

# Introduction to differential forms

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The calculus of differential forms give an alternative to vector calculus which is ultimately simpler and more flexible. Unfortunately it is rarely encountered at the undergraduate level. However, the last few times I taught undergraduate advanced calculus I decided I would do it this way. So I wrote up this brief supplement which explains how to work with them, and what they are good for. By the time I got to this topic, I had covered a certain amount of standard material, which is briefly summarized at the end of these notes.

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## 1 1-forms

A differential 1-form (or simply a differential or a 1-form) on an open subset of  $\mathbb{R}^2$  is an expression  $F(x, y)dx + G(x, y)dy$  where  $F, G$  are  $\mathbb{R}$ -valued functions on the open set. A very important example of a differential is given as follows: If  $f(x, y)$  is  $C^1$   $\mathbb{R}$ -valued function on an open set  $U$ , then its total differential (or exterior derivative) is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

It is a differential on  $U$ .

In a similar fashion, a differential 1-form on an open subset of  $\mathbb{R}^3$  is an expression  $F(x, y, z)dx + G(x, y, z)dy + H(x, y, z)dz$  where  $F, G, H$  are  $\mathbb{R}$ -valued functions on the open set. If  $f(x, y, z)$  is a  $C^1$  function on this set, then its total differential is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

At this stage, it is worth pointing out that a differential form is very similar to a vector field. In fact, we can set up a correspondence:

$$F\mathbf{i} + G\mathbf{j} + H\mathbf{k} \leftrightarrow Fdx + Gdy + Hdz$$

where  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are the standard unit vectors along the  $x, y, z$  axes. Under this set up, the gradient  $\nabla f$  corresponds to  $df$ . Thus it might seem that all we are doing is writing the previous concepts in a funny notation. However, the notation is

very suggestive and ultimately quite powerful. Suppose that that  $x, y, z$  depend on some parameter  $t$ , and  $f$  depends on  $x, y, z$ , then the chain rule says

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}$$

Thus the formula for  $df$  can be obtained by canceling  $dt$ .

## 2 Exactness in $\mathbb{R}^2$

Suppose that  $Fdx + Gdy$  is a differential on  $\mathbb{R}^2$  with  $C^1$  coefficients. We will say that it is *exact* if one can find a  $C^2$  function  $f(x, y)$  with  $df = Fdx + Gdy$ . Most differential forms are not exact. To see why, note that the above equation is equivalent to

$$F = \frac{\partial f}{\partial x}, \quad G = \frac{\partial f}{\partial y}.$$

Therefore if  $f$  exists then

$$\frac{\partial F}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial G}{\partial x}$$

But this equation would fail for most examples such as  $ydx$ . We will call a differential *closed* if  $\frac{\partial F}{\partial y}$  and  $\frac{\partial G}{\partial x}$  are equal. So we have just shown that if a differential is to be exact, then it had better be closed.

Exactness is a very important concept. You've probably already encountered it in the context of differential equations. Given an equation

$$\frac{dy}{dx} = \frac{F(x, y)}{G(x, y)}$$

we can rewrite it as

$$Fdx - Gdy = 0$$

If  $Fdx - Gdy$  is exact and equal to say,  $df$ , then the curves  $f(x, y) = c$  give solutions to this equation.

These concepts arise in physics. For example given a vector field  $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$  representing a force, one would like find a function  $P(x, y)$  called the potential energy, such that  $\mathbf{F} = -\nabla P$ . The force is called *conservative* (see section 8) if it has a potential energy function. In terms of differential forms,  $\mathbf{F}$  is conservative precisely when  $F_1dx + F_2dy$  is exact.

## 3 Parametric curves

Before discussing line integrals, we have to say a few words about parametric curves. A parametric curve in the plane is vector valued function  $C : [a, b] \rightarrow \mathbb{R}^2$ . In other words, we let  $x$  and  $y$  depend on some parameter  $t$  running from  $a$  to  $b$ . It is not just a set of points, but the trajectory of particle travelling along the

curve. To begin with, we will assume that  $C$  is  $C^1$ . Then we can define the velocity or tangent vector  $\mathbf{v} = (\frac{dx}{dt}, \frac{dy}{dt})$ . We want to assume that the particle travels without stopping,  $\mathbf{v} \neq 0$ . Then  $\mathbf{v}$  gives a direction to  $C$ , which we also refer to as its *orientation*. If  $C$  is given by

$$x = f(t), y = g(t), a \leq t \leq b$$

then

$$x = f(-u), y = g(-u), -b \leq u \leq -a$$

will be called  $-C$ . This represents the same set of points, but traveled in the opposite direction.

Suppose that  $C$  is given depending on some parameter  $t$ ,

$$x = f(t), y = g(t)$$

and that  $t$  depends in turn on a new parameter  $u = h(u)$  such that  $\frac{dt}{du} \neq 0$ . Then we can get a new parametric curve  $C'$

$$x = f(h(u)), y = g(h(u))$$

If the derivative  $\frac{dt}{du}$  is everywhere positive, we want to view the oriented curves  $C$  and  $C'$  as the equivalent. If this derivative is everywhere negative, then  $-C$  and  $C'$  are equivalent. For example, the curves

$$C : x = \cos \theta, y = \sin \theta, 0 \leq \theta \leq 2\pi$$

$$C' : x = \sin t, y = \cos t, 0 \leq t \leq 2\pi$$

represent going once around the unit circle counterclockwise and clockwise respectively. So  $C'$  should be equivalent to  $-C$ . We can see this rigorously by making a change of variable  $\theta = \pi/2 - t$ .

It will be convenient to allow piecewise  $C^1$  curves. We can treat these as unions of  $C^1$  curves, where one starts where the previous one ends. We can talk about parametrized curves in  $\mathbb{R}^3$  in pretty much the same way.

## 4 Line integrals

Now comes the real question. Given a differential  $Fdx + Gdy$ , when is it exact? Or equivalently, how can we tell whether a force is conservative or not? Checking that it's closed is easy, and as we've seen, if a differential is not closed, then it can't be exact. The amazing thing is that the converse statement is often (although not always) true:

**THEOREM 4.1** *If  $F(x, y)dx + G(x, y)dy$  is a closed form on all of  $\mathbb{R}^2$  with  $C^1$  coefficients, then it is exact.*

To prove this, we would need solve the equation  $df = Fdx + Gdy$ . In other words, we need to undo the effect of  $d$  and this should clearly involve some kind of integration process. To define this, we first have to choose a parametric  $C^1$  curve  $C$ . Then we define:

**DEFINITION 4.2**

$$\int_C Fdx + Gdy = \int_a^b [F(x(t), y(t)) \frac{dx}{dt} + G(x(t), y(t)) \frac{dy}{dt}] dt$$

If  $C$  is piecewise  $C^1$ , then we simply add up the integrals over the  $C^1$  pieces. Although we've done everything at once, it is often easier, in practice, to do this in steps. First change the variables from  $x$  and  $y$  to expressions in  $t$ , then replace  $dx$  by  $\frac{dx}{dt} dt$  etc. Then integrate with respect to  $t$ . For example, if we parameterize the unit circle  $\mathbf{c}$  by  $x = \cos \theta, y = \sin \theta, 0 \leq \theta \leq 2\pi$ , we see

$$-\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy = -\sin \theta (\cos \theta)' d\theta + \cos \theta (\sin \theta)' d\theta = d\theta$$

and therefore

$$\int_C -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy = \int_0^{2\pi} d\theta = 2\pi$$

From the chain rule, we get

**LEMMA 4.3**

$$\int_{-C} Fdx + Gdy = - \int_C Fdx + Gdy$$

If  $C$  and  $C'$  are equivalent, then

$$\int_C Fdx + Gdy = \int_{C'} Fdx + Gdy$$

While we're at it, we can also define a line integral in  $\mathbb{R}^3$ . Suppose that  $Fdx + Gdy + Hdz$  is a differential form with  $C^1$  coefficients. Let  $C : [a, b] \rightarrow \mathbb{R}^3$  be a piecewise  $C^1$  parametric curve, then

**DEFINITION 4.4**

$$\int_C Fdx + Gdy + Hdz = \int_a^b [F(x(t), y(t), z(t)) \frac{dx}{dt} + G(x(t), y(t), z(t)) \frac{dy}{dt} + H(x(t), y(t), z(t)) \frac{dz}{dt}] dt$$

The notion of exactness extends to  $\mathbb{R}^3$  automatically: a form is exact if it equals  $df$  for a  $C^2$  function. One of the most important properties of exactness is its path independence:

**PROPOSITION 4.5** *If  $\omega$  is exact and  $C_1$  and  $C_2$  are two parametrized curves with the same endpoints (or more accurately the same starting point and ending point), then*

$$\int_{C_1} \omega = \int_{C_2} \omega$$

It's quite easy to see why this works. If  $\omega = df$  and  $C_1 : [a, b] \rightarrow \mathbb{R}^3$  then

$$\int_{C_1} df = \int_a^b \frac{df}{dt} dt$$

by the chain rule. Now the fundamental theorem of calculus shows that the last integral equals  $f(C_1(b)) - f(C_1(a))$ , which is to say the value of  $f$  at the endpoint minus its value at the starting point. A similar calculation shows that the integral over  $C_2$  gives same answer. If the  $C$  is closed, which means that the starting point is the endpoint, then this argument gives

**COROLLARY 4.6** *If  $\omega$  is exact and  $C$  is closed, then  $\int_C \omega = 0$ .*

Now we can prove theorem 4.1. If  $Fdx + Gdy$  is a closed form on  $\mathbb{R}^2$ , set

$$f(x, y) = \int_C Fdx + Gdy$$

where the curve is indicated below:



We parameterize both line segments separately by  $x = t, y = 0$  and  $x = x(\text{constant}), y = t$ , and sum to get

$$f(x, y) = \int_0^x F(t, 0) dt + \int_0^y G(x, t) dt$$

Then we claim that  $df = Fdx + Gdy$ . To see this, we differentiate using the fundamental theorem of calculus. The easy calculation is

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} \int_0^y G(x, t) dt \\ &= G(x, y) \end{aligned}$$

Slightly trickier is

$$\begin{aligned}
 \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} \int_0^x F(x,0)dt + \frac{\partial}{\partial x} \int_0^y G(x,t)dt \\
 &= F(x,0) + \int_0^y \frac{\partial G(x,t)}{\partial x} dt \\
 &= F(x,0) + \int_0^y \frac{\partial F(x,t)}{\partial t} dt \\
 &= F(x,0) + F(x,y) - F(x,0) \\
 &= F(x,y)
 \end{aligned}$$

The same proof works if we replace  $\mathbb{R}^2$  by an open rectangle. However, it will fail for more general open sets. For example,

$$-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

is  $C^1$  1-form on the open set  $\{(x,y) \mid (x,y) \neq (0,0)\}$  which is closed. But it is not exact, since its integral along the unit circle is not 0.

## 5 Work

Line integrals have many important uses. One very direct application in physics comes from the idea of work. If you pick up a rock off the ground, or perhaps roll it up a ramp, it takes energy. The energy expended is called work. If you're moving the rock in straight line for a short distance, then the displacement can be represented by a vector  $\mathbf{d} = (\Delta x, \Delta y, \Delta z)$  and the force of gravity by a vector  $\mathbf{F} = (F_1, F_2, F_3)$ . Then the work done is simply

$$-\mathbf{F} \cdot \mathbf{d} = -(F_1 \Delta x + F_2 \Delta y + F_3 \Delta z).$$

On the other hand, if you decide to shoot a rocket up into space, then you would have to take into account that the trajectory  $\mathbf{c}$  may not be straight nor can the force  $\mathbf{F}$  be assumed to be constant (it's a vector field). However as the notation suggests, for the work we would now need to calculate the integral

$$-\int_{\mathbf{c}} F_1 dx + F_2 dy + F_3 dz$$

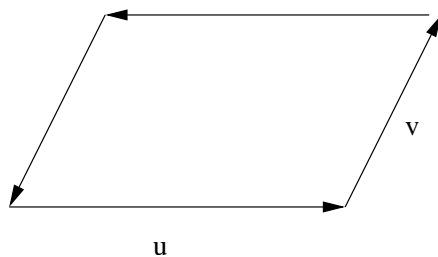
One often writes this as

$$-\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$$

(think of  $d\mathbf{s}$  as the "vector"  $(dx, dy, dz)$ .)

## 6 2-forms

The cross product of vectors  $\mathbf{u} \times \mathbf{v}$  is a very useful operation in 3 dimensional geometry. Its length gives the area of the parallelogram spanned by  $\mathbf{u}, \mathbf{v}$ , and it determines the plane containing this parallelogram. There is no direct analogue of the cross product in higher dimensions. However, there are certain features which do generalize. The essential idea is to define new objects called 2-vectors which are to planes what ordinary vectors are to lines. A vector in the usual sense is a directed line segment. It is determined by its length, the line it lies on, and a choice of one of two possible directions along the line. The basic building block of a 2-vector is an oriented parallelogram (we usually omit the modifier “oriented”). It has three attributes: its area, the plane on which it lies, and the orientation which is a choice of direction for walking around its perimeter. Two oriented parallelograms are considered equal if the areas are equal, the planes are parallel, and the orientations match. A parallelogram is equal to zero precisely when its area is, in which case the other attributes can be arbitrary. Given a parallelogram  $P$  and a number  $c$ , we define  $cP$  to be a parallelogram with the same plane, and area given by  $|c| \cdot \text{area}(P)$  and the same orientation if  $c > 0$  and the opposite orientation if  $c < 0$ . Given two vectors  $\mathbf{u}, \mathbf{v}$  we define the parallelogram  $\mathbf{u} \wedge \mathbf{v}$  to be given as follows ( $\wedge$  is pronounced “wedge”).



Evidently

$$\mathbf{v} \wedge \mathbf{u} = -\mathbf{u} \wedge \mathbf{v}$$

$$\mathbf{u} \wedge \mathbf{u} = 0$$

and

$$c(\mathbf{u} \wedge \mathbf{v}) = (c\mathbf{u}) \wedge \mathbf{v} = \mathbf{u} \wedge (c\mathbf{v})$$

The sum of vectors can be defined geometrically using the parallelogram law. Unfortunately there is no simple geometric rule for adding parallelograms, so we just add them formally without worrying about the meaning. A 2-vector is a finite sum of oriented parallelograms. The set of 2-vectors with these operations form a *vector space*. In other words, all the expected rules for  $+$  and  $\cdot$  apply. In addition, we have a distributive law:

$$\mathbf{u} \wedge (\mathbf{v} + \mathbf{w}) = \mathbf{u} \wedge \mathbf{v} + \mathbf{u} \wedge \mathbf{w}$$

We want to emphasize that the formalism of 2-vectors works in any dimension. In  $\mathbb{R}^3$ , we can identify 2-vectors with vectors via  $\mathbf{u} \wedge \mathbf{v} \rightarrow \mathbf{u} \times \mathbf{v}$ . However, we will usually refrain from doing this.

A 2-form is like a 2-vector but built using forms. On  $\mathbb{R}^3$ , this would be an expression:

$$F(x, y, z)dx \wedge dy + G(x, y, z)dy \wedge dz + H(x, y, z)dz \wedge dx$$

where  $F, G$  and  $H$  are functions defined on an open subset of  $\mathbb{R}^3$ . Any wedge product of two 1-forms can be put in this format. For example, using the above rules, we can see that

$$(3dx + dy) \wedge (dx + 2dy) = 6dx \wedge dy + dy \wedge dx = 5dx \wedge dy$$

The real significance of 2-forms will come later when we do surface integrals. A 2-form will be an expression that can be integrated over a surface in the same way that a 1-form can be integrated over a curve.

In order to make the comparison with traditional vector calculus, we note that we can convert vector fields to 2-forms and back

$$F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \leftrightarrow F_1dy \wedge dz + F_2dz \wedge dx + F_3dx \wedge dy,$$

Earlier, we learned how to convert a vector field to a 1-form:

$$F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \leftrightarrow F_1dx + F_2dy + F_3dz$$

To complete the triangle, we can interchange 1-forms and 2-forms using the so called Hodge star operator.

$$*(F_1dx + F_2dy + F_3dz) = F_1dy \wedge dz + F_2dz \wedge dx + F_3dx \wedge dy$$

$$*(F_1dy \wedge dz + F_2dz \wedge dx + F_3dx \wedge dy) = F_1dx + F_2dy + F_3dz$$

## 7 “d” of a 1-form and the curl

Given a 1-form  $F(x, y, z)dx + G(x, y, z)dy + H(x, y, z)dz$ . We want to define its derivative  $d\omega$  which will be a 2-form. The rules we use to evaluate it are:

$$d(\alpha + \beta) = d\alpha + d\beta$$

$$d(f\alpha) = (df) \wedge \alpha + f d\alpha$$

$$d(dx) = d(dy) = d(dz) = 0$$

where  $\alpha$  and  $\beta$  are 1-forms and  $f$  is a function. Putting these together yields a formula

$$d(Fdx + Gdy + Hdz) = (G_x - F_y)dx \wedge dy + (H_y - G_z)dy \wedge dz + (F_z - H_x)dz \wedge dx$$

where  $F_x = \frac{\partial F}{\partial x}$  and so on.

A 2-form can be converted to a vector field by replacing  $dx \wedge dy$  by  $\mathbf{k} = \mathbf{i} \times \mathbf{j}$ ,  $dy \wedge dz$  by  $\mathbf{i} = \mathbf{j} \times \mathbf{k}$  and  $dz \wedge dx$  by  $\mathbf{j} = \mathbf{k} \times \mathbf{i}$ . If we start with a vector field  $\mathbf{V} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$ , replace it by a 1-form  $Fdx + Gdy + Hdz$ , apply  $d$ , then convert it back to a vector field, we end up with the curl of  $\mathbf{V}$

$$\nabla \times \mathbf{V} = (H_y - G_z)\mathbf{i} + (G_x - F_y)\mathbf{k} + (F_z - H_x)\mathbf{j}$$

(In practice, one often writes this as a determinant

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F & G & H \end{vmatrix}.$$

)

## 8 Exactness in $\mathbb{R}^3$ and conservation of energy

A  $C^1$  1-form  $\omega = Fdx + Gdy + Hdz$  is called exact if there is a  $C^2$  function (called a potential) such that  $\omega = df$ .  $\omega$  is called closed if  $d\omega = 0$ , or equivalently if

$$F_y = G_x, F_z = H_x, G_z = H_y$$

Then exact 1-forms are closed.

**THEOREM 8.1** *If  $\omega = Fdx + Gdy + Hdz$  is a closed form on  $\mathbb{R}^3$  with  $C^1$  coefficients, then  $\omega$  is exact. In fact if  $f(x_0, y_0, z_0) = \int_C \omega$ , where  $C$  is any piecewise  $C^1$  curve connecting  $(0, 0, 0)$  to  $(x_0, y_0, z_0)$ , then  $df = \omega$ .*

This can be rephrased in the language of vector fields. If  $\mathbf{F} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$  is  $C^1$  vector field representing a force, then it is called conservative if there is a  $C^2$  real valued function  $P$ , called potential energy, such that  $\mathbf{F} = -\nabla P$ . The theorem implies that a force  $\mathbf{F}$ , which is  $C^1$  on all of  $\mathbb{R}^3$ , is conservative if and only if  $\nabla \times \mathbf{F} = 0$ .  $P(x, y, z)$  is just the work done by moving a particle of unit mass along a path connecting  $(0, 0, 0)$  to  $(x, y, z)$ .

To appreciate the importance of this concept, recall from physics that the kinetic energy of a particle of constant mass  $m$  and velocity

$$\mathbf{v} = \left( \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$$

is

$$K = \frac{1}{2}m\|\mathbf{v}\|^2 = \frac{1}{2}m\mathbf{v} \cdot \mathbf{v}.$$

Also one of Newton's laws says

$$m \frac{d\mathbf{v}}{dt} = \mathbf{F}$$

If  $\mathbf{F}$  is conservative, then we can replace it by  $-\nabla P$  above, move it to the other side, and then dot both sides by  $\mathbf{v}$  to obtain

$$m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \mathbf{v} \cdot \nabla P = 0$$

which simplifies<sup>1</sup> to

$$\frac{d}{dt}(K + P) = 0.$$

This implies that the total energy  $K + P$  is constant.

## 9 “d” of a 2-form and divergence

A 3-form is simply an expression

$$f(x, y, z)dx \wedge dy \wedge dz = -f(x, y, z)dy \wedge dx \wedge dz = f(x, y, z)dy \wedge dz \wedge dx = \dots$$

These are things that will eventually get integrated over solid regions. The important thing for the present is an operation which takes 2-forms to 3-forms once again denoted by “d”.

$$d(Fdy \wedge dz + Gdz \wedge dx + Hdx \wedge dy) = (F_x + G_y + H_z)dx \wedge dy \wedge dz$$

It’s probably easier to understand the pattern after converting the above 2-form to the vector field  $\mathbf{V} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}$ . Then the coefficient of  $dx \wedge dy \wedge dz$  is the divergence

$$\nabla \cdot \mathbf{V} = F_x + G_y + H_z$$

So far we’ve applied  $d$  to functions to obtain 1-forms, and then to 1-forms to get 2-forms, and finally to 2-forms. The real power of this notation is contained in the following simple-looking formula

**PROPOSITION 9.1**  $d^2 = 0$

What this means is that given a  $C^2$  real valued function defined on an open subset of  $\mathbb{R}^3$ , then  $d(df) = 0$ , and given a 1-form  $\omega = Fdx + Gdy + Hdz$  with  $C^2$  coefficients defined on an open subset of  $\mathbb{R}^3$ ,  $d(d\omega) = 0$ . Both of these are quite easy to check:

$$d(df) = (f_{yx} - f_{xy})dx \wedge dy + (f_{zy} - f_{yz})dy \wedge dz + (f_{xz} - f_{zx})dz \wedge dx = 0$$

$$d(d\omega) = [G_{xz} - F_{yz} + H_{yx} - G_{zx} + F_{zy} - H_{xy}]dx \wedge dy \wedge dz = 0$$

In terms of standard vector notation this is equivalent to

$$\nabla \times (\nabla f) = 0$$

$$\nabla \cdot (\nabla \times \mathbf{V}) = 0$$

The analogue of theorem 8.1 holds:

**THEOREM 9.2** *If  $\omega$  is a 2-form on  $\mathbb{R}^3$  such that  $d\omega = 0$ , then there exists a 1-form  $\xi$  such that  $d\xi = \omega$ .*

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<sup>1</sup>This takes a bit of work that I’m leaving as an exercise. It’s probably easier to work backwards. You’ll need the product rule for dot products and the chain rule.

## 10 Parameterized Surfaces

Recall that a parameterized curve is a  $C^1$  function from an interval  $[a, b] \subset \mathbb{R}^1$  to  $\mathbb{R}^3$ . If we replace the interval by a subset of the plane  $\mathbb{R}^2$ , we get a parameterized surface. Let's look at a few of examples

1) The upper half sphere of radius 1 centered at the origin can be parameterized using cartesian coordinates

$$\begin{cases} x = u \\ y = v \\ z = \sqrt{1 - u^2 - v^2} \\ u^2 + v^2 \leq 1 \end{cases}$$

2) The upper half sphere can be parameterized using spherical coordinates

$$\begin{cases} x = \sin(\phi) \cos(\theta) \\ y = \sin(\phi) \sin(\theta) \\ z = \cos(\phi) \\ 0 \leq \phi \leq \pi/2, 0 \leq \theta < 2\pi \end{cases}$$

3) The upper half sphere can be parameterized using cylindrical coordinates

$$\begin{cases} x = r \cos(\theta) \\ y = r \sin(\theta) \\ z = \sqrt{1 - r^2} \\ 0 \leq r \leq 1, 0 \leq \theta < 2\pi \end{cases}$$

An orientation on a curve is a choice of a direction for the curve. For a surface an orientation is a choice of "up" or "down". The easiest way to make this precise is to view an orientation as a choice of (an upward or outward pointing) unit normal vector field  $\mathbf{n}$  on  $S$ . A parameterized surface  $S$

$$\begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \\ (u, v) \in D \end{cases}$$

is called *smooth* provided that  $f, g, h$  are  $C^1$ , the function that they define from  $D \rightarrow \mathbb{R}^3$  is one to one, and the tangent vector fields

$$\mathbf{T}_u = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$$

$$\mathbf{T}_v = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$$

are linearly independent. In this case, once we pick an ordering of the variables (say  $u$  first,  $v$  second) an orientation is determined by the normal

$$\mathbf{n} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|}$$

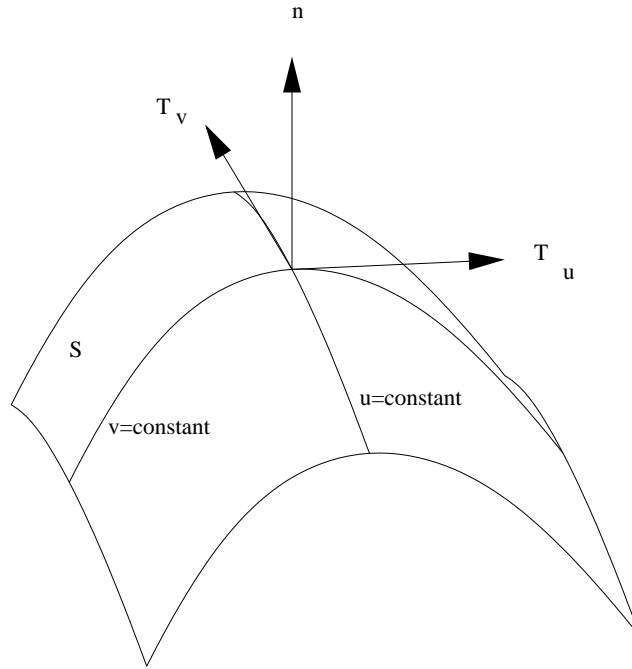


FIGURE 1

If we look at the examples given earlier. (1) is smooth. However there is a slight problem with our examples (2) and (3). Here  $\mathbf{T}_\theta = 0$ , when  $\phi = 0$  in example (2) and when  $r = 0$  in example (3). To deal with scenario, we will consider a surface smooth if there is at least one smooth parameterization for it.

Let  $C$  be a closed  $C^1$  curve in  $\mathbb{R}^2$  and  $D$  be the interior of  $C$ .  $D$  is an example of a surface with a boundary  $C$ . In this case the surface lies flat in the plane, but more general examples can be constructed by letting  $S$  be a parameterized surface

$$\begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \\ (u, v) \in D \subset \mathbb{R}^2 \end{cases}$$

then the image of  $C$  in  $\mathbb{R}^3$  will be the boundary of  $S$ . For example, the boundary of the upper half sphere  $S$

$$\begin{cases} x = \sin(\phi) \cos(\theta) \\ y = \sin(\phi) \sin(\theta) \\ z = \cos(\phi) \\ 0 \leq \phi \leq \pi/2, 0 \leq \theta < 2\pi \end{cases}$$

is the circle  $C$  given by

$$x = \cos(\theta), y = \sin(\theta), z = 0, 0 \leq \theta \leq 2\pi$$

In what follows, we will need to match up the orientation of  $S$  and its boundary curve. This will be done by the *right hand rule*: if the fingers of the right hand point in the direction of  $C$ , then the direction of the thumb should be “up”.

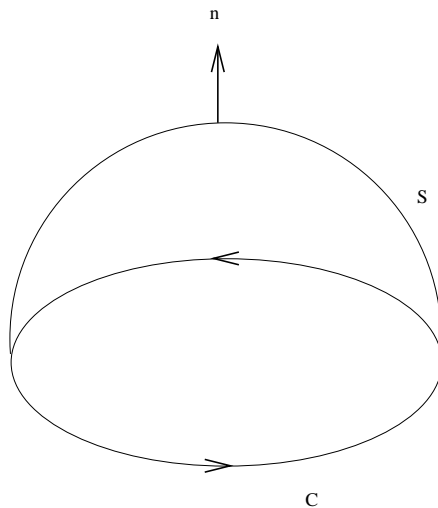


FIGURE 2

## 11 Surface Integrals

Let  $S$  be a smooth parameterized surface

$$\begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \\ (u, v) \in D \end{cases}$$

with orientation corresponding to the ordering  $u, v$ . The symbols  $dx$  etc. can be converted to the new coordinates as follows

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$$

$$\begin{aligned}
dx \wedge dy &= \left( \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right) \wedge \left( \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \right) \\
&= \left( \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) du \wedge dv \\
&= \frac{\partial(x, y)}{\partial(u, v)} du \wedge dv
\end{aligned}$$

In this way, it is possible to convert any 2-form  $\omega$  to  $uv$ -coordinates.

**DEFINITION 11.1** *The integral of a 2-form on  $S$  is given by*

$$\int \int_S F dx \wedge dy + G dy \wedge dz + H dz \wedge dx = \int \int_D \left[ F \frac{\partial(x, y)}{\partial(u, v)} + G \frac{\partial(y, z)}{\partial(u, v)} + H \frac{\partial(z, x)}{\partial(u, v)} \right] du dv$$

In practice, the integral of a 2-form can be calculated by first converting it to the form  $f(u, v) du \wedge dv$ , and then evaluating  $\int \int_D f(u, v) du dv$ .

Let  $S$  be the upper half sphere of radius 1 oriented with the upward normal parameterized using spherical coordinates, we get

$$dx \wedge dy = \frac{\partial(x, y)}{\partial(\phi, \theta)} d\phi \wedge d\theta = \cos(\phi) \sin(\phi) d\phi \wedge d\theta$$

So

$$\int \int_S dx \wedge dy = \int_0^{2\pi} \int_0^{\pi/2} \cos(\phi) \sin(\phi) d\phi d\theta = \pi$$

On the other hand if use the same surface parameterized using cylindrical coordinates

$$dx \wedge dy = \frac{\partial(x, y)}{\partial(r, \theta)} dr \wedge d\theta = r dr \wedge d\theta$$

Then

$$\int \int_S dx \wedge dy = \int_0^{2\pi} \int_0^1 r dr d\theta = \pi$$

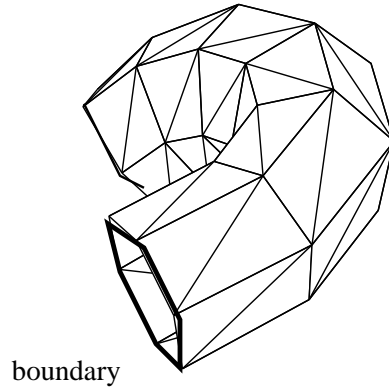
which leads to the same answer as one would hope. The general result is:

**THEOREM 11.2** *Suppose that a oriented surface  $S$  has two different smooth  $C^1$  parameterizations, then for any 2-form  $\omega$ , the expression for the integrals of  $\omega$  calculated with respect to both parameterizations agree.*

(This theorem needs to be applied to the half sphere with the point  $(0, 0, 1)$  removed in the above examples.)

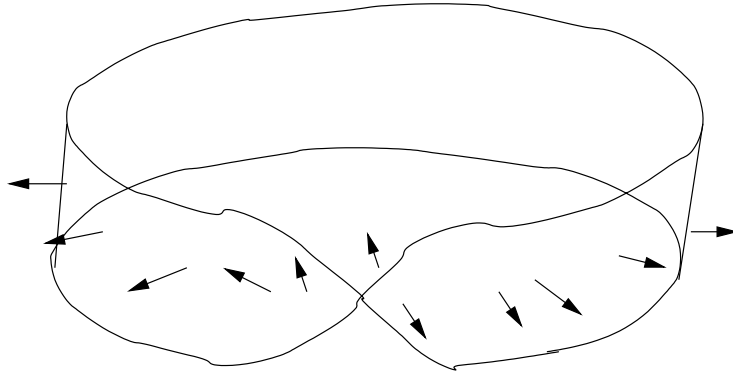
## 12 Surface integrals (continued)

Complicated surfaces may be divided up into nonoverlapping patches which can be parameterized separately. The simplest scheme for doing this is to *triangulate* the surface, which means that we divide it up into triangular patches as depicted below. Each triangle on the surface can be parameterized by a triangle on the plane.



We will insist that if any two triangles touch, they either meet only at a vertex, or they share an entire edge. We define the *boundary* of a surface to be the union of all edges which are not shared. The surface is called *closed* if the boundary is empty.

Given a surface which has been divided up into patches, we can integrate a 2-form on it by summing up the integrals over each patch. However, we require that the orientations match up, which is possible if the surface has “two sides”. Below is a picture of a one sided, or nonorientable, surface called the Mobius strip.



Once we have pick and orientation of  $S$ , we get one for the boundary using the right hand rule.

In many situations arising in physics, one needs to integrate a vector field  $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$  over a surface. The resulting quantity is often called a

*flux*. We will simply define this integral, which is usually written as  $\int \int_S \mathbf{F} \cdot d\mathbf{S}$  or  $\int \int_S \mathbf{F} \cdot \mathbf{n} dS$ , to mean

$$\int \int_S F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$$

It is probably easier to view this as a two step process, first convert  $\mathbf{F}$  to a 2-form as follows:

$$F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k} \leftrightarrow F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy,$$

then integrate. As a typical example, consider a fluid such as air or water. Associated to this, there is a scalar field  $\rho(x, y, z)$  which measures the density, and a vector field  $\mathbf{v}$  which measures the velocity of the flow (e.g. the wind velocity). Then the rate at which the fluid passes through a surface  $S$  is given by the flux integral  $\int \int_S \rho \mathbf{v} \cdot d\mathbf{S}$

### 13 Length and Area

It is important to realize some line and surface integrals are not expressible as integrals of differential forms in general. Two notable examples are the arclength and area integrals.

**DEFINITION 13.1** *The arclength of  $C : [a, b] \rightarrow \mathbb{R}^3$  is given by*

$$\int_C ds = \int_a^b \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt} + \frac{dz^2}{dt}} dt$$

The symbol  $ds$  is not a 1-form in spite of the notation. For example

$$\int_{-C} ds = \int_C ds$$

whereas for 1-forms the integral would change sign. Nevertheless,  $ds$  (or more accurately its square) is a sort of generalization of a differential form called a *tensor*. To get a sense what this means, let us calculate the arclength of a curve lying on a surface. Suppose that  $S$  is a parameterized surface given by

$$\begin{cases} x = f(u, v) \\ y = g(u, v) \\ z = h(u, v) \\ (u, v) \in D \end{cases}$$

and suppose that  $C$  lies on  $S$ . This means that there are functions  $k, \ell : [a, b] \rightarrow \mathbb{R}$  such that  $x = f(k(t), \ell(t)), \dots$  determines  $C$ . We can calculate the arclength of  $C$  by applying the chain to the above integral all at once. Instead, we want to break this down into a series of steps.

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$\begin{aligned}
dy &= \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \\
dz &= \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv
\end{aligned}$$

For the next step, we introduce a new product (indicated by juxtaposition) which is distributive and unlike the wedge product is commutative. We square the previous formulas and add them up. (The objects that we are getting are tensors.)

$$\begin{aligned}
dx^2 &= \left(\frac{\partial x}{\partial u}\right)^2 du^2 + 2\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} dudv + \left(\frac{\partial x}{\partial v}\right)^2 dv^2 \\
dy^2 &= \left(\frac{\partial y}{\partial u}\right)^2 du^2 + 2\frac{\partial y}{\partial u}\frac{\partial y}{\partial v} dudv + \left(\frac{\partial y}{\partial v}\right)^2 dv^2 \\
dz^2 &= \left(\frac{\partial z}{\partial u}\right)^2 du^2 + 2\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} dudv + \left(\frac{\partial z}{\partial v}\right)^2 dv^2 \\
dx^2 + dy^2 + dz^2 &= Edu^2 + 2Fdudv + Gdv^2 \tag{1}
\end{aligned}$$

where

$$E = \|\mathbf{T}_u\|^2, \quad F = \mathbf{T}_u \cdot \mathbf{T}_v, \quad G = \|\mathbf{T}_v\|^2$$

in the notation of section 10. The expression in (1) is called the metric tensor of the surface. We can easily deduce a formula for arclength in terms of it:

$$\int_C ds = \int_a^b \sqrt{E \frac{du^2}{dt} + 2F \frac{du}{dt} \frac{dv}{dt} + G \frac{dv^2}{dt}} dt$$

The area of  $S$  can also be expressed in terms of the metric tensor. First recall that

**DEFINITION 13.2** *The area of  $S$  is given by*

$$\iint_S dS = \iint_D \|\mathbf{T}_u \times \mathbf{T}_v\| dudv$$

**THEOREM 13.3** *The area is given by*

$$\iint_S dS = \iint_D \sqrt{EG - F^2} dudv$$

The proof is as follows

$$\begin{aligned}
\|\mathbf{T}_u \times \mathbf{T}_v\|^2 &= \|\mathbf{T}_u\|^2 \|\mathbf{T}_v\|^2 \sin^2 \theta \\
&= \|\mathbf{T}_u\|^2 \|\mathbf{T}_v\|^2 (1 - \cos^2 \theta) \\
&= \|\mathbf{T}_u\|^2 \|\mathbf{T}_v\|^2 - (\mathbf{T}_u \cdot \mathbf{T}_v)^2 \\
&= EG - F^2
\end{aligned}$$

If  $S$  is sphere of radius 1 parameterized by spherical coordinates, a straight forward calculation gives the metric tensor as

$$\sin^2 \phi d\theta^2 + d\phi^2$$

which yields

$$\text{area}(S) = \int_0^{2\pi} \int_0^\pi \sin \phi d\phi d\theta = 4\pi$$

as expected.

## 14 Green's and Stokes' Theorems

Stoke's theorem is really the fundamental theorem of calculus of surface integrals. We assume that surfaces can be triangulated.

**THEOREM 14.1 (Stokes' theorem)** *Let  $S$  be an oriented smooth surface with smooth boundary curve  $C$ . If  $C$  is oriented using the right hand rule, then for any  $C^1$  1-form  $\omega$  on an open set of  $\mathbb{R}^3$  containing  $S$ ,*

$$\int \int_S d\omega = \int_C \omega$$

If the surface lies in the plane, it is possible make this very explicit:

**THEOREM 14.2 (Green's theorem)** *Let  $C$  be a closed  $C^1$  curve in  $\mathbb{R}^2$  oriented counterclockwise and  $D$  be the interior of  $C$ . If  $P(x, y)$  and  $Q(x, y)$  are both  $C^1$  functions then*

$$\int_C Pdx + Qdy = \int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

As an application, Green's theorem shows that the area of  $D$  can be computed as line integral on the boundary

$$\int \int_D dx dy = \int_C y dx$$

If  $S$  is a closed oriented surface in  $\mathbb{R}^3$ , such as the surface of a sphere, Stoke's theorem show that any exact 2-form integrates to 0. To see this write  $S$  as the union of two surfaces  $S_1$  and  $S_2$  with common boundary curve  $C$ . Orient  $C$  using the right hand rule with respect to  $S_1$ , then orientation coming from  $S_2$  goes in the opposite direction. Therefore

$$\int \int_S d\omega = \int \int_{S_1} d\omega + \int \int_{S_2} d\omega = \int \int_C \omega - \int \int_C \omega = 0$$

In vector notation, Stokes' theorem is written as

$$\int \int_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS = \int_C \mathbf{F} \cdot d\mathbf{s}$$

where  $\mathbf{F}$  is a  $C^1$ -vector field.

In physics, there are two fundamental vector fields, the electric field  $\mathbf{E}$  and the magnetic field  $\mathbf{B}$ . They're governed by Maxwell's equations, one of which is

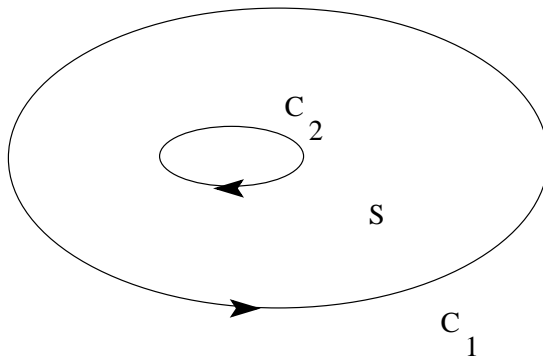
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

where  $t$  is time. If we integrate both sides over  $S$ , apply Stokes' theorem and simplify, we obtain Faraday's law of induction:

$$\int_C \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \mathbf{n} dS$$

To get a sense of what this says, imagine that  $C$  is a wire loop and that we are dragging a magnet through it. This action will induce an electric current; the left hand integral is precisely the induced voltage and the right side is related to the strength of the magnet and the rate at which it is being dragged through.

Stokes' theorem works even if the boundary has several components. However, the inner and outer components would have opposite directions.



**THEOREM 14.3 (Stokes' theorem II)** *Let  $S$  be an oriented smooth surface with smooth boundary curves  $C_1, C_2, \dots$ . If  $C_i$  is oriented using the right hand rule, then for any  $C^1$  1-form  $\omega$  on an open set of  $\mathbb{R}^3$  containing  $S$ ,*

$$\iint_S d\omega = \int_{C_1} \omega + \int_{C_2} \omega + \dots$$

## 15 Cauchy's theorem

Recall that a *complex number* is an expression  $z = a + bi$  where  $a, b \in \mathbb{R}$  and  $i = \sqrt{-1}$ , so that  $i^2 = -1$ . The components  $a$  and  $b$  are called the real and imaginary parts of  $z$ . We can identify the set of complex numbers  $\mathbb{C}$  with the plane  $\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}$ . Addition and subtraction of complex numbers correspond to the usual vector operations:

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$$

However, we can do more, such as multiplication and division:

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$$

The next step is calculus. The power of complex numbers is evident in the beautiful formula of Euler

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

which unifies the basic functions of calculus. Given a function  $f : \mathbb{C} \rightarrow \mathbb{C}$ , we can write it as

$$f(z) = f(x + yi) = u(x, y) + iv(x, y)$$

where  $x, y$  are real and imaginary parts of  $z \in \mathbb{C}$ , and  $u, v$  are the real and imaginary parts of  $f$ .  $f$  is continuous at  $z = a + bi$  if  $u$  and  $v$  are continuous at  $(a, b)$  in the usual sense. So far there are no surprises. However, things get more interesting when we define the complex derivative

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z + h) - f(z)}{h}$$

Notice that  $h$  is a complex number. For the limit to exist, we should get the same value no matter how it approaches 0. If  $h = \Delta x$  approaches along the  $x$ -axis, we get

$$\begin{aligned} f'(z) &= \lim_{\Delta x \rightarrow 0} \frac{[u(x + \Delta x, y) - u(x, y)] + i[v(x + \Delta x, y) - v(x, y)]}{\Delta x} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \end{aligned}$$

If  $h = \Delta y i$  approaches along the  $y$ -axis, then

$$\begin{aligned} f'(z) &= \lim_{\Delta y \rightarrow 0} \frac{[u(x, y + \Delta y) - u(x, y)] + i[v(x, y + \Delta y) - v(x, y)]}{i\Delta y} \\ &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \end{aligned}$$

Setting these equal leads to the *Cauchy-Riemann* equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These equations have to hold when the complex derivative  $f'(z)$  exists, and in fact  $f'(z)$  exists when they do.  $f$  is called *analytic* at  $z = a + bi$  when these hold at that point. For example,  $z^2 = x^2 + y^2 + 2xyi$  and

$$e^z = e^x \cos(y) + ie^x \sin(y)$$

are analytic everywhere. But  $f(z) = \bar{z} = x - iy$  is not analytic anywhere.

A complex differential form is an expression  $\alpha + i\beta$  where  $\alpha, \beta$  are differential forms in the usual sense. Complex 1-forms can be integrated by the rule

$$\int_C \alpha + i\beta = \int_C \alpha + i \int_C \beta$$

Suppose that  $f$  is analytic. Then expanding

$$f(z)dz = (u + iv)(dx + idy) = [udx - vdy] + i[vdx + udy]$$

Differentiating and applying the Cauchy-Riemann equations shows

$$d(f(z)dz) = -\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)dx \wedge dy + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx \wedge dy = 0$$

Therefore Stokes' theorem implies what may be thought of as the fundamental theorem of complex analysis:

**THEOREM 15.1 (Cauchy's theorem)** *If  $f(z)$  is analytic on a region with boundary  $C$  then*

$$\int_C f(z)dz = 0$$

Suppose we replace  $f(z)$  by  $g(z) = \frac{f(z)}{z}$ . This is analytic away from 0. Therefore the theorem applies to the boundary of any region not containing 0. If  $C$  is a closed positively curve whose interior  $U$  contains 0. Then applying Cauchy's theorem to a  $U - D_r$ , where  $D_r$  is a disk of small radius  $r$  in  $U$ , shows that

$$\int_C g(z)dz = \int_{C_r} g(z)dz \tag{2}$$

Here  $C_r$  is a circle of radius  $r$  around 0. We can parameterize this with the help of Euler's formula by

$$z = r \cos(\theta) + ri \sin(\theta) = re^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

Then  $dz = rie^{i\theta} d\theta$ , so that

$$\int_{C_r} g(z)dz = ri \int_0^{2\pi} \frac{f(re^{i\theta})}{re^{i\theta}} e^{i\theta} d\theta = i \int_0^{2\pi} f(re^{i\theta}) d\theta$$

As  $r \rightarrow 0$ ,  $f(re^{i\theta}) \rightarrow f(0)$ , therefore the above integral approaches  $2\pi r$ . Since (2) holds for all small  $r$ , it follows that this equality holds on the nose. Therefore:

**THEOREM 15.2 (Cauchy's Integral Formula)** *If  $f(z)$  is analytic in the interior of positively oriented closed curve  $C$ , then*

$$f(0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z} dz$$

Using a change of variable  $z \rightarrow z - a$ , we get a more general formula:

**COROLLARY 15.3 (Cauchy's Integral Formula II)** *With the same assumptions, for any point  $a$  in the interior of  $C$*

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - a} dz$$

This formula has many uses. Among other things, it can be used to evaluate complicated definite integrals. This and more can be found in any book on complex analysis.

## 16 Triple integrals and the divergence theorem

A 3-form is an expression  $f(x, y, z)dx \wedge dy \wedge dz$ . Given a solid region  $V \subset \mathbb{R}^3$ , we define

$$\int \int \int_V f(x, y, z)dx \wedge dy \wedge dz = \int \int \int_V f(x, y, z)dxdydz$$

**THEOREM 16.1 (Divergence theorem)** *Let  $V$  be the interior of a smooth closed surface  $S$  oriented with the outward pointing normal. If  $\omega$  is a  $C^1$  2-form on an open subset of  $\mathbb{R}^3$  containing  $V$ , then*

$$\int \int \int_V d\omega = \int \int_S \omega$$

In standard vector notation, this reads

$$\int \int \int_V \nabla \cdot \mathbf{F} dV = \int \int_S \mathbf{F} \cdot \mathbf{n} dS$$

where  $\mathbf{F}$  is a  $C^1$  vector field.

As an application, consider a fluid with density  $\rho$  and velocity  $\mathbf{v}$ . If  $S$  is the boundary of a solid region  $V$  with outward pointing normal  $\mathbf{n}$ , then the flux  $\int \int_S \rho \mathbf{v} \cdot \mathbf{n} dS$  is the rate at which matter flows out of  $V$ . In other words, it is minus the rate at which matter flows in, and this equals  $-\partial/\partial t \int \int \int_V \rho dV$ . On the other hand, by the divergence theorem, the above double integral equals  $\int \int \int_S \nabla \cdot (\rho \mathbf{v}) dV$ . Setting these equal and subtracting yields

$$\int \int \int_V [\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t}] dV = 0.$$

The only way this can hold for all possible regions  $V$  is that the integrand

$$\nabla \cdot (\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} = 0 \tag{3}$$

This is one of the basic laws of fluid mechanics.

We can extend the divergence theorem to solids with disconnected boundary. Suppose that  $S_2$  is a smooth closed oriented surface contained inside another such surface  $S_1$ . We use the outward pointing normal  $S_1$  and the inner pointing normal on  $S_2$ . Let  $V$  be region in between  $S_1$  and  $S_2$ . Then,

**THEOREM 16.2 (Divergence theorem II)** *If  $\omega$  is a  $C^1$  2-form on an open subset of  $\mathbb{R}^3$  containing  $V$ , then*

$$\int \int \int_V d\omega = \int \int_{S_1} \omega + \int \int_{S_2} \omega$$

## 17 Gravitational Flux

Place a “point particle” of mass  $m$  at the origin of  $\mathbb{R}^3$ , then this generates a force on any particle of unit mass at  $\mathbf{r} = (x, y, z)$  given by

$$\mathbf{F} = \frac{-m\mathbf{r}}{r^3}$$

where  $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$ . This has singularity at 0, so it is a vector field on  $\mathbb{R}^3 - \{0\}$ . The corresponding 2-form is given by

$$\omega = -m \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3}$$

Let  $B_R$  be the ball of radius  $R$  around 0, and let  $S_R$  be its boundary. Since the outward unit normal to  $S_R$  is just  $\mathbf{n} = \mathbf{r}/r$ . One might expect that the flux

$$\begin{aligned} \int \int_{S_R} \mathbf{F} \cdot \mathbf{n} dS &= -\frac{m}{R^2} \int \int_{S_R} dS \\ &= -\frac{m}{R^2} \text{area}(S_R) = -\frac{m}{R^2} (4\pi R^2) \end{aligned}$$

However, this is really just a proof by notation at this point. To justify it, we work in spherical coordinates.  $S_R$  is given by

$$\begin{cases} x = R \sin(\phi) \cos(\theta) \\ y = R \sin(\phi) \sin(\theta) \\ z = R \cos(\phi) \\ 0 \leq \phi \leq \pi, 0 \leq \theta < 2\pi \end{cases}$$

Rewriting  $\omega$  in these coordinates and simplifying:

$$\omega = -mR^3 \sin(\phi) d\phi \wedge d\theta$$

Therefore,

$$\int \int_{S_R} \mathbf{F} \cdot \mathbf{n} dS = \int \int_{S_R} \omega = -4\pi m$$

as hoped. We claim that if  $S$  is any closed surface not containing 0

$$\int \int_S \omega = \begin{cases} -4\pi m & \text{if 0 lies in the interior of } S \\ 0 & \text{otherwise} \end{cases}$$

By direct calculation, we see that  $d\omega = 0$ . Therefore, the divergence theorem yields

$$\int \int_S \omega = \int \int \int_V d\omega = 0$$

if the interior  $V$  does not contain 0. On the other hand, if  $V$  contains 0, let  $B_R$  be a small ball contained in  $V$ , and let  $V - B_R$  denote part of  $V$  lying outside of  $B_R$ . We use the second form of the divergence theorem

$$\int \int \int_S \omega - \int \int \int_{S_R} \omega = \int \int_{V - B_R} \omega = 0$$

We are subtracting the second surface integral, since we are suppose to use the inner normal for  $S_R$ . Thus

$$\int \int_S \omega = -4\pi m$$

Incidentally, this shows that  $\omega$  is not exact. Thus theorem 9.2 fails for  $\mathbb{R}^3 - \{0\}$ .

From here, we can easily extract an expression for the flux for several particles or even a continuous distribution of matter. If  $\mathbf{F}$  is the force of gravity associated to some mass distribution, then for any closed surface  $S$  oriented by the outer normal, then the flux

$$\int \int_S \mathbf{F} \cdot \mathbf{n} dS$$

is  $-4\pi$  times the mass inside  $S$ . For a continuous distribution with density  $\rho$ , this is given by  $\int \int \int \rho dV$ . Applying the divergence theorem again, in this case, yields

$$\int \int \int_V (\nabla \cdot \mathbf{F} + 4\pi\rho) dV = 0$$

for all regions  $V$ . Therefore

$$\nabla \cdot \mathbf{F} = -4\pi\rho \tag{4}$$

## 18 Laplace's equation

The *Laplacian* is a partial differential operator defined by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

This can expressed using previous operators as  $\Delta f = \nabla \cdot (\nabla f)$ . As an example of where this arises, suppose that  $\mathbf{F}$  is a gravitational force, this is known to be conservative so that  $\mathbf{F} = -\nabla P$ . Substituting into (4) yields the Poisson equation

$$\Delta P = 4\pi\rho$$

In a vacuum, this reduces to Laplace's equation

$$\Delta P = 0$$

A solution to Laplace's equation is called a *harmonic* function. These are of fundamental importance both in pure and applied mathematics. If we write  $r = \sqrt{x^2 + y^2 + z^2}$ , then

$$P = -\frac{m}{r} + \text{Const.}$$

is the potential energy associated to particle of mass  $m$  at  $\mathbf{0}$ . This is harmonic away from the singularity  $\mathbf{0}$ .

We express  $\Delta$  in terms of forms as

$$\Delta f dx \wedge dy \wedge dz = d * df$$

or simply

$$\Delta f = *d * df$$

once we define  $*(gdx \wedge dy \wedge dz) = g$ . This last formula also works in the *plane* provided we define

$$\begin{aligned} *(f dx + g dy) &= f dy - g dx \\ *f dx \wedge dy &= f \end{aligned}$$

(The  $*$ -operator in  $n$  dimensions always takes  $p$ -forms to  $(n - p)$ -forms.)

As an exercise, let us work out the Laplace equation in polar coordinates, and use this to determine the radially symmetric harmonic functions on the plane. The key is the determination of the  $*$ -operator:

$$dr = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy = \frac{x}{r} dx + \frac{y}{r} dy$$

Similarly

$$d\theta = -\frac{y}{r^2} dx + \frac{x}{r^2} dy$$

So that

$$\begin{aligned} *dr &= \frac{x}{r} dy - \frac{y}{r} dx = r d\theta \\ *d\theta &= -\frac{y}{r^2} dy - \frac{x}{r^2} dx = -\frac{1}{r} dr \\ *dr \wedge d\theta &= * \frac{1}{r} dx \wedge dy = \frac{1}{r} \end{aligned}$$

Thus

$$\begin{aligned} \Delta f &= *d * \left( \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta \right) \\ &= *d \left( r \frac{\partial f}{\partial r} d\theta - \frac{1}{r} \frac{\partial f}{\partial \theta} dr \right) \\ &= \frac{1}{r} \frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} \end{aligned}$$

If  $f$  is radially symmetric, then it depends only on  $r$  so we obtain

$$\frac{1}{r} \frac{df}{dr} + \frac{d^2 f}{dr^2} = \frac{1}{r} \frac{d}{dr} \left( r \frac{df}{dr} \right) = 0$$

This differential equation can be solved using standard techniques to get

$$f(r) = C + D \log r$$

for constants  $C, D$ . By a similar, but more involved, calculation we find that

$$f(r) = C + \frac{D}{r}$$

are the only radially symmetric harmonic functions in  $\mathbb{R}^3$ , where as above we write  $r$  instead of  $\rho$  for the distance from the origin. These are precisely the physical solutions written at the beginning of this section.

## 19 Beyond 3 dimensions

It is possible to do calculus in  $\mathbb{R}^n$  with  $n > 3$ . Here the language of differential forms comes into its own. While it would be impossible to talk about the curl of a vector field in, say,  $\mathbb{R}^4$ , the derivative of a 1-form or 2-form presents no problems; we simply apply the rules we've already learned. For example, if  $x, y, z, t$  are the coordinates of  $\mathbb{R}^4$ , then a 1-form is a linear combination of the 4 basic 1-forms

$$dx, dy, dz, dt$$

a forms is a linear combination of the 6 basic 2-forms

$$dx \wedge dy = -dy \wedge dx$$

$$dx \wedge dz = -dz \wedge dx$$

...

$$dz \wedge dt = -dt \wedge dz$$

and a 3 form is a linear combination of

$$dx \wedge dy \wedge dz = -dy \wedge dx \wedge dz = -dx \wedge dz \wedge dy = dy \wedge dz \wedge dx = \dots$$

...

$$dy \wedge dz \wedge dt = -dz \wedge dy \wedge dt = \dots$$

The higher dimensional analogue of a surface is a *k-manifold*. A parameterized *k*-manifold  $M$  in  $\mathbb{R}^n$  is given by a collection of  $C^1$  functions

$$\begin{cases} x_1 = f_1(u_1, \dots, u_k) \\ x_2 = f_2(u_1, \dots, u_k) \\ \dots \\ x_n = f_n(u_1, \dots, u_k) \\ (u_1, \dots, u_k) \in D \subseteq \mathbb{R}^k \text{ open} \end{cases}$$

such that the map  $D \rightarrow \mathbb{R}^n$  is one to one and the tangent vectors  $(\frac{\partial x_1}{\partial u_1}, \dots, \frac{\partial x_n}{\partial u_1}), \dots, (\frac{\partial x_1}{\partial u_k}, \dots, \frac{\partial x_n}{\partial u_k})$  are linearly independent for all values of the coordinates  $(u_1, \dots, u_k)$ . Given a  $k$ -form  $\alpha$  on  $\mathbb{R}^n$ , we can express it as a linear combination of  $k$ -fold wedges of  $dx_1 \dots dx_n$ , and then rewrite it as  $g(u_1, \dots, u_k) du_1 \wedge \dots \wedge du_k$ . The “surface” integral is defined as

$$\int_M \alpha = \int \dots \int_D g(u_1 \dots u_k) du_1 \dots du_k \quad (5)$$

Notice that number of integrations is usually suppressed in the notation on the left, since it gets too cumbersome after a while. In practice, for computing integrals, it’s convenient to relax the conditions a bit by allowing  $D$  to be nonopen and allowing some degenerate points where the map  $D \rightarrow \mathbb{R}^n$  isn’t one to one.

More generally, a  $k$ -manifold  $M$  is obtained by gluing several parameterized manifolds as we did for surfaces. To be more precise, a closed set  $M \subset \mathbb{R}^n$  is a  $k$ -manifold, if each point of  $M$  lies in the image of a parameterized  $k$ -manifold called a chart. As with curves and surfaces, it is important to specify orientations. Things are a little trickier since we can no longer rely on our geometric intuition to tell us which way is “up” or “down”. Instead we can think that an orientation is a rule for specifying whether a coordinate system on a chart is right or left handed. We’ll spell this out in an example below. The integral  $\int_M \alpha$  can be defined by essentially summing up (5) over various non-overlapping right handed charts (we can use left handed charts provided we use the opposite sign). A  $k$ -manifold with boundary  $M$  is a closed set which can be decomposed as a union of a  $(k-1)$ -manifold  $\partial M$ , called the boundary, and a  $k$ -manifold  $M - \partial M$ . We orient this by the rule that a coordinate system  $u_2, \dots, u_k$  of  $\partial M$  is right handed if it can be completed to right handed coordinate system  $u_1, u_2, \dots, u_k$  of  $M$  such that the tangent vector  $(\frac{\partial x_1}{\partial u_1}, \dots, \frac{\partial x_n}{\partial u_1})$  “points out”.

Then the ultimate form of Stokes’ theorem is:

**THEOREM 19.1 (Generalized Stokes’ theorem)** *If  $M$  is an oriented  $k$  manifold with boundary  $\partial M$  and if  $\alpha$  is a  $(k-1)$ -form defined on (an open set containing)  $M$ , then*

$$\int_M d\alpha = \int_{\partial M} \alpha$$

In order to get a feeling for how this works, let’s calculate the “volume”  $V$  of the 4-dimensional ball  $B = \{(x, y, z, t) \mid x^2 + y^2 + z^2 + t^2 \leq R\}$  of radius  $R$  in  $\mathbb{R}^4$  in two ways. This is a 4-manifold with boundary  $S = \{(x, y, z, t) \mid x^2 + y^2 + z^2 + t^2 = R\}$ .

$V$  can be expressed as the integral

$$V = \iiint \int_B dx dy dz dt = \int_B dx \wedge dy \wedge dz \wedge dt$$

We use extended spherical coordinates  $\sigma, \rho, \phi, \theta$ , where  $\sigma$  measures the distance of  $(x, y, z, t)$  to the origin in  $\mathbb{R}^4$ , and  $\psi$  the angle to the  $t$ -axis. So that

$$t = \sigma \cos \psi$$

and

$$\rho = \sigma \sin \psi$$

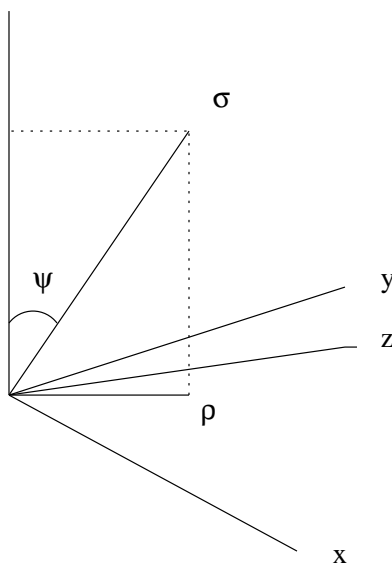
is the distance from from the projection  $(x, y, z)$  to the origin. Then letting  $\phi, \theta$  be the remaining spherical coordinates gives

$$x = \rho \sin \phi \cos \theta = \sigma \sin \psi \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta = \sigma \sin \psi \sin \phi \sin \theta$$

$$z = \rho \cos \phi = \sigma \sin \psi \cos \phi$$

t



In these coordinate  $B$  is described as

$$\left\{ \begin{array}{l} 0 \leq \psi \leq \pi \\ 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \sigma \leq R \end{array} \right.$$

To simplify computations, we note that form will get multiplied by the Jacobian when we change coordinates:

$$\begin{aligned} dx \wedge dy \wedge dz \wedge dt &= \frac{\partial(x, y, z, t)}{\partial(\sigma, \psi, \theta, \phi)} d\sigma \wedge d\psi \wedge d\theta \wedge d\phi \\ &= \sigma^3 \sin^2 \psi \sin \phi d\sigma \wedge d\psi \wedge d\theta \wedge d\phi \end{aligned}$$

Note that the Jacobian is positive, and this what it means to say the coordinate system  $\sigma, \psi, \theta, \phi$  is right handed or positively oriented. The volume is now easily

computed

$$\int_0^R \sigma^3 d\sigma \int_0^\pi \sin^2 \psi d\psi \int_0^{2\pi} d\theta \int_0^\pi \sin \phi d\phi = \frac{1}{2} \pi^2 R^4$$

Alternatively, we can use Stokes' theorem, to see that

$$V = \int_B dx \wedge dy \wedge dz \wedge dt = - \int_S t dx \wedge dy \wedge dz$$

The parameter  $x, y, z$  gives a left hand coordinate system on the upper hemisphere  $U = S \cap \{t > 0\}$ . It is left handed because  $t, x, y, z$  is left handed on  $\mathbb{R}^4$ . For similar reasons,  $x, y, z$  gives a right handed system on the lower hemisphere  $L$  where  $t < 0$ . Therefore

$$\begin{aligned} V &= - \int_U t dx \wedge dy \wedge dz - \int_L t dx \wedge dy \wedge dz \\ &= 2 \int \int \int_{x^2+y^2+z^2 \leq R} \sqrt{R^2 - x^2 - y^2 - z^2} dx dy dz \\ &= 2 \int_0^R \rho^2 \sqrt{R^2 - \rho^2} d\rho \int_0^\pi \sin \phi d\phi \int_0^{2\pi} d\theta \\ &= 8\pi \int_0^{\pi/2} R^4 \sin^2 \alpha \cos^2 \alpha d\alpha \\ &= \frac{1}{2} \pi^2 R^4 \end{aligned}$$

## 20 Maxwell's equations in $\mathbb{R}^4$

As exotic as higher dimensional calculus sounds, there are many applications of these ideas outside of mathematics. For example, in relativity theory one needs to treat the electric  $\mathbf{E} = E_1 \mathbf{i} + E_2 \mathbf{j} + E_3 \mathbf{k}$  and magnetic fields  $\mathbf{B} = B_1 \mathbf{i} + B_2 \mathbf{j} + B_3 \mathbf{k}$  as part of a single "field" on space-time. In mathematical terms, we can take space-time to be  $\mathbb{R}^4$  - with the fourth coordinate as time  $t$ . The electromagnetic field can be represented by a 2-form

$$F = B_3 dx \wedge dy + B_1 dy \wedge dz + B_2 dz \wedge dx + E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt$$

If we compute  $dF$  using the analogues of the rules we've learned:

$$\begin{aligned} dF &= \left( \frac{\partial B_3}{\partial x} dx + \frac{\partial B_3}{\partial y} dy + \frac{\partial B_3}{\partial z} dz + \frac{\partial B_3}{\partial t} dt \right) \wedge dx \wedge dy + \dots \\ &= \left( \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} \right) dx \wedge dy \wedge dz + \left( \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} + \frac{\partial B_3}{\partial t} \right) dx \wedge dy \wedge dt + \dots \end{aligned}$$

Two of Maxwell's equations

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}$$

can be expressed very succinctly in this language as  $dF = 0$ . The analogue of theorem 9.2 holds for  $\mathbb{R}^n$ , and shows that

$$F = d(A_1 dx + A_2 dy + A_3 dz + A_4 dt)$$

for some 1-form called the potential. Thus we've reduced the 6 quantities to just 4. In terms of vector analysis this amounts to the more complicated looking equations

$$\mathbf{B} = \nabla \times (A_1 \mathbf{i} + A_2 \mathbf{j} + A_3 \mathbf{k}), \quad \mathbf{E} = \nabla A_4 - \frac{\partial A_1}{\partial t} \mathbf{i} + \frac{\partial A_2}{\partial t} \mathbf{j} + \frac{\partial A_3}{\partial t} \mathbf{k}$$

There are two remaining Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + 4\pi\mathbf{J}$$

where  $\rho$  is the electric charge density, and  $\mathbf{J}$  is the electric current. The first law is really an analog of (4) for the electric field. After applying the divergence theorem, it implies that the electric flux through a closed surface equals  $(-4\pi)$  times the electric charge inside it. These last two Maxwell equations can also be replaced by the single equation  $d*F = 4\pi J$  of 3-forms. Here

$$*F = E_3 dx \wedge dy + E_1 dy \wedge dz + E_2 dz \wedge dx - B_1 dx \wedge dt - B_2 dy \wedge dt - B_3 dz \wedge dt$$

and

$$J = \rho dx \wedge dy \wedge dz - J_3 dx \wedge dy \wedge dt - J_1 dy \wedge dz \wedge dt - J_2 dz \wedge dx \wedge dt$$

(We have been relying on explicit formulas to avoid technicalities about the definition of the  $*$ -operator. In principle however, it involves a metric, and in this case we use the so called Lorenz metric.)

Let's see how the calculus of differential forms can be used to extract a physically meaningful consequence of these laws. Proposition 9.1 (in extended form) implies that  $dJ = \frac{1}{4\pi} d^2 *F = 0$ . Expanding this out yields

$$\begin{aligned} \frac{\partial \rho}{\partial t} dt \wedge dx \wedge dy \wedge dz - \frac{\partial J_3}{\partial z} dz \wedge dx \wedge dy \wedge dt + \dots = \\ -\left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J}\right) dx \wedge dy \wedge dz \wedge dt = 0 \end{aligned}$$

Thus the expression in brackets is zero. This really an analog of the equation (3). To appreciate the meaning integrate  $\frac{\partial \rho}{\partial t}$  over a solid region  $V$  with boundary  $S$ . Then this equals

$$-\int \int \int_V \nabla \cdot \mathbf{J} dV = -\int \int_S \mathbf{J} \cdot \mathbf{n} dS$$

In other words, the rate of change of the electric charge in  $V$  equals minus the flux of the current across the surface. This is the law of conservation of electric charge.

## 21 Further reading

For more information about differential forms, see the books [Fl, S]. All the physics background can be found in [Fe]. A standard reference for complex analysis is [A]. The material of the appendix can be found in any book on advanced calculus. For a rigorous treatment, see [R, S].

## References

- [A] L. Ahlfors, *Complex Analysis*
- [Fe] R. Feynman et. al., *Lectures on Physics*
- [Fl] H. Flanders, *Differential forms and applications to the physical sciences*
- [R] W. Rudin, *Principles of mathematical analysis*
- [S] M. Spivak, *Calculus on manifolds*

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## A Essentials of multivariable calculus

### A.1 Differential Calculus

To simplify the review, we'll stick to two variables, but the corresponding statements hold more generally. Let  $f(x, y)$  be a real valued function defined on open subset of  $\mathbb{R}^2$ . Recall that the limit

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

means that  $f(x, y)$  is approximately  $L$  whenever  $(x, y)$  is close to  $(a, b)$ . The precise meaning is as follows. If we specified  $\epsilon > 0$  (say  $\epsilon = 0.0005$ ), then we could pick a tolerance  $\delta > 0$  which would guarantee that  $|f(x, y) - L| < \epsilon$  (i.e.  $f(x, y)$  agrees with  $L$  up to the first 3 digits for  $\epsilon = 0.0005$ ) whenever the distance between  $(x, y)$  and  $(a, b)$  is less than  $\delta$ . A function  $f(x, y)$  is *continuous* at  $(a, b)$  if

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y)$$

exists and equals  $f(a, b)$ . It is continuous if it is so at each point of its domain.

We say that  $f$  is *differentiable*, if near any point  $p = (x_0, y_0, f(x_0, y_0))$  the graph  $z = f(x, y)$  can be approximated by a plane passing through  $p$ . In other words, there exists quantities  $A = A(x_0, y_0), B = B(x_0, y_0)$  such that we may write

$$f(x, y) = f(x_0, y_0) + A(x - x_0) + B(y - y_0) + \text{remainder}$$

with *remainder*  $\rightarrow 0$  as  $(x, y) \rightarrow (x_0, y_0)$ . We can see that the coefficients are nothing but the partial derivatives

$$A(x, y) = \frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$B(x, y) = \frac{\partial f}{\partial y} = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

There is a stronger condition which is generally easier to check.  $f$  is called continuously differentiable or  $C^1$  if it and its partial derivatives exist and are continuous. Consider the following example

$$f(x, y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

This is continuous, however

$$\frac{\partial f}{\partial x} = \frac{3x^2}{x^2+y^2} - \frac{2x^4}{(x^2+y^2)^2}$$

has no limit as  $(x, y) \rightarrow (0, 0)$ . To see this, note that along the  $x$ -axis  $y = 0$ , we have  $\frac{\partial f}{\partial x} = 1$ . So the limit would have to be 1 if it existed. On the other hand, along the  $y$ -axis  $x = 0$ ,  $\frac{\partial f}{\partial x} = 0$ , which shows that there is no limit. So  $f(x, y)$  is not  $C^1$ .

Partial derivatives can be used to determine maxima and minima.

**THEOREM A.2** *If  $(a, b)$  is local maximum or minimum of a  $C^1$  function  $f(x, y)$ , then  $(a, b)$  is a critical point, i.e.*

$$\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0$$

**THEOREM A.3 (Chain Rule)** *If  $f, g, h : \mathbb{R}^2 \rightarrow \mathbb{R}$  are  $C^1$  functions, then  $f(g(u, v), h(u, v))$  is also  $C^1$  and if  $z = f(x, y)$ ,  $x = g(u, v)$ ,  $y = h(u, v)$  then*

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$

A function  $f(x, y)$  is twice continuously differentiable or  $C^2$  if is  $C^1$  and if its partial derivatives are also  $C^1$ . We have the following basic fact:

**THEOREM A.4** *If  $f(x, y)$  is  $C^2$  then the mixed partials*

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \frac{\partial f}{\partial x}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \frac{\partial f}{\partial y}$$

*are equal.*

If  $f$  is  $C^2$ , then we have a Taylor approximation

$$f(x, y) \approx f(a, b) + \left[ \frac{\partial f}{\partial x}(a, b) \right] (x - a) + \left[ \frac{\partial f}{\partial y}(a, b) \right] (y - b) + \left[ \frac{\partial^2 f}{\partial x^2}(a, b) \right] (x - a)^2 + 2 \left[ \frac{\partial^2 f}{\partial y \partial x}(a, b) \right] (x - a)(y - b) + \left[ \frac{\partial^2 f}{\partial y^2}(a, b) \right] (y - b)^2$$

Since it is relatively easy to determine when quadratic polynomials have maxima or minima, this leads to the second derivative test.

**THEOREM A.5** *A critical point  $(a, b)$  of a  $C^2$  function  $f(x, y)$  is a local minimum (respectively maximum) precisely when the matrix*

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(a, b) & \frac{\partial^2 f}{\partial y \partial x}(a, b) \\ \frac{\partial^2 f}{\partial y \partial x}(a, b) & \frac{\partial^2 f}{\partial y^2}(a, b) \end{pmatrix}$$

*is positive (respectively negative) definite.*

The above conditions are often formulated in a more elementary but ad hoc way in calculus books. Positive definiteness is equivalent to requiring

$$\frac{\partial^2 f}{\partial x^2}(a, b) > 0$$

$$\left[ \frac{\partial^2 f}{\partial x^2}(a, b) \right] \left[ \frac{\partial^2 f}{\partial y^2}(a, b) \right] - \left[ \frac{\partial^2 f}{\partial y \partial x}(a, b) \right]^2 > 0$$

## A.6 Integral Calculus

Integrals can be defined using Riemann's method. This has some limitations but it's the easiest to explain. Given a rectangle  $R = [a, b] \times [c, d] \subset \mathbb{R}^2$ , choose integers  $m, n > 0$  and let  $\Delta x = \frac{b-a}{m}$ ,  $\Delta y = \frac{d-c}{n}$ . Choose a set of sample points  $P = \{(x_1, y_1), \dots, (x_m, y_n)\} \subset R$  with

$$(x_i, y_j) \in R_{ij} = [a + (i-1)\Delta x, a + i\Delta x] \times [b + (j-1)\Delta y, b + j\Delta y]$$

The Riemann sum

$$S(m, n, P) = \sum_{i,j} f(x_i, y_j) \Delta x \Delta y$$

Then the double integral is

$$\iint_R f(x, y) dx dy = \lim_{m,n \rightarrow \infty} S(m, n, P)$$

This definition is not really that precise because we need to choose  $P$  for each pair  $m, n$ . For the integral to exist, we really have to require that the limit exists for any choice of  $P$ , and that any two choices lead to the same answer.

The usual way to resolve the above issues is to make the two extreme choices. Define upper and lower sums

$$U(m, n) = \sum_{i,j} M_{ij} \Delta x \Delta y$$

$$L(m, n) = \sum_{i,j} m_{ij} \Delta x \Delta y$$

where

$$M_{ij} = \max\{f(x, y) \mid (x, y) \in R_{ij}\}$$

$$m_{ij} = \min\{f(x, y) \mid (x, y) \in R_{ij}\}$$

In the event that the maxima or minima don't exist, we should use the greatest lower bound and least upper bound instead. As  $m, n \rightarrow \infty$  the numbers  $L(m, n)$  tend to increase. So their limit can be understood as the least upper bound, i.e. the smallest number  $L \geq L(m, n)$ . Likewise we define the limit  $U$  as the largest number  $U \leq U(m, n)$ . If these limits coincide, the common value is taken to be

$$\iint_R f(x, y) dx dy = L = U$$

otherwise the (Riemann) integral is considered to not exist.

**THEOREM A.7**  $\iint_R f(x, y) dx dy$  exists if  $f$  is continuous.

The integral of

$$f(x, y) = \begin{cases} 1 & \text{if } (x, y) \text{ has rational coordinates} \\ 0 & \text{otherwise} \end{cases}$$

would be undefined from the present point of view, because  $L = 0$  and  $U = 1$ . Although, in fact the integral can be defined using the more powerful Lebesgue theory [R]; in this example the Lebesgue integral is 0.

For more complicated regions  $D \subset R$ , set

$$\iint_D f(x, y) dx dy = \iint_R f(x, y) \chi_D(x, y) dx dy$$

where  $\chi_D = 1$  inside  $D$  and 0 elsewhere. The key result is

**THEOREM A.8 (Fubini)** If  $D = \{(x, y) \mid a \leq x \leq b, g(x) \leq y \leq h(x)\}$  with  $f, g, h$  continuous. Then the double integral exists and

$$\iint_D f(x, y) dx dy = \int_a^b \left( \int_{g(x)}^{h(x)} f(x, y) dy \right) dx$$

A similar statement holds with the roles of  $x$  and  $y$  interchanged.

This allows one to compute these integrals in practice.

The final question to answer is how double integrals behave under change of variables. Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a transformation given by  $C^1$  functions

$$x = f(u, v), \quad y = g(u, v)$$

We think of the first  $\mathbb{R}^2$  as the  $uv$ -plane and the second as the  $xy$ -plane. Given a region  $D$  in the  $uv$ -plane, we can map it to the  $xy$ -plane by

$$T(D) = \{(f(u, v), g(u, v)) \mid (u, v) \in D\}$$

The Jacobian

$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} &= \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \\ &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}. \end{aligned}$$

**THEOREM A.9** *If  $T$  is a one to one function,  $h$  is continuous and  $D$  a region of the type occurring in Fubini's theorem, then*

$$\int \int_{T(D)} h(x, y) dx dy = \int \int_D h(f(u, v), g(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$