INTRODUCTION TO COMPLEX ALGEBRAIC GEOMETRY/HODGE THEORY

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I assume that everyone has some familiarity with basic algebraic geometry. For our purposes, the main objects are complex quasiprojective algebraic varieties (henceforth called varieties). These are solutions to

$$f_i(x_0,\ldots,x_n) = 0, g_j(x_0,\ldots,x_n) \neq 0$$

in complex projective space $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$, where f_i, g_j are finite collection of homogeneous polynomials. To simplify matters, assume that the variety X is projective, which means that there are no g_j's. For local questions, it is more convenient to dehomogenize the equations and work in $\mathbb{C}^n$. In particular, we say that X is nonsingular or smooth (at p) if the Jacobian matrix, i.e. the matrix of partial derivatives , of the dehomogenized equations has expected rank everywhere (or just at p). When X is nonsingular, then X is a manifold, and in fact a complex manifold by the usual (or holomorphic) implicit function theorem. The dimension of X as an algebraic variety equals the dimension as a complex manifold or twice the dimension as a real manifold. Unless otherwise stated, you should assume I am talking about complex dimension. Also, at least initially, I will assume that everything is nonsingular.

1. Algebraic curves

If X is a nonsingular projective algebraic curve, then it is a compact one dimensional complex manifold by the above discussion. (Note these are usually called Riemann surfaces, but I think it will be less confusing to adhere to the above rule about dimensions.) Here we recall one of the basic theorems:

Theorem 1.1. *As a topological space X is determined by a single integer $g \geq 0$, called the genus, where $g = 0$ corresponds to the 2-sphere S^2, $g = 1$ to the 2-torus, ...*

However, for each $g > 0$, there are infinitely many different (i.e. nonisomorphic) curves. This begs the question, how do you actually tell them apart? The classical answer is via the period matrix. Before getting to this we recall a few things. The first homology group $H_1(X,\mathbb{Z})$ is very roughly the set of $\mathbb{Z}$-linear combinations of *real* closed curves on X modulo the equivalence relation that their difference is the boundary of a (real) surface embedded in X. A basic course in algebraic topology will tell you how to make this precise, and also how to compute it: $H_1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$. Next we need to talk holomorphic 1-forms. These are expressions locally given by $f(z)dz$ where z is an analytic coordinate and $f(z)$ a holomorphic. Again this is not very precise, since I haven't said what this means exactly, but never mind. $H^0(X,\Omega^1_X)$ will denote the complex vector space of holomorphic 1-forms. Here is a first miracle in the subject.

Theorem 1.2 (Riemann, Weyl). [1] *The dimension of $H^0(X, \Omega^1_X)$ equals the genus.*

Assuming this, we can proceed to define the period matrix. First, choose $2g$ closed curves $\gamma_i \subset X$ which forms a basis for the first homology $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$. (Actually, we need to choose a particular kind of basis called a symplectic basis.) Then choose a basis ω_j of $H^0(X, \Omega^1_X)$. The period matrix is the $g \times 2g$ matrix $(\int_{\gamma_i} \omega_j)$.

Theorem 1.3 (Torelli). *The period matrix (modulo changes of basis) determines X.*

Theorem 1.2 is a special case of the Hodge theorem, which well will get to shortly. A proof of theorem 1.3 can be found in [GH].

2. Algebraic surfaces

By an algebraic surface X, we mean a 2 dimensional nonsingular projective variety. Remember, dimension is complex dimension, so this has 4 real dimensions. The subject originated with the work of Castelnuovo, Enriques, Max Noether (Emmy's father), and Picard in the late 19th century. In a sense, they were ahead of there time, so they did not have the tools to rigorously justify a lot of their arguments. But this came later. The first result from this period, that I want to discuss is the finiteness of the Picard number. Recall that a divisor $D = \sum n_i D_i$ is a formal linear combination of irreducible curves $D_i \subset X$. For example, if f is a nonzero rational function on X, $div(f)$ is the sum of the zeros of f (counted with multiplicity) minus the sum of poles. The set of divisors, which forms a group, modulo the ones of the form $div(f)$ is called the divisor class group $Cl(X)$.

Theorem 2.1. *There exists a symmetric bilinear form $\cdot : Cl(X) \times Cl(X) \to \mathbb{Z}$ such that if D and E are irreducible curves which meet transversally, then $D \cdot E$ is the number of points of intersection.*

A proof can be found in [H]. We define two divisors D, D' to be numerically equivalent (or $D \equiv D'$) if $D \cdot E = D' \cdot E$ for all E.

Theorem 2.2. *The rank, called the Picard number, of $Cl(X)/\equiv$ is finite.*

Outline. The second homology $H_2(X, \mathbb{Z})$ is the set of 2-cycles in X modulo the equivalence relation modulo that differences should bound a 3-chain. Moreover, it is known to be finite dimensional from topology. Thus any divisor $D = \sum n_i D_i$ gives an element $[D] = \sum n_i [D_i] \in H_2(X, \mathbb{Z})$. If $D - D' = div(f)$, the preimage under f of a real curve joining 0 to ∞ in $\mathbb{P}^1$ gives a 3-chain with $D - D'$ as boundary. Thus $[D] = [D']$. Therefore we have a well defined map $Cl(X) \to H_2(X, \mathbb{Z})$. The pairing on $Cl(X)$ is also compatible with the intersection pairing on H_2 (which dual the cup product pairing on H^2). Thus $Cl(X)/\equiv$ injects into $H_2(X, \mathbb{Z})$. □

Given the proof, the next natural question is every element of $H_2(X, \mathbb{Z})$ represented by divisors. We will see that the answer is usually no. But first we need to discuss the Hodge decomposition.

[1] While I'm not a historian, I think this was certainly known to and used by Riemann, but I suspect Weyl gave the first rigorous proof along with the modern definition of Riemann surface in his 1913 book *Der Idee die Riemannfläche* [= The idea of a Riemann surface].

3. Hodge decomposition

Let us return briefly to the case of X a smooth projective algebraic curve of genus g. We want to understand the basic idea behind the proof of theorem 1.2. A C^∞ 1-form α is an expression given locally by $f(x,y)dx + g(x,y)dy$, where x, y are the real and imaginary parts of a complex coordinate z. We say it is closed if $d\alpha = (g_x - f_y)dx \wedge dy = 0$, and exact if $\alpha = dh = h_x dx + h_y dy$ for some h. Clearly, exact implies closed, but not the converse in general. The failure is measured by the first de Rham group

$$H^1(X, \mathbb{C}) = \frac{\{\text{ closed } \mathbb{C}\text{-valued 1-forms}\}}{\{\text{exact 1-forms}\}}$$

Note that we have a map

$$H^1(X, \mathbb{C}) \to Hom(H_1(X, \mathbb{Z}), \mathbb{C})$$

sending a form α to the functional $\gamma \mapsto \int_\gamma \alpha$.

Theorem 3.1 (de Rham). *This is an isomorphism; in particular* $\dim H^1(X, \mathbb{C}) = 2g$.

The key point is that we have a good set of representatives. Call a 1-form α harmonic if *locally* it equals dh, where h is harmonic in the usual sense. Then clearly harmonic forms are closed, and holomorphic forms as well as their complex conjugates (called antiholomorphic forms) are harmonic. What Riemann understood, Weyl proved, is this.

Theorem 3.2 (Riemann-Weyl). *Under the obvious map,*

$$\{\textit{harmonic 1-forms}\} \cong H^1(X, \mathbb{C})$$

Theorem 1.2 is a pretty immediate corollary, because we can decompose a harmonic 1-form uniquely into sum of a holomorphic and antiholomorphic forms.

When we go to higher dimensions, the naive definition of harmonic form we used above doesn't work. The correct thing to do is to first define the Laplacian. First, we need choose a Riemannian metric $g(-,-)$ which is C^∞ family of inner products on the tangent spaces. The choice of g will allow us to define inner products on the spaces of k-forms. (So for the moment, we may let X be a compact oriented Riemannian manifold.) Let d^* denote the adjoint to the exterior derivative d with respect to this inner product. Then set $\Delta = dd^* + d^*d$ (you should compare this to $\Delta = div\, grad$ that we learn in calculus).

Theorem 3.3 (Hodge theorem I). *The kth de Rham group $H^k(X, \mathbb{C})$, which* $\ker d$ *on k-forms modulo* $\operatorname{im} d$, *is isomorphic to* $\ker \Delta$ *acting on k-forms.*

A self contained stripped down proof can be found in [Wa]. Other proofs can be found in [GH, W]. I tried to explain the heat equation proof in [A2] but it's only an outline. Now let us return to the case where X is a smooth projective variety of arbitrary dimension. It turns out, that if we use an arbitrary Riemannian metric, then harmonic theory will not interact well with the holomorphic structure, which is the whole point for us. The solution is to use a special kind of metric called a Kähler metric. There are several ways to say what this means. Here is one. Since X is a complex manifold, it comes with an operator J which corresponds to multiplication by i on the real tangent spaces (more explicitly $\frac{\partial}{\partial x_j} \to \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_j} \to -\frac{\partial}{\partial x_j}$). We first of all require that J is orthogonal with respect to g, i.e. $g(\xi, \eta) = g(J\xi, J\eta)$. Then $\omega(\xi, \eta) = g(\xi, J\eta)$ is skew symmetric matrix, i.e. a 2-form. We also require $d\omega = 0$. At first glance, this looks a crazy condition, but here is another way to understand it: In Riemannian geometry, it is always possible to choose C^∞ coordinates about any point so that g becomes Euclidean up to second order. But if we insist on doing this with analytic coordinates then it is not always possible. In fact this is equivalent to the Kähler condition [GH]. These special coordinates can be used to help verify the Kähler identities discussed below. Here are the key points:

(1) Every smooth projective variety carries a Kähler metric inherited from its embedding into $\mathbb{P}^n$. Note the choice is far from unique.
(2) On any Kähler manifold, we get a magic identity (called a Kähler identity) that $\Delta = 2(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)$, where $\bar\partial$ is the Cauchy-Riemann operator and $\bar\partial$ is the adjoint. In essence this says exactly what were hoping that the harmonic and holomorphic aspects mesh well.

Details can be found in [GH, W]. Putting this together, we get

Theorem 3.4 (Hodge theorem II). *If X is compact Kähler manifold, then there is a bigrading*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{pq}(X)$$

such that

$$H^{pq} = \bar{H}^{qp}$$

Proof. Let $H^{pq}(X)$ be the space of harmonic forms of type (p, q), that is which are locally expressible as

$$\sum f_{i_1\ldots i_p, j_1\ldots j_q} dz_{i_1} \wedge \ldots dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots d\bar{z}_{j_p}$$

The first Hodge theorem, and the Kähler identity does the rest. □

Corollary 3.5. *The odd Betti numbers of a compact Kähler manifold, e.g. smooth projective variety, are even.*

The notion of harmonic form depends on a choice of metric, which not unique. However, the decomposition itself can be made independent of the metric. As a first step, we recall [GH, W]

Theorem 3.6 (Dolbeault). *$H^{pq}(X)$ is isomorphic to the qth sheaf cohomology of the sheaf of holomorphic p-forms $H^q(X, \Omega^p_X)$*

Thus we have an abstract isomorphism

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X, \Omega^p_X)$$

where the right side is independent of the metric, but this isn't quite good enough. A Hodge structure of weight k is a finitely generated abelian group $H_\mathbb{Z}$ with a bigrading on $H = H_Z \otimes \mathbb{C}$ as in theorem 3.4. This can be made into a category in an obvious way.

Theorem 3.7 (Deligne). *There is functor from the category of smooth projective varieties to Hodge structures which is abstractly isomorphic to the one above.*

Deligne [D2] proved a much more general result, that I say a few words about later.

4. Back to surfaces

We now apply the above story to an algebraic surface X. When D is a divisor, we have associated a class $[D] \in H_2(X, \mathbb{Z})$. It is convenient to switch to cohomology, we can do this because of

Theorem 4.1 (Poincaré duality)**.** *Let X be a compact oriented 4 (real) dimensional manifold (e.g. a surface), then there is a (specific) isomorphism $H^i(X, \mathbb{Z}) \cong H_{4-i}(X, \mathbb{Z})$. In particular, $H^4(X, \mathbb{Z}) \cong H_0(X, \mathbb{Z}) = \mathbb{Z}$ and $H^2(X, \mathbb{Z}) \cong H_2(X, \mathbb{Z})$.*

See [Hr] for a proof. Given a divisor D, we can therefore associate a cohomology class, with the same symbol, $[D] \in H^2(X, \mathbb{Z})$. We can now map this into de Rham cohomology. If $\alpha \in H^2(X, \mathbb{C})$ is another class represented by a 2-form. Then we can cup α with $[D]$ to a class in $H^4(X, \mathbb{C}) = \mathbb{C}$ and hence a number after integrating over X. This works out to

$$\int_X [D] \cup \alpha = \int_D \alpha$$

(if D is singular, pull it back to the normalization and then integrate). It follows that if α is an (anti)holomorphic 2-form, then this is zero. With a bit more playing around, this leads to

Proposition 4.2. $[D] \in H^{11}(X)$.

Alternatively, one prove this using the first Chern class. If D is given locally on an open set U_i by equations $f_i = 0$. Then f_i/f_j gives a cocycle, which you can use to build a line bundle $\mathcal{O}_X(D)$. The key point is that this cocycle defines a class in $H^1(X, \mathcal{O}_X^*)$, and an element of $c_1(\mathcal{O}_X(D)) \in H^2(X, \mathbb{Z})$ under the connecting map

$$c_1 : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$$

associated to the exponential sequence.

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp 2\pi i} \mathcal{O}_X^* \to 1$$

We see almost immediately that the composite

$$H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathcal{O}_X) \cong H^{02}(X)$$

is zero. Thus $\operatorname{im} c_1 \subset H^{11} \oplus H^{02}$. But it is invariant under conjugation, so it lies in the first factor. The only thing left to observe is

Theorem 4.3. $c_1(\mathcal{O}(D)) = [D]$.

(I wrote up a proof in [A2].) The advantage of doing it with c_1 is that the argument can be run backwards to obtain:

Theorem 4.4 (Lefschetz $(1,1)$ theorem)**.** *An integral class whose image lies in $H^{11}(X)$ must come from a divisor.*

Initially, we started out with the class group $Cl(X)$, and then we passed to $Cl(X)/\equiv$. We can ask, how different are they?

Theorem 4.5. *$Cl(X) \cong Cl(X)/\equiv$ if the first Betti number of X is zero.*

Sketch. It turns out that $Cl(X) \cong H^1(X, \mathcal{O}_X^*)$. Therefore we have a sequence

$$H^1(X, \mathcal{O}_X) \to Cl(X) \to Cl(X)/\equiv \to 0$$

It follows that if $H^1(X, \mathcal{O}_X) = 0$, then. Dolbeault's theorem tells us that this is the same has $H^{01}(X) = 0$. By the Hodge theorem this is equivalent to the vanishing of b_1. □

5. Problems and Directions

This a pretty incomplete list of where to go from here. While some of the low hanging fruit has been picked in these areas, I think there is a lot more.

5.1. **Vanishing theorems.** This is probably *the* most useful topic in Hodge theory to algebraic geometers working outside the area. The point is that sheaf cohomology often contains the obstructions to proving what you want, so it's good if you can get rid of it. The classic vanishing theorem, which goes back to Kodaira in the 1950's, says that an ample line bundle tensored with the canonical line bundle has no higher cohomology. The proof can be found in the standard textbooks [GH, W]. Since the 1980's a number of important descendants of this theorem have been found by Kawamata, Viehweg and others. A good account is in [L]. There is a really general vanishing theorem due to Saito [S] that is starting to get applied only now, probably because few people understood what it said. I think there some interesting possibilities here that I would be happy to discuss it further with anyone.

5.2. **Noether-Lefschetz.** The classical NL theorem says if you take a sufficiently general surface of degree ≥ 4 in $\mathbb{P}^3$ then the Picard number is 1 (which is the smallest it can be). "Sufficiently general" means that you have to throw away a countable union of proper Zariski closed sets from the moduli space of hypersurfaces. There has been some work on refining it, e.g. in finding roughly what is being thrown away, and also in generalizing it. See [CMP, V] for further discussion.

5.3. **Period map and Torelli theorems.** One way to phrase the classical Torelli theorem is that the "universal cover" of the moduli space of algebraic curves of genus g injects into the moduli of polarized Hodge structures of type $H = H^{10} \oplus H^{01}, \dim H^{10} = g$; this can be identified with the Siegel upper half plane of $g \times g$ matrices with positive definite imaginary part. In the early 1970's Griffiths generalized this picture to arbitrary Hodge structures. There has been a great deal of work trying to study these spaces (called Griffiths period domains), and in finding generalizations of the classical Torelli theorem from curves to other classes of varieties. Period domains are homogeneous manifolds, so not surprisingly the techniques involve a lot of representation theory. Again see [CMP, V] for more.

5.4. **Topology of varieties.** Hodge theory leads to many interesting and deep restrictions on the topology of algebraic varieties. I have already given one example about the Betti numbers, but there is much more. For example, Deligne, Griffiths, Morgan and Sullivan proved smooth projective varieties have the simplest possible structure from the point of view of rational homotopy theory (they are *formal* [GM]). One special problem that has attracted a certain amount of attention is understanding what fundamental groups can arise [ABC]. Note that by the Lefschetz hyperplane theorem [M], any such group would have to arise from a surface. So in particular, this is an obstruction to classifying algebraic surfaces up to homeomorphism, unlike the case of curves which is trivial. The problem is hard, nevertheless, it is possible to do things here. (This is an old stomping ground for me, and I have had at least two students write theses in this area.)

5.5. **Hodge conjecture and friends.** Solve it and you get a million bucks plus any job you want, but seriously there are realistic and interesting things that can be done in this area. The conjecture is that higher dimensional version of the Lefschetz $(1,1)$ should hold. Here is the precise statement:

Conjecture 5.6 (Hodge conjecture)**.** *If X is a smooth projective variety, and $\alpha \in H^{2p}(X,\mathbb{Q}) \cap H^{pp}(X)$, then α is the fundamental class of an algebraic cycle.*

Let me make a few comments.

(1) It is easy to see that this is a necessary condition to be an algebraic cycle.
(2) You may wonder why I didn't use $\mathbb{Z}$ as coefficients. In fact, Hodge [Ho] originally formulated it this way, but this was quickly shown to be false by Atiyah and Hirzebruch [AH] who showed that there are additional obstructions coming from Steenrod operations. These obstructions were refined clarified and Totaro [T]. Later Kollár, Soulé and Voisin found a totally different set of obstructions. So the fact that it's false is actually pretty interesting!
(3) To appreciate how hard the conjecture is, start with a smooth projective n dimensional variety X. Take $X \times X$, and decompose the diagonal $[\Delta] \in H^n(X \times X, \mathbb{Q})$ into its components under the Kunneth decomposition
$$H^n(X \times X, \mathbb{Q}) \cong \bigoplus_{i+j=n} H^i(X,\mathbb{Q}) \otimes H^j(X,\mathbb{Q})$$
These components satisfy the conditions of the Hodge conjecture, but it's far from obvious why they are algebraic in general. Incidentally, this special case is one of Grothendieck's standard conjectures [Gr].
(4) A good question is why is there so much fuss. Grothendieck reformulated the question in a way that seems more fundamental (at least to me). Given smooth projective varieties X and Y, a correspondence from X to Y is a cohomology class of an algebraic cycle on $X \times Y$. Think of it as multivalued function from X to Y. In particular, like functions these can be composed in a natural way. So we get a category. Grothendieck's idea was to complete this to an abelian category of so called (pure) motives, which would be the *universal* cohomology theory. Grothendieck did in fact construct the category of motives, but it fell short of what he wanted.[2]. He needed his standard conjectures to be true to prove it was abelian. This would all follow (in char 0) from the Hodge conjecture. From the point of view of motives, the Hodge conjecture is equivalent to the statement that the category of pure motives has a fully faithful embedding into the category of Hodge structures.
(5) A reasonable follow up question is what is the essential image of category of motives in Hodge structures? I don't think anyone even has a clue – not even a conjecture. By contrast, Fontaine and Mazur [FM] have at least made a conjecture on the p-adic Hodge theory side.
(6) There are alternative theories of motives, which lead to their own Hodge-like conjectures. I don't want to say much more about this, except to say that some of these seem to be easier to prove things about (e.g. see André [An]).

[2]Grothendieck never published anything about motives, but it was clear he was thinking about them since the early 1960's [GS]

(7) I realize I haven't talked about the things that one can realistically do. The point is that although proving these conjectures for *all* varieties would (presumably) require totally new ideas. Proving them for some varieties is doable, and frequently a lot of fun. There is a body of techniques for doing this in some cases. Some details can be found [A1, Le, V]. The kind of technique often depends on the nature of the variety. There has been a lot of interesting work for abelian varieties, where the main tool involves Mumford-Tate groups which to be really vague are what corresponds to Galois groups in Hodge theory. Another *really* interesting class to look at (for people with a background in automorphic forms) are Shimura varieties. Some nontrivial cases of the Hodge conjecture have been checked for these [R].

5.7. **Mixed Hodge structures etc.** I have said next to nothing about this, but this is pretty fundamental. Deligne [D2] generalized classical Hodge theory to arbitrary complex varieties (and even if one is interested primarily in smooth projective varieties, this is useful to understand). He showed that the kth cohomology of any variety carries a so called mixed Hodge structure which may be built up from pure Hodge structures of several different weights. As a simple example, take an elliptic curve E and pinch two points together to get a nodal curve X, then $H^1(X)$ is 3 dimensional, with one part of weight 1 coming from E and the remaining part of weight 0 comes from the singularity. Deligne was thinking about the Weil conjectures around the same time, and these ideas are connected [D1]. At about the same time, Griffiths [G] introduced the notion of variations of Hodge structure which is the right notion of a family of Hodge structures. These ideas were merged in the work of Saito [S] whose notion of mixed Hodge module includes both earlier notions as special cases. It was meant to be the mirror image in Hodge theory of the work of Beilinson, Bernstein, Deligne and Gabber [BBD] in étale cohomology. This stuff is incredibly technical, so I am a bit hesitant to recommend people to jump into this area, but if you want to look at it, probably the best place to start is [PS].

References

[ABC] Amoros, Burger, Corlette, Kotschick, Toledo, *Fundamental groups of compact Kähler manifolds*, AMS (1996)

[An] André, *Pour théorie inconditionelle des motifs*, IHES (1996)

[A1] Arapura, *Motivation for Hodge cycles*, Adv. Math. (2006)

[A2] Arapura, *Algebraic geometry over the complex numbers*, Springer. (2012)

[AH] Atiyah, Hirzebruch, *Analytic cycles on complex manifolds*, Topology (1961)

[BBD] Beilinson, Bernstein, Deligne, *Faisceux pervers*, Asterisque (1982)

[CMP] Carlson, Müller-Stach, Peters, *Period mappings and period domains.* Cambridge (2003)

[D1] Deligne, *Théorie de Hodge I*, ICM (1970)

[D2] Deligne, *Théorie de Hodge II, III*, IHES (1972, 74)

[FM] Fontaine, Mazur, *Geometric Galois representations*, (1993)

[G] Griffiths, *Periods of integrals of algebraic manifolds* IHES (1970)

[GH] Griffiths, Harris, *Principles of algebraic geometry*, Wiley (1978)

[GM] Griffiths, Morgan, *Rational homotopy theory and differential forms*, Birkhauser (2013)

[Gr] Grothendieck, *Standard conjectures on algebraic cycles*, Tata Inst. (1968)

[GS] Grothendieck, Serre, *Grothendieck-Serre correspondence* AMS (2004)

[H] Hartshorne, *Algebraic geometry*, Springer (1978)

[Hr] Hatcher, *Algebraic topology*, Cambridge (2002)

[Ho] Hodge, *Topological invariants of algebraic varieties,* ICM (1950)

[L] Lazarsfeld, *Positivity in algebraic geometry I, II*, Springer (2004)

[Le] Lewis, *A survey of the Hodge conjecture*, AMS (1999)
[M] Milnor, *Morse theory*, Princeton (1963)
[PS] Peters, Steenbrink, *Mixed Hodge structures*, Springer (2008)
[Wa] Warner, *Foundations of differentiable manifolds and Lie groups*, Springer
[R] Ramakrishnan, *Algebraic cycles on Hilbert modular fourfolds and poles of L-functions*, Tata (2004)
[S] Saito, *Modules de Hodge Polarizables; Mixed Hodge modules* RIMS (1988, 90)
[T] Totaro, *Torsion algebraic cycles and complex coobordism* JAMS (1997)
[V] Voisin, *Hodge theory and complex algebraic geometry I, II*, Cambridge (2003)
[W] Wells, *Differential analysis on complex manifolds*, Springer (2008)