw, pp 202-214]). For the converse statement, we use the fact that $h^{n0}(X) = 0$ for uniruled varieties.
(2) More generally Fano varieties are uniruled. In fact, they are rationally connected (below).

**Proposition 3.1** (Conte-Murre [CM])**.** *A smooth projective uniruled four dimensional variety satisfies the Hodge conjecture.*

*Proof.* Suppose $X$ is a projective uniruled 4-fold. Then any desingularization is also uniruled. Therefore we can assume that $X$ is smooth by proposition 2.3. There exists a surjective map $X' \to X$ where $X'$ is a smooth projective variety birational to the product of $\mathbb{P}^1$ and a smooth threefold. Therefore $X'$ and consequently $X$ satisfies the conjecture by the above exercises. □

A variety is rationally connected if any two general points can be joined by a rational curve i.e. a curve birational to $\mathbb{P}^1$. This is a much stronger condition than uniruledness. Rational varieties are clearly rationally connected. A smooth projective variety $X$ is said to be Fano if the dual of the canonical bundle $\omega_X = \Omega_X^{\dim X}$ is ample. This means that there is an embedding into projective space $X \subset \mathbb{P}^N$ such that the restriction of $\mathcal{O}(1)$ is isomorphic to a multiple of $\omega_X^{-1} = \omega_X^*$. The adjunction formula implies that if $X \subset \mathbb{P}^{n+1}$ is a degree hypersurface, then $\omega_X \cong \mathcal{O}_X(d-n-2)$. Thus the hypersurface $X$ is Fano if and only if $d \leq n+1$.

**Theorem 3.2** (Kollár, Miyaoka, Mori)**.** *Any Fano variety is rationally connected.*

The proof uses Mori's "bend and break" technique which involves working over a field of positive characteristic! See [Ko] for details of the proof.

**Theorem 3.3** (Laterveer)**.** *The Hodge conjecture holds for a smooth projective rationally connected variety of dimension at most five.*

The proof uses the method of Bloch-Srinivas [BS].

**Theorem 3.4** (Bloch-Srinivas)**.** *Let $X$ be a smooth rationally connected variety over an algebraically closed field $K$, then the fundamental class of the diagonal $\Delta \subset X \times X$ is equal to a sum of algebraic cycles $[\xi \times X] + [\Gamma]$, where $\xi$ is the* *multiple of the class of a point, and* $\Gamma$ *is supported on* $X \times Z$ *for some proper closed subset* $Z \subset X$.

First, we point out that the rational connectedness of $X$ implies the existence of a morphism $R : \mathbb{P}^1 \times F \to X$, such that

$$R^{(2)} : \mathbb{P}^1 \times \mathbb{P}^1 \times F \to X \times X,\ R^{(2)}(t_1, t_2, f) = (R(t_1, f), R(t_2, f))$$

is dominant, by [Ko, thm II 2.8, IV def 3.2].

*Proof.* Fix a general point $x_0 \in X$. Consider the maps $F \to X \times X$ given by the composition of $0 \times \infty \times id_F$ and $R$, and $X \to X \times X$ given by $x_0 \times id_X$. Let $U = F \times_{X \times X} X$ be the fibre product. The restriction of $R$ gives a morphism $r : \mathbb{P}^1 \times U \to X$, with $r(0, u) = x_0$ such that $g : U \to X$ given by $g(u) = r(\infty, u)$ is dominant. After replacing $U$ by the normalization of $\mathbb{C}(X)$ in $\mathbb{C}(U)$, we can assume that $\dim U = \dim X$. Thus we have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}^1 \times U & \xrightarrow{q} & X \times X \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} \\ U & \xrightarrow{g} & X \end{array}$$

where $\pi_2$ are projections to the second factor, and $q(t, u) = (r(t, u), g(u))$. We see that $q(\infty \times U) = \Delta \cap (X \times g(U))$ and $q(0 \times U) = x_0 \times g(U)$ as sets. Therefore the difference $deg(g)[\Delta - x_0 \times X]$ is supported on $X \times \overline{(X - g(U))}$. □

*Proof of theorem 3.3.* The theorem is automatic when the dimension is less than or equal to three. Rationally connected fourfolds are uniruled, so the result follows from proposition 3.1 in this case. Thus we may assume that $X$ is a rationally connected fivefold. In this case, it suffices to prove the that any Hodge cycle $\alpha \in H^4_{hodge}(X, \mathbb{Q})$ is algebraic.

Applying theorem 3.4, we see that $[\Delta] = [\xi \times X] + [\Gamma]$, where $\xi$ is the multiple of the class of a point, and $\Gamma$ is supported on $X \times Z$ for some proper closed subset $Z \subset X$. We can assume that $Z = D$ is a divisor. Let $\gamma$ denote the fundamental class of $\Gamma$ in $X \times D$. Let $\iota : D \to X$ and $id \times \iota : X \times D \to X \times X$ denote the inclusions, and let $p_1 : X \times D \to X$ and $p_2 : X \times D \to D$ denote the projections.

A standard calculation gives $\alpha = \pi_{2*}([\Delta] \cup \pi_1^* \alpha)$. Therefore, we can decompose this as

$$\alpha = \pi_{2*}([\xi \times X] \cup \pi_1^* \alpha) + \pi_{2*}([\Gamma] \cup \pi_1^* \alpha)$$

The first summand equals

$$\pi_{2*}(\pi_1^*([\xi] \cup \alpha)) = 0,$$

and the second summand can be written as

$$\pi_{2*}((id \times \iota)_* \gamma \cup \pi_1^* \alpha) = \iota_* p_{2*}(\gamma \cup p_1^* \alpha)$$

Since $\gamma \cup p_1^* \alpha$ is a Hodge cycle $H^2(D)$, it is algebraic by the the Lefschetz $(1, 1)$ theorem. Consequently $\alpha$ is algebraic.

□

## 4. Mumford-Tate groups

It is sometimes possible to reduce the verification of the Hodge conjecture to a problem of invariant theory using a somewhat remarkable group that we will define presently. The unit circle $U(1) \subset \mathbb{C}^*$ is a real form given as the fixed points under the conjugation $z \mapsto \bar{z}^{-1}$. Given a pure Hodge $H$ of weight $w$, $U(1)$ acts on $H_{\mathbb{C}} = H \otimes \mathbb{C}$ by $z^{p-q}$ on the $H^{pq}$. This action defines a homomorphism $h : U(1) \to GL(H_{\mathbb{R}})$. The the special Mumford-Tate group or Hodge group $Hdg(H) \subset GL(H)$ is the smallest $\mathbb{Q}$-algebraic group whose real points contains the image of $U(1)$. We note the following properties.

(1) This is a connected algebraic group defined over $\mathbb{Q}$.
(2) If $H$ has a polarization $\psi$, then $Hdg(H)$ is a reductive subgroup of the group preserving $\psi$ (which is orthogonal or symplectic according to the parity of $w$).
(3) If $\mathcal{H}$ is a variation of Hodge structure over $S$, then $Hdg(\mathcal{H}_s)$ contains a finite index subgroup of the monodromy group $im[\pi_1(S,s) \to GL(\mathcal{H}_s)]$ for all $s$ outside a countable union of proper subvarieties.

Let us give an alternative description (assuming $H$ is polarizable) that explains the significance. Let

$$T^{m,n}(H) = H^{\otimes m} \otimes (H^*)^{\otimes n}$$

This is a Hodge structure of weight $w(m-n)$. When this is even, say $2p$, call an element a Hodge cycle if it rational and lies in the $(p,p)$ part. The general linear group $GL(H)$ acts on this $T^{m,n}(H)$. For each tensor $t$ in this space let

$$Fix(t) = \{g \in GL(H) \mid g(t) = t\}$$

We define

$$Hdg_2(H) = \bigcap Fix(t)$$

as $t$ runs over all Hodge cycles in all possible $T^{mn}(H)$.

**Theorem 4.1.** *$Hdg(H) = Hdg_2(H)$.*

We prove the easy inclusion of the theorem. $Hdg_2(H)$ is clearly an algebraic group defined over $\mathbb{Q}$. Since the action of $U(1)$ fixes Hodge cycles, we therefore have $Hdg(H) \subset Hdg_2(H)$.

**Corollary 4.2.** *The invariants of $Hdg(H)$ on $T^{mn}(H)$ are precisely the Hodge cycles.*

It is worth noting that there is a third characterization of this group as the group associated to the tannakian category generated by $H$. We can use the previous results to compute $Hdg$ in certain generic cases. For example $X$ is a general curve of genus $g$ or a general principally polarized abelian variety of dimension $g$, then $Hdg(H^1(X)) = Sp(2g,\mathbb{Q})$. The point is that the monodromy groups for the universal families are Zariski dense in $Sp(2g)$, which forces $Hdg \supset Sp(2g)$. While (2) forces the opposite inclusion.

**Exercise:** Show that if $X$ is an elliptic curve, then $Hdg(H^1(E)) = Sp(2,\mathbb{Q}) = SL_2(\mathbb{Q})$ if and only if $X$ does not have complex multiplication.

Using this it's fairly straight forward to verify the Hodge conjecture in these examples. First we recall the following result from invariant theory [FH, appendix].

**Theorem 4.3** (Weyl)**.** *Let $V$ a finite dimensional $\mathbb{Q}$-vector space with a nondegenerate symplectic form $\psi$. The invariants of $Sp(2g, V)$ on the space of tensors $(V^*)^{\otimes n}$ are generated by $\psi \in (V^*)^{\otimes 2}$ by tensor product and permutation of factors.*

**Proposition 4.4.** *If $X$ is a curve with $Hdg(H^1(X)) = Sp(2g)$, then the Hodge conjecture holds for $X \times X \times \ldots X$.*

*Proof.* Here's a sketch. By Künneth's formula

$$H^i(X^n) = \bigoplus_{i_1+\ldots i_n=i} H^{i_1}(X) \otimes \ldots \otimes H^{i_n}(X)$$

Since $H^0(X)$ and $H^2(X)$ are represented by algebraic cycles, we can rewrite this as

$$\bigoplus H^1(X) \otimes \ldots \otimes H^1(X) \otimes [\text{alg. cycle}]$$

The Hodge cycles are the $Sp(2g)$ invariants. By Weyl these are generated by pullbacks of invariants under projections $X^n \to X^2$. These are all algebraic cycles. □

A more refined argument shows that GHC holds under the above assumption, c.f. [A]. For more sophisticated examples see [Grd, Mu2].

## 5. Motivation

Fix a smooth projective variety $Y$. Let

- $M_{hom}$ (resp. $M_{hom}(Y)$) denote the tensor category of homological motives (generated by $Y$).
- $M_A$ (resp. $M_A(Y)$) denote the tensor category of motives (modeled on $Y$) constructed by André [An]. The rough idea is to modify the construction of $M_{hom}$ by throwing in correspondences of the form $p_*(\alpha \cup *\beta)$ for projections $p : Y^n \to Y^m$, and algebraic cycles $\alpha, \beta$. Note that such classes would be algebraic assuming the standard conjectures. In particular, $M_A = M_{hom}$ if these hold.

We will say that a smooth projective variety $X$ is *motivated* by $Y$ iff $X \in M_A(Y)$. In hindsight, many earlier results can be reinterpreted in this language. (The attributions should be understood in this sense.)

**Example 5.1** (Bloch-Srinivas)**.** *A rationally connected $n$-fold is motivated by $n-2$-fold.*

**Example 5.2** (Biyalnicki-Birula)**.** *A smooth projective variety with a $\mathbb{C}^*$-action is motivated by its fixed point set.*

**Example 5.3** (Shioda)**.** *Fermat hypersurfaces are motivated by Fermat curves.*

The motivation for motivation is the following:

**Proposition 5.4.** *If one of the above conjectures (LSC, HC, GHC) holds for $Y$ and all its powers, then it holds for any variety motivated by it.*

**Corollary 5.5.** *LSC holds for a variety motivated by a curve or surface.*

*Proof.* LSC obviously holds in dimension $\leq 2$, and is stable under products [Kl]. □

## 6. Moduli Spaces

**Theorem 6.1** (A-). *If $M$ is a projective moduli space of stable parabolic bundles over a smooth projective curve $X$, then $X$ motivates $M$.*

For vector bundles, this result (in a more precise form) goes back to del Baño. From a previous result, we deduce

**Corollary 6.2** (Biswas-Narasimhan). *LSC holds for $M$.*

Putting the theorem together with known instances of HC, we get

**Corollary 6.3.** *If $X$ is*

(1) *A curve of genus* $2, 3$
(2) *A Fermat curve of prime degree.*
(3) *A modular curve.*

*then the Hodge conjecture holds for $M$. If $X$ is sufficiently general then GHC*[5] *holds for $M$.*

**Theorem 6.4** (A-). *If $M$ is a projective moduli space of stable bundles over an Abelian or K3 surface $X$, then $X$ motivates $M$.*

**Corollary 6.5.** *The Hodge conjecture holds for $M$, when $X$ is Abelian.*

The proof is based on:

**Lemma 6.6.** *If there are algebraic correspondences on $X \times M$ whose Künneth components generate $H^*(M)$ then $X$ motivates $M$.*

*Proof.* Each correspondence yield morphisms of motives

$$[X](*) \to [M]$$

Taking products of these maps and composing with diagonals yields an epimorphism

$$\oplus [X]^{\otimes *}(*) \to \oplus [M]^{\otimes *} \to [M]$$

□

To prove theorem 6.1 in the special case where $M$ is the moduli of vector bundles, let $E$ be the universal vector bundle ( $M$ would be fine by the assumptions). Atiyah-Bott implies that $c_i(E)$ satisfies the assumptions of the above lemma. For the general case of theorem 6.1, and for theorem 6.1 we use results of Biswas-Ragavendra and Markman in place of Atiyah-Bott. The details can be found in [A].

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