These course notes from spring 2010 are extremely rough, so *caveat lector*.

An Introduction to Etale Cohomology

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Part I

Introduction

Chapter 0

Review of schemes

In the classical approach to algebraic geometry, one considers the basic objects, called varieties, as sets of solutions to a polynomial equations in affine or projective space. There are however some limitations:

- One would like to be able to construct spaces by gluing simpler pieces together, as one does in other parts of geometry and topology.
- When working over nonalgebraically closed fields – which is important for applications to number theory and also within algebraic geometry – there may not be any points at all.
- Even in classical arguments, one sometimes has to keep track of multiplicities. For example, one might view the double line $y^2 = 0$ as the different from the x-axis. So one would like a framework where this kind of distinction makes sense.

Grothendieck's theory schemes gave a solution to all of these problems. Of course there is a price to be paid in the extra abstraction. Here is a quick overview. See Hartshorne's text [H] for more details. The ultimate source is [EGA]. In a nutshell, a scheme is built by gluing together simpler pieces called affine schemes. Affine schemes are determined by the rings of functions on them.

0.1 Coordinate rings

In classical algebraic geometry, the basic objects can be thought of sets of points. Over an algebraically closed field this is reasonable, but consider the equations

$$X : x^2 + y^2 = -1$$

$$Y : x^2 + y^2 = 3$$

These have no solutions in $\mathbb{A}^2_{\mathbb{R}}$ and $\mathbb{A}^2_{\mathbb{Z}/4\mathbb{Z}}$ respectively, and therefore no solutions over $\mathbb{Z}$. Since they both define the empty set in $\mathbb{A}^2_{\mathbb{Z}}$, should they be regarded as the same? The answer is no. An isomorphism *should* be given by an invertible transformation $x \mapsto f(x,y), y \mapsto g(x,y)$, given by a pair of integer polynomials, which takes the first equation to the second. In other words, what we really want is an isomorphism of rings

$$\mathcal{O}(X) = \mathbb{Z}[x,y]/(x^2+y^2+1) \cong \mathbb{Z}[x,y]/(x^2+y^2-3) = \mathcal{O}(Y)$$

Such an isomorphism would guarantee a bijection between the sets $X(R) \cong Y(R)$ of solutions to $x^2+y^2+1=0$ and $x^2+y^2+3=0$ respectively, for any ring R. Taking $R=\mathbb{R}$ shows that this impossible.

In general, the affine "variety" $X \subseteq \mathbb{A}^n_R$, over some ring R, defined by polynomials $f_i = 0$ is completely determined by the coordinate ring

$$S = \mathcal{O}(X) = R[x_1, \ldots x_n]/(f_1, \ldots f_N)$$

The set of solutions $X(R')$ in any R-algebra R' can be identified with

$$Hom_{R\text{-alg}}(S, R')$$

because any homomorphism $h : S \to R'$ is uniquely determined by the point $(h(x_1), \ldots h(x_n)) \in X(R')$. When $R = R'$ is an algebraically closed field, we can identify this with the set of maximal ideals in S by Hilbert's nullstellensatz. All the other familiar constructions: dimension, tangent spaces, ... can be read off from S as well.

0.2 Spectrum

Before going further, it may be useful to have a basic model from analysis/topology. Given a compact Hausdorff space X, let $C(X)$ denote the set of continuous complex valued functions. This is a commutative ring (always with 1 for us). When endowed with the *sup* norm, it becomes a so called unital C^*-algebra. (Since this is just for motivation, we won't bother writing down the axioms.)

A special case of Gelfand duality says that we can reconstruct X from $C(X)$. This is what we want to understand. Given a commutative ring R, let $Max\, R$ denote the set of maximal ideals of R. This is the maximal ideal spectrum. We make this into a topological space by equipping it with the Zariski topology: the closed sets are the sets

$$\{m \mid I \subseteq m\}$$

for ideals $I \subset R$. To every $x \in X$, define

$$m_x = \{f \in C(X) \mid f(x) = 0\}$$

This is a maximal ideal such that $C(X)/m_x \cong \mathbb{C}$. The reconstruction result now follows from the next theorem.

Theorem 0.2.1. *The map $X \to Max\, C(X)$ is a homeomorphism.*

Given a continuous map of spaces $f : X \to Y$, we get a homomorphism $C(Y) \to C(X)$ given by $g \mapsto g \circ f$. Thus $C(X)$ can be regarded as contravariant functor, or equivalently a covariant functor of the category of C^*-algebras with arrows reversed. This is called the *opposite* category. The full form of Gelfand duality says

Theorem 0.2.2 (Gelfand)**.** *The functor $X \mapsto C(X)$ induces an equivalence between the category of compact Hausdorff spaces and the opposite category of commutative unital C^*-algebras.*

Now let us return to algebraic geometry. Following Grothendieck, we can make the bold leap

$$\text{Affine Algebraic Geometry} = (\text{Commutative Algebra})^{opposite}$$

We need to understand how to get a geometric object out of a general commutative ring. Given a such ring R, we have already defined $Max\, R$. The problem is that given a homomorphism of rings $h : R \to S$, there is in general no natural way to get a map $Max\, S \to Max\, R$. If however, we relax the conditions from maximal ideals to prime ideals, then there is a way. Let Spec R denote the set of prime ideals of R. It is called the spectrum of R. We do get a map $\operatorname{Spec} S \to \operatorname{Spec} R$ given by $p \mapsto h^{-1}p$.

So now we have a set $\operatorname{Spec} R$. We need to give it more structure.

Proposition 0.2.3. *There is a topology, called the Zariski topology, on* $\operatorname{Spec} R$ *whose basic open sets are $D(f) = \{p \in \operatorname{Spec} R \mid f \notin p\}$ with $f \in R$. The closed sets are $V(I) = \{p \in \operatorname{Spec} R \mid I \subset P\}$ for ideals $I \subset R$.*

The map $\operatorname{Spec} S \to \operatorname{Spec} R$ defined above is easily seen to be continuous. We make an observation that will be useful below. If $R[1/f] = R[x]/(xf - 1)$ is the localization of R at f. Then

Lemma 0.2.4. *The maps* $\operatorname{Spec} R[1/f] \to \operatorname{Spec} R$ *and* $\operatorname{Spec} R/I \to \operatorname{Spec} R$ *are injective and the images are $D(f)$ and $V(I)$ respectively.*

We now have a contravariant functor from the category of commutative rings to the category of topological spaces. However, there is still not enough structure to set up the equivalence given above. The missing ingredient is the structure sheaf.

0.3 Affine schemes

The intuition coming form Gelfand duality is bit too coarse for what happens next. A better model for us comes from manifolds. The basic example is $X = \mathbb{R}^n$. Consider the collection C^∞ real valued functions on X and its open subsets. The key thing to observe is that the C^∞ condition is local. This means that a function is C^∞ if and only if its restriction to the neighbourhood of every point is C^∞. Or equivalently this means that $f \in C^\infty(X)$ if and only for any open cover $\{U_i\}$, $f|_{U_i} \in C^\infty(U_i)$. There are plenty of other classes of functions with this property. Abstracting this leads to the notion of a sheaf.

Given a topological space X, a presheaf of sets (groups, rings...), $\mathcal{F}$, is a collection of sets (groups, rings...) $\mathcal{F}(U)$ for each open set $U \subseteq X$, together with maps (homomorphisms...) $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for each pair $U \subseteq V$ such that

1. $\rho_{UU} = id$
2. $\rho_{WV} \circ \rho_{VU} = \rho_{WU}$ whenever $U \subseteq V \subseteq W$.

We usually denote $\rho_{UV}(f) = f|_V$.

A presheaf is called a sheaf if for any open cover $\{U_i\}$ of an open $U \subseteq X$ and sections $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, there exists a unique $f \in \mathcal{F}(U)$ satisfying $f_i = f|_{U_i}$.

It will help to think of a scheme, when we get to it, as something like a C^∞ manifold. In additional to a topological space, we have a sheaf of commutative rings. The building blocks are given as follows:

Theorem 0.3.1. *Given a commutative ring R, there exists a sheaf of commutative rings $\mathcal{O}_{\operatorname{Spec} R}$ on* $\operatorname{Spec} R$ *such that*

1. $\mathcal{O}_{SpecR}(D(f)) \cong R[1/f]$.
2. *The diagram*

$$\begin{tikzcd} \mathcal{O}_{\operatorname{Spec} R}(D(f)) \arrow[r, "\rho"] \arrow[d, "\cong"] & \mathcal{O}_{\operatorname{Spec} R}(D(fg)) \arrow[d, "\cong"] \\ R[1/f] \arrow[r, "k"] & R[1/fg] \end{tikzcd}$$

commutes, where the map labelled k is the canonical localization map.

This sheaf is characterized up to isomorphism by these properties.

A pair $(X, \mathcal{O}_X)$ consisting of a topological space and a sheaf of commutative rings is called a ringed space. For example, $(\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$ is a ringed space. The pair $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is called the affine scheme associated to R. The symbol $\operatorname{Spec} R$ is also used to denote the whole thing. Given $\operatorname{Spec} R$, we can recover the ring by taking $\mathcal{O}(\operatorname{Spec} R)$.

0.4 Schemes

The collection of (pre)sheaves on a topological space X form a category $(PSh(X))$ $Sh(X)$. A morphism $\eta : \mathcal{F} \to \mathcal{G}$ is collection of maps or homomorphisms

$$\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

compatible with restriction i.e. such that

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\eta_U} & \mathcal{G}(U) \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} \\ \mathcal{F}(V) & \xrightarrow{\eta_V} & \mathcal{G}(V) \end{array}$$

commutes whenever $V \subseteq U$. We say that η is an isomorphism, if each η_U is an isomorphism. An isomorphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a homeomorphism $f : X \to Y$, and a collection of isomorphisms $\eta_U : \mathcal{O}_Y(U) \cong \mathcal{O}_X(f^{-1}U)$ compatible with restriction.

A scheme is a ringed space $(X, \mathcal{O}_X)$ which is locally an affine scheme. This means that there exists an open cover $\{U_i\}$ and rings R_i along with isomorphisms $(U_i, \mathcal{O}_X|_{U_i}) \cong (\operatorname{Spec} R_i, \mathcal{O}_{\operatorname{Spec} R_i})$. $\{U_i\}$ is called an affine open cover. Here the restriction is defined by $\mathcal{O}_X|_{U_i}(V) = \mathcal{O}_X(V)$ for $V \subseteq U_i$. Examples of nonaffine schemes can be constructed taking open subsets of a given affine scheme with the restriction of the sheaf. For example, if k is field then $\mathbb{A}^2_k - \{0\}$ is not affine.

More interesting examples can be constructed by gluing. If X_1 and X_2 are schemes with opens sets $X_{12} \subset X_1$, $X_{21} \subset X_2$ together with an isomorphism $\phi : X_{12} \to X_{21}$, we can construct a new scheme $X = X_1 \cup_\phi X_2$ as follows. As a space X is obtained by first taking the disjoint union of X_1 and X_2 and then identifying $x \in X_{12}$ with $\phi(x)$. A section of $\mathcal{O}_X(U)$ is given by a pair of section $f_i \in \mathcal{O}_{X_i}(U_i)$ such that $\phi(f_1) = \phi(f_2)$. A basic example is the projective line $\mathbb{P}^1_R$ over a ring R. It can be constructed by gluing the two lines $\operatorname{Spec} R[x]$ and $\operatorname{Spec} R[x^{-1}]$ along the common open set $\operatorname{Spec} R[x, x^{-1}]$. One can glue several X_i via isomorphisms

$$X_i \supset X_{ij} \xrightarrow{\phi_{ij}} X_{ji} \subset X_j$$

subject to appropriate compatibilities [H, p 80]. We can construct $\mathbb{P}^n_R$ gluing the $n+1$ affine spaces

$$\operatorname{Spec} R\left[\frac{x_0}{x_i}, \dots \frac{x_n}{x_i}\right] \supset \operatorname{Spec}\left[\frac{x_0}{x_i}, \dots \frac{x_n}{x_i}, \frac{x_i}{x_j}\right] \cong \operatorname{Spec}\left[\frac{x_0}{x_j}, \dots \frac{x_n}{x_j}, \frac{x_j}{x_i}\right] \subset \operatorname{Spec} R\left[\frac{x_0}{x_j}, \dots \frac{x_n}{x_j}\right]$$

This can also be described using the *Proj* construction [H]. Note that all quasiprojective varieties in the classical sense can be viewed as schemes.

0.5 Localization

Given a property P of rings (e.g. the property of being an integral domain), we may ask when does it extend to schemes. Clearly we want to say that a scheme X has P if it has open affine cover by spectra of rings satisfying P. Of course, we need to make sure that this does not depend on the cover, and we would also know that when applied to affine schemes we recover the original definition.

Given a commutative ring R, a subset $S \subset R$ is called multiplicative if it contains 1 and is closed under multiplication. The localization $S^{-1}R = R[S^{-1}]$ is obtained by formally adjoining inverses the elements of S. For example, $\{f^n \mid n \in \mathbb{N}\}$ is multiplicative, and $S^{-1}R = R[1/f]$.

Let us say that a property P is local if

1. P holds for $S^{-1}R$ whenever it holds for R.
2. P holds for R if there exists $f_1, \ldots f_n$ generating the trivial ideal such that P holds for $R[1/f_i]$.

We can state the following basic principle:

Principle: *A property of commutative rings extends to schemes if it is local*

For example, a ring is reduced if it has no nilpotent elements. This property is easily seen to be local.

Given a prime ideal $p \subset R$, the complement $S = R - p$ is multiplicative. Define $R_p = S^{-1}R$. This is a local ring in the sense that it has a unique maximal ideal $m = pR_p$. It is easy to see that a property P is local whenever P holds for $R \Leftrightarrow P$ holds for all R_p.

The localization at p has a direct interpretation in terms of sheaf theory Given a point $x \in X$ on a space with a sheaf of rings $\mathcal{F}$, we get a system of rings

$$\begin{tikzcd}
 & \mathcal{F}(U) \arrow[dr] & \\
\mathcal{F}(X) \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{F}(V) \arrow[r] & \mathcal{F}(U\cap V) \\
 & \ldots &
\end{tikzcd}$$

where $U, V, \ldots$ run over neighbourhoods of x. The stalk $\mathcal{F}_x$, which is the direct limit, gives local behaviour of this sheaf near x. When X is a scheme, x has a neighbourhood isomorphic to the spectrum of a ring, say R. The set of the form $D(f)$ with $f \notin x$ (regarded as prime ideal) form a cofinal system in the above, so that the direct limit is the same. Therefore

Lemma 0.5.1. *The stalk $\mathcal{O}_{X,x}$ is isomorphic to $\varinjlim R[1/f]$ which is isomorphic to the localization R_x.*

0.6 Morphisms

A scheme is locally the spectrum of ring, so a morphism of schemes should be locally determined by the map on spectra induced by a homomorphism of rings. However, making this into a precise definition requires some work.

Given a continuous map $f : X \to Y$ and a sheaf $\mathcal{F}$ on X, the direct image $f_*\mathcal{F}$ is a sheaf of Y defined by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}U)$. A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of schemes consists of

1. A continuous map $f : X \to Y$
2. A morphism of sheaves of rings $f^\# : \mathcal{O}_Y \to \mathcal{O}_X$ such that for each $x \in X$, the induced homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,f(x)}$ is local in the sense that the preimage of $m_{f(x)}$ is m_x.

Here is the basic example. Given a homomorphism of rings $h : R \to S$. There exists a continuous map $f : \operatorname{Spec} S \to \operatorname{Spec} R$ given by $f(p) = h^{-1}p$. We have a map of sheave $f^\# : \mathcal{O}_{\operatorname{Spec} S} \to \mathcal{O}_{\operatorname{Spec} S}$ which on basic open sets is given by

$$\begin{tikzcd} \mathcal{O}(D(r)) \arrow[r] \arrow[d, "\cong"] & \mathcal{O}(f^{-1}D(r)) \arrow[d, "\cong"] \\ R[1/r] \arrow[r] & S[1/h(r)] \end{tikzcd}$$

where the bottom map is the canonical map given by $x/r^n \mapsto f(x)/h(r)^n$. It is easy see that the map $R_{h^{-1}p} \to S_p$ is local, because the preimage of maximal for S_p is pS_p is precisely $h^{-1}pR_{h^{-1}p}$. Conversely, any morphism of schemes $\operatorname{Spec} S \to \operatorname{Spec} R$ arises from the homomorphism of $R \to S$ given by $\mathcal{O}(\operatorname{Spec} R) \to \mathcal{O}(\operatorname{Spec} S)$.

For general schemes X and Y a morphism can be described as follows. A continuous map $f : X \to Y$, an affine open cover $U_i = \operatorname{Spec} R_i$ of Y, an affine open cover $\operatorname{Spec} S_{ij}$ of each $f^{-1}U_i$, and homomorphisms $h_{ij} : R_i \to S_{ij}$ subject to the appropriate compatibilities. The principle stated in the previous section can be extend to morphisms: A property P of homomorphisms of rings $h : R \to S$ extends to morphisms of schemes if P can be characterized in terms of the local homomorphisms $R_{h^{-1}p} \to S_p$.

Given two morphisms of schemes $X \to S$ and $Y \to S$, there exists a third morphism $X \times_S Y$ called the fibre product. It fits into a commutative diagram

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

and for any Z fitting into a similar diagram, there is a unique morphism f rendering the following diagram commutative

$$\begin{tikzcd} Z \arrow[dr, "f"] \arrow[drr] \arrow[ddr] & & \\ & X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}$$

For affine schemes, the fibre product is just the spectrum of the tensor product of the corresponding rings. As a simple example, we have the fibre product

$$\begin{aligned}
&\operatorname{Spec}\mathbb{Z}[x_1,\dots x_n]/(f_1,\dots f_N)\times_{\operatorname{Spec}\mathbb{Z}}\operatorname{Spec}\mathbb{Z}/p\mathbb{Z}\\
&\qquad = \operatorname{Spec}(\mathbb{Z}[x_1,\dots x_n]/(f_1,\dots f_N)\otimes_{\mathbb{Z}}\mathbb{Z}/p\mathbb{Z})\\
&\qquad = \operatorname{Spec}\mathbb{Z}/p\mathbb{Z}[x_1,\dots x_n]/(f_1,\dots f_N)
\end{aligned}$$

is the reduction mod p.

0.7 Schemes as functors

We made a big deal that the set of solutions over a fixed ring need not determine the geometric object. However, if we allow the rings to vary then it does. Given the affine scheme

$$X = \operatorname{Spec}\mathbb{Z}[x_1,\dots x_n]/(f_1,\dots f_N)$$

we can form the set

$$X(R) = \{(a_1\dots a_n)\in R^n \mid f_i(a_1,\dots a_n) = 0\}$$

of solutions in any commutative ring. If $h : R \to S$ is a homomorphism, then we get an induced map $X(R)\to X(S)$ given by $(a_1,\dots)\mapsto(h(a_1),\dots)$. Thus X defines a functor from commutative rings to sets. The following is straight forward

Lemma 0.7.1. *$(a_i)\in X(R)$ iff $x_i\mapsto a_i$ defines a homomorphism from $\mathbb{Z}[x_1,\dots x_n]/(f_1,\dots f_N)$ to R. Therefore $X(R) = Hom(\mathbb{Z}[x_1,\dots x_n]/(f_1,\dots f_N), R)$*

In view of this, we define $X(R) = Hom(S,R)$ whenever $X = \operatorname{Spec} R$. A familiar example is the scheme

$$SL_n = \operatorname{Spec}\mathbb{Z}[x_{11},\dots x_{nn}]/(\det(x_{ij})-1)$$

Then $SL_n(R)$ is the set of $n\times n$ matrices over R with determinant 1, exactly as the notation suggests.

Given affine schemes X, Y, a collection of maps $\phi_R : X(R)\cong Y(R)$ is a *natural* if

$$\begin{tikzcd}
X(R) \arrow[r] \arrow[d, "\phi_R"] & X(S) \arrow[d, "\phi_R"] \\
Y(R) \arrow[r] & Y(S)
\end{tikzcd}$$

commute for every homomorphism $R\to S$. Here is basic fact.

Theorem 0.7.2 (Yoneda's lemma)**.** *Given affine schemes $X = \operatorname{Spec} S, Y = \operatorname{Spec} T$, if a natural bijection $X(R)\cong Y(R)$ exists, then $X\cong Y$. More generally, a natural map is induced by a morphisms of affine schemes $Y\to X$, or equivalently a homomorphism $S\to T$.*

In fact, this result has nothing to do with rings or schemes, but is a general result of category theory [Ma].

Given an arbitrary scheme X, we define

$$X(R) = Hom_{schemes}(\operatorname{Spec} R, X)$$

The above theorem generalizes to these functors. Thus schemes can be identified with certain kinds of functors on the category of commutative rings. I know of at least one book [DG] that introduces schemes this way, but it is in my opinion not a good starting point for learning the theory. Nevertheless, this point of view is very powerful, and we will use it from time to time. Functors that arise this way, are pretty special. We shall see later that they are sheaves in the appropriate sense.

Chapter 1

Differential Calculus of Schemes

Given an affine variety $X = V(f_1, \dots f_N) \subset \mathbb{A}^n_k$ over a field, by imitating the usual constructions from calculus, we define the tangent space $T_{X,p}$ at $p \in X$ as the set of vectors $(v_i) \in k^n$ such that

$$\sum_i \frac{\partial f_j}{\partial x_i}(p) v_i = 0$$

Given a morphism $h : X \to Y \subseteq \mathbb{A}^m$ given by polynomials h_i the Jacobian matrix $\frac{\partial h_i}{\partial x_j}(p)$ can be seen (with some work) to define a linear transformation $T_{X,p} \to T_{Y,h(p)}$ called the derivative. We want to refine this in the context of scheme theory. It turns out that the dual object is the more fundamental, and we start with that.

1.1 Étale maps

An affine variety $X \subset \mathbb{A}^n_k$ is smooth if the Jacobian has the expected rank. Before giving the precise definition, recall that the Krull dimension of a ring/scheme gives measure of size. For example

$$\dim A^n_k = \dim k[x_1, \dots x_n] = n$$

for any field.

Definition 1.1.1. *A domain $R = k[x_1, \dots x_n]/(f_1, \dots f_N)$ or $\operatorname{Spec} R$ is smooth if and only if the rank of $\left(\frac{\partial f_j}{\partial x_i}(p)\right)$ is $n - \dim R$ for all $p \in Max\, R$. (The expression $f(p)$ denotes the image of f in R/p.)*

When k is algebraically closed, then this condition is also referred to as nonsingularity. This condition is equivalent to the regularity of the local rings

[E], and in general this is so after extending scalars to $\bar{k}$. So smoothness is independent of the equations. We can define the tangent space $T_{X,p}$ of $\operatorname{Spec} R$ at p as the kernel of the matrix $\left(\frac{\partial f_j}{\partial x_i}(p)\right)$. Smoothness is equivalent to the equality $\dim T_{X,p} = \dim R$ for all p. A point is called nonsmooth or singular if equality fails.

Example 1.1.2. *If the equations f_i are linear polynomials, then R is smooth.*

Example 1.1.3. *A hypersurface $R = k[x_1, \dots x_n]/(f)$ is smooth if and only if the partials $\frac{\partial f}{\partial x_j}(p)$ are not simultaneously zero for any $p \in V(f)$. This follows from the Krull's principal ideal theorem* $\dim R = n - 1$.

We come now to the basic notion. An étale map should be something like a covering space in topology. Such a map is a continuous function $f : X \to Y$ for which every point of X has neighbourhood U such that $U \to f(U)$ is a homeomorphism. For manifolds, we can use the inverse function theorem to test this condition. We can take a suitable algebrization of the hypothesis of the inverse function theorem as our definition.

Definition 1.1.4. *An R-algebra $S = R[x_1, \dots x_n]/(f_1, \dots f_n)$ is called étale if it is smooth of relative dimension* 0, *if* $\det\left(\frac{\partial f_i}{\partial x_j}\right)$ *is a unit in S*

Using a different characterization to be given later, étaleness turns out to be independent of the choice of presentation i.e. if the Jacobian is a unit for one set of equations then it is for any set of equations. If S is étale, then for every prime ideal $p \in \operatorname{Spec} S$ we have an isomorphism $T_p \operatorname{Spec} S \cong T_{p\cap R} \operatorname{Spec} R$ of tangent spaces via the derivative. For smooth varieties, this is an equivalent condition.

Example 1.1.5. *A seperable field extension is given by $L = k[x]/(f(x))$, where f is an irreducible polynomial such that $f'(x)$ and $f(x)$ are coprime. It follows that L is étale.*

Example 1.1.6. *Let k have characteristic $p > 0$ and suppose $a \in k$ is not a pth power. Then the inseperable field extension $k(a^{1/p}) = k[x]/(x^p - a)$ is* **not** *étale. This is clear with the given presentation.*

These examples can be refined as follows:

Proposition 1.1.7. *If k is a field, an algebra over k is étale if and only if it is a finite cartesian product of seperable field extensions.*

Example 1.1.8. *Suppose n is an integer not divisible by the characteristic of a field k. Let $R = k[x, x^{-1}]$, then $S = R[y]/(y^n - x)$ is étale.*

Example 1.1.9. *$R[1/f]$ is an étale algebra.*

Proposition 1.1.10. *Then tensor product of two étale algebras is étale.*

Proof. If $S = R[x_1, \dots x_n]/(f_1, \dots f_n)$ and $T = R[y_1, \dots y_m]/(g_1, \dots g_m)$ are étale, then

$$S \otimes_R T = R[x_1, \dots x_n, y_1, \dots y_m]/(f_1(x), \dots f_n(x), g_1(y), \dots g_m(y))$$

where x, y denote strings of variables. The Jacobian determinant for the latter is the product of the determinants $\det\left(\frac{\partial f_i}{\partial x_j}\right) \det\left(\frac{\partial g_i}{\partial y_j}\right)$ so it's a unit. $\square$

Proposition 1.1.11. *If S is étale over R, and T is étale over S, then T is étale over R.*

Proof. The proof is similar to the one above. Write $S = R[x_1, \dots x_n]/(f_1, \dots f_n)$ and $T = S[y_1, \dots y_m]/(g_1, \dots g_m)$. Then

$$T = R[x_1, \dots x_n, y_1 \dots y_m]/(f_1(x), \dots f_n(x), g_1(x, y) \dots g_m(x, y))$$

and the Jacobian determinant is again the product $\det\left(\frac{\partial f_i}{\partial x_j}\right) \det\left(\frac{\partial g_i}{\partial y_j}\right)$ $\square$

1.2 Kähler differentials

Given a commutative ring R and an R-algebra S and an S-module M, an R-linear derivation from S to M is a map $\delta : S \to M$ satisfying

$$\delta(s_1 + s_2) = \delta(s_1) + \delta(s_2)$$

$$\delta(s_1 s_2) = s_1 \delta(s_2) + s_2 \delta(s_1)$$

$$\delta(r) = 0$$

for $r \in R, s_i \in S$. (Combining these rules shows that δ is in fact R-linear.) For example, the derivatives $\frac{\partial}{\partial x_i}$ are R-linear derivations from $S = R[x_1, \dots x_n]$ to itself.

Proposition 1.2.1. *There exists an S-module $\Omega_{S/R}$ with a universal R-linear derivation $d : S \to \Omega_{S/R}$. Universality means that given any derivation $\delta : S \to M$ there exists a unique S-module map $u : \Omega_{S/R} \to M$ such that $\delta = u \circ d$.*

Proof. For the sake of expedience, we do this by generators and relations. However, this is a bit less useful than the usual construction. Let F be free S-module generated by symbols $\{ds \mid s \in S\}$. Let K be the submodule of F generated by the elements in the set

$$\{d(s_1 + s_2) - d(s_1) - d(s_2), d(s_1 s_2) - s_1 d(s_2) + s_2 d(s_1), d(r) = 0 \mid s_i \in S, r \in R\}$$

Then set $\Omega_{S/R} = F/K$. The map $s \mapsto ds$ is easily seen to satisfy the above conditions. $\square$

Example 1.2.2. *Let $S = R[x_1, \dots x_n]$. Then $\Omega_{S/R}$ is the free S-module generated by the symbols $dx_1, \dots dx_n$ and*

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

Proof. To see the above description is correct, it is enough to check that for any (M, δ) $\delta(f)$ satisfies a formula similar to the one above. Therefore the map $dx_i \mapsto \delta x_i$ on the free module takes df to $\delta(f)$. □

Given $p \in \operatorname{Spec} S$, let $(\Omega_{S/R})_p = S_p \otimes_S \Omega_{S/R}$ denote the localization of the module of differentials. This is a rank n free module over the local ring S_p. Let $k(p) = S_p/pS_p$ denote the residue field at this prime. The cotangent space, which is by definition

$$T^*_{S/R,p} = k(p) \otimes_S \Omega_{S/R}$$

is therefore a vector space of dimension n for each p.

Proposition 1.2.3. *Let $T = S/I$, then there is an exact sequence*

$$I/I^2 \to T \otimes_S \Omega_{S/R} \to \Omega_{T/R} \to 0$$

where $f \mapsto 1 \otimes df$ in the first map.

Example 1.2.4. *Let $S = R[x_1, \dots x_n]/(f_1, \dots f_N)$. Then $\Omega_{S/R}$ is the quotient of the free module generated by the symbols $dx_1, \dots dx_n$ by the submodule generated by $df_1, \dots df_N$. In particular, when $R = k$ is a field, we see that the cotangent space*

$$T^*_{S/k,p} \cong \operatorname{coker} \left(\frac{\partial f_i}{\partial x_j}(p) \right)$$

is dual to the space given in the introduction.

The module $\Omega_{S/R}$ is almost never free. There is however a weaker property which often holds. An R-module M is locally free (of rank n) if for any if there exist an open cover of $\operatorname{Spec} R$ by basic open sets $\{D(f_i)\}$ such that $M[1/f_i] = R[1/f_i] \otimes_R M$ is free (of rank n).

Theorem 1.2.5. *If M is a finitely generated module over a noetherian ring R, then the following are equivalent*

(1) M is locally free.

(2) $M_p = R_p \otimes M$ is free for each $p \in SpecR$

(3) M is projective (i.e. a direct summand of a free module).

When R is reduced, these are equivalent to

(4) The function $m \mapsto \dim k(m) \otimes M$ f from $Max\, R \to \mathbb{Z}$ is locally constant.

Proof. [E, p 475]. For the last statement, see [Mu, III§2] □

Since $k(m) \otimes \Omega_{R/k}$ is the dimension of the (co)tangent space at m, we obtain an intrinsic characterization of smoothness.

Proposition 1.2.6. *A domain $R = k[x_1, \ldots x_n]/(f_1, \ldots f_N)$ is smooth if and only if $\Omega_{R/k}$ is locally free of rank equal to the Krull dimension $d = \dim R$.*

1.3 Flat maps

" The concept of flatness is a riddle that comes from algebra, but which technically is the answer to many prayers."

- D. Mumford

Given a module M over a commutative ring R. The functor $N \mapsto M \otimes N$ is right exact, i.e. given an exact sequence of modules

$$0 \to N_1 \xrightarrow{\iota} N_2 \to N_3 \to 0$$

we have an exact sequence

$$M \otimes N_1 \xrightarrow{M \otimes \iota} M \otimes N_2 \to M \otimes N_3 \to 0$$

The map $M \otimes \iota$ need not be injective. M is called flat if $M \otimes \iota$ is injective for any ι. Suppose that R is a domain and that M has a element such that $rm = 0$ with $r \neq 0$. If we take ι to be multiplication by r on R, then $M \otimes \iota$ is multiplication by r on M. This is not injective. Therefore, we conclude that *a flat module is torsion free*. When R is a PID, the converse is true as well.

Example 1.3.1. *A locally free module is flat. In fact, a finitely generated module over a noetherian ring is flat if and only if it is locally free.*

Example 1.3.2. *The polynomial ring $R[x_1, \ldots x_n]$ viewed as an R-module is flat.*

Proposition 1.3.3. *Let S be an R-algebra. Suppose that M is an S-module, $f \in S$ an element such that multiplication by f is injective on $M \otimes k(m)$ for all $m \in Max\, R$, and M is flat over R. Then M/fM is flat over R.*

We omit the proof which follows easily from properties of the Tor functors.

An R-algebra is called smooth of relative dimension m if it is has a presentation

$$S = R[x_1, \ldots x_{n+m}]/(f_1 \ldots f_n)$$

with $rank\left(\frac{\partial f_i}{\partial x_j}\right) = n$. Note that étale is the same as smooth of relative dimension 0. From the Jacobian criterion for nonsingularity, we obtain

Lemma 1.3.4. *A smooth algebra of relative dimension m over an algebraically closed field is the coordinate ring of a disjoint union of m-dimensional varieties.*

Proposition 1.3.5. *A smooth algebra is flat.*

Proof. Set $R_i = R[x_1, \dots x_n]/(f_1, \dots f_i)$. We use induction to prove that each R_i is flat. For R_0 this clear. Let $m \subset R$ be maximal ideal, and $k = R/m$. Suppose R_i is flat. Then we can use proposition 1.3.3 to check flatness of $R_{i+1} = R_i/f_{i+1}R_i$, provided that we can show that f_{i+1} is a nonzero divisor in $k[x_1, \dots x_n]/(f_1, \dots f_i)$. Then by proposition 1.3.5, $V_i = \operatorname{Spec} \bar{k}[x_1, \dots x_n]/(f_1, \dots f_i)$ is a union of $n - i$ dimensional varieties. In particular, $\bar{k}[x_1, \dots x_n]/(f_1, \dots f_i)$ is a product of integral domains. Since all components of V_{i+1} have lower dimension, it follows that f_{i+1} does not vanish on any component of V_i. So in particular, f_{i+1} is a nonzero divisor in $\bar{k}[x_1, \dots x_n]/(f_1, \dots f_i)$ and therefore in $k[x_1, \dots x_n]/(f_1, \dots f_i)$. □

We have the following coordinate free characterization of étale maps.

Theorem 1.3.6. *Suppose R is noetherian. A homomorphism $R \to S$ is étale if and only if the following hold*

(1) S is finitely generated as an algebra.

(2) S is flat as an R-module.

(3) $\Omega_{S/R} = 0$.

Proof. Suppose that $S = R[x_1, \dots x_n]/(f_1, \dots f_n)$ is étale. Then finite generation is automatic. Flatness follows from proposition 1.3.5. $\Omega_{S/R}$ vanishes because it is the cokernel of the Jacobian matrix.

The converse can be found in [A, chap I] or [Mu, chap III§10]. □

One application of this is that the Jacobian condition for étaleness is independent of the presentation. Smoothness has a similar characterization where the last property is replaced by is locally freeness of $\Omega_{S/R}$.

The next example shows that flatness cannot be omitted.

Example 1.3.7. *Let $R = \mathbb{C}[x, y]/(y^2 - x^2(x - 1))$ be the coordinate ring of a nodal curve. Its normalization is $\tilde{R} = \mathbb{C}[t]$ with $R \to \tilde{R}$ given by*

$$x \mapsto t^2 - 1, \quad y \mapsto t(t^2 - 1)$$

With the classical topology $\operatorname{Spec} \tilde{R} \to \operatorname{Spec} R$ is not a covering space, so it shouldn't be étale. To justify this, write

$$\tilde{R} = R[t]/(x - (t^2 - 1), y - t(t^2 - 1))$$

From this it follows that it is finitely generated and that $\Omega_{\tilde{R}/R} = 0$. $\tilde{R}$ is also finitely generated as an R-module. However, it is not locally free. Therefore $\tilde{R}$ is not flat, and so not étale.

We give a refinement of an earlier result.

Proposition 1.3.8. *If S is a flat (resp. étale) R-algebra, and T is a flat (resp. étale) S-algebra, then T is a flat (resp. étale) R-algebra.*

Proof. Writing $T \otimes_R N = T \otimes_S S \otimes_R N$ makes it clear that T is flat over R under the above conditions. The Kähler differentials fit into an exact sequence

$$\Omega_{S/R} \to \Omega_{T/R} \to \Omega_{T/S} \to 0$$

Therefore the middle term vanishes if the outer two do. Finally, fnite generation of T over R is clear. $\square$

1.4 Extension to schemes

Given an R-module M, recall we can form a sheaf $\tilde{M}$ on $\operatorname{Spec} R$ such that $\tilde{M}(D(f)) = M[1/f] = M \otimes_R R[1/f]$. Sheaves which arise this way are called quasi-coherent. More generally, a sheaf on a scheme is quasi-coherent if it is with respect to some (equivalently any) affine open cover.

Lemma 1.4.1. *Given a morphism $f : X \to Y$, there exists a quasi-coherent sheaf $\Omega_{X/Y}$ such that $\Omega_{X/Y}|_{\operatorname{Spec} S_{ij}} = \tilde{\Omega}_{S_{ij}/R_i}$ for open affine covers $\operatorname{Spec} R_i = U_i$ of Y and $\operatorname{Spec} S_{ij}$ of $f^{-1}U_i$.*

Flatness is a local condition. Therefore we can extend it to schemes: A morphism $X \to Y$ is called flat if $\mathcal{O}_{X,x}$ is a flat module $\mathcal{O}_{Y,f(x)}$ for each $x \in X$. For étale, we take theorem 1.3.6 as the definition. A morphism of noetherian schemes is étale if the following condition holds

(1) It is locally of finite type

(2) It is flat.

(3) $\Omega_{X/Y} = 0$.

Putting together earlier results shows that

- The class of étale morphisms includes open immersions
- The class of étale morphisms is stable under composition.
- The class of étale morphisms is stable under fibre products.

Chapter 2

Étale fundamental group

2.1 Profinite groups and Galois theory

Given a sequence of groups and homomorphisms

$$\ldots \to G_1 \xrightarrow{f_1} G_0$$

their inverse limit is the group

$$\varprojlim_n G_n = \{(g_n) \in \prod G_n \mid f_n(g_n) = g_{n-1}\}$$

One can consider more general inverse limits parametrized by ordered sets. The construction is similar. A group is called profinite if it is an inverse limit of finite groups.

Example 2.1.1. *The additive group of p-adic integers*

$$\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$$

is profinite.

Example 2.1.2. *Given a group G, the profinite completion is*

$$\hat{G} = \varprojlim (\textit{finite quotients of } G)$$

If each finite group is given the discrete topology, then the limit becomes a topological group, i.e. it carries a topology such that the group operations are continuous.

Proposition 2.1.3. *A topological group is profinite if and only if it is compact Hausdorff and totally disconnected. The last condition means that there is a basis given by sets which are both open and closed.*

An infinite field extension L/K is Galois if it is a union of finite Galois extensions. The Galois group G can be identified with the inverse limit of the Galois groups of all the finite intermediate extensions. So in particular, it is profinite. The fundamental theorem of Galois theory for infinite extensions says

Theorem 2.1.4. *There is a one to one correspondence between closed subgroups of $Gal(L/K)$ and intermediate field extensions: $H \subseteq Gal(L/K)$ corresponds to $L^H = \{f \in L \mid \forall h \in H, hf = f\}$. Galois extensions correspond to closed normal subgroups.*

2.2 Étale fundamental group

Recall that the topological fundamental group $\pi_1(X)$ of a connected topological space is the group of homotopy classes of loops in X with a given base point. There is another characterization which is more convenient for us. A map $\pi : Y \to X$ is called a covering space if it is a locally a homeomorphism. The universal cover $\pi : \tilde{X} \to X$ is a covering space such that $\tilde{X}$ is connected and simply connected $\pi_1(\tilde{X}) = \{1\}$. Then $\pi_1(X)$ can be identified with the group of self-homeomorphisms of $\tilde{X}$ commuting with π.

When X is a scheme, we would like an algebraic geometric analogue of the fundamental group. If X is say a quasi-projective variety defined over $\mathbb{C}$, we can give X give the usual topology (induced from the Euclidean topology on projective space) and then form the fundamental group as above. This does not generalize. The root of the problem is that the universal cover has no meaning in algebraic geometry. However, we can consider finite approximations to it. The key point is

Theorem 2.2.1 ("Riemann's existence theorem")**.** *Any étale morphism $Y \to X$ is a finite to one covering space of X with the usual topology. Conversely, every finite to one covering space arises this way from a unique étale cover.*

Proof. The correspondence between étale covers of X and of the associated analytic space is given in [SGA1, XII, 5.2], and the latter covers are easily seen to correspond to covering spaces in the topological sense. $\square$

Thus we should be able to build a kind of fundamental group from the collection of étale covers. To simplify the discussion assume that X is a normal scheme. This means that X is irreducible as a topological space, and it is covered by spectra of normal (= integrally closed) rings. If $X = \cup \operatorname{Spec} R_i$, then the field of fractions of each R_i is independent of i. It is called the function field of X and denoted by $K(X)$. If $L \supset K(X)$ is a finite extension. Let $\tilde{R}_i$ denote the integral closure of R_i in L. Then we can glue $\operatorname{Spec} \tilde{R}_i$ to get a new normal scheme $X_L \to X$ with function field L. We define the maximal unramified extension

$$K(X)_{unr} = \bigcup \{L \supset K(X) \mid \ X_L \to X \text{ is étale}\}$$

Define the étale fundamental group

$$\pi_1^{et}(X) = Gal(K(X)_{unr}/K(X))$$

Example 2.2.2. *If $X = \operatorname{Spec} k$, where k is a field. $K(X)_{unr} = k^{sep}$ is the seperable closure. Therefore $\pi_1^{et}(X) = Gal(k) = Gal(k^{sep}/k)$ is the absolute Galois group.*

As a corollary to the previous theorem, we can deduce.

Theorem 2.2.3 (Grothendieck)**.** *If X a scheme of finite type over $\mathbb{C}$, then $\pi_1^{et}(X)$ is the profinite completion of $\pi_1(X)$*

This allows the computation of these groups by topological methods.

Example 2.2.4. *Let X be a smooth complex projective curve of genus g. There is a well known presentation for the fundamental group*

$$\pi_1(X) = \langle a_1, \dots a_{2g} \mid [a_1, a_2] \dots [a_{2g-1}, a_{2g}] = 1 \rangle$$

where $[a, b] = aba^{-1}b^{-1}$ is the commutator. Thus $\pi_1^{et}(X)$ is given as the profinite completion of this group.

When X is a smooth curve defined over a field of characteristic $p > 0$, the structure of $\pi_1^{et}(X)$ is somewhat mysterious. Even for the line, $\pi_1^{et}(\mathbb{A}^1)$ is not well understood. It is clear that it is nontrivial since the Artin-Scheier covers $y^p - y - f(x) = 0$ of $\mathbb{A}^1 = \operatorname{Spec} k[x]$ are étale. However, we can understand a part of this quite well. Define the tame fundamental group

$$\pi_1^{tame}(X) = Gal(\bigcup\{L \mid L \supset K(X) \text{ unramified of deg prime to } p\}/K(X)$$

Theorem 2.2.5 (Grothendieck)**.** *If X is a smooth curve over an algebraically closed field of characteristic $p > 0$, $\pi_1^{tame}(X)$ is the the pro-prime to p completion*

$$\varprojlim \Gamma/N, \quad N \triangleleft \Gamma \text{ of index coprime to } p$$

of the corresponding topological fundamental group Γ. That is $\Gamma = \langle a_1, \dots a_{2g} \mid [a_1, a_2] \dots [a_{2g-1}, a_{2g}] = 1 \rangle$ if X is projective of genus g, and Γ is free on $2g+r-1$ generators if X is the complement of r points in a genus g curve.

2.3 First homology

Returning to the topological case, recall that Hurewic's theorem says that the abelianization of the fundamental group $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ is isomorphic to the first homology $H_1(X, \mathbb{Z})$. Combining this with the universal coefficient theorem, we see that the first cohomology with coefficients in an abelian group A is

$$H^1(X, A) \cong Hom(H_1(X, \mathbb{Z}), A) \cong Hom(\pi_1(X), A)$$

By imitating this, we can define the first étale cohomology

$$H^1(X_{et}, A) = Hom(\pi_1^{et}(X), A)$$

for finite A. When X is defined over $\mathbb{C}$, this does coincide with the usual cohomology. This is because any homomorphism from $\pi_1(X)$ to A factors uniquely through the profinite completion, so that

$$Hom(\pi_1(X), A) = Hom(\hat{\pi}_1(X), A)$$

This isn't true when A is infinite, and this is part of the reason for making the finiteness restriction.

Example 2.3.1. *When X is complex smooth projective curve of genus g, we see*

$$H^1(X_{et}, A) = Hom(\langle a_1, \dots a_{2g} \mid [a_1, a_2] \dots [a_{2g-1}, a_{2g}] = 1\rangle, A) = A^{2g}$$

When X is a smooth projective curve defined over any algebraically closed field k, and A is a finite abelian group of order coprime to the characteristic, then

$$H^1(X_{et}, A) = Hom(\pi_1^{tame}(X), A) \cong A^{2g}$$

still holds. So it does give a good generalization of usual cohomology. In some ways, it is better because there is more structure. If X defined over a non algebraically closed field k with seperable closure k^{sep}. Then there is a tower $k \subset k^{sep} \subset K(X)_{unr}$ giving rise to an exact sequence

$$1 \to Gal(K(X)_{unr}/k^{sep}) \to \pi_1^{et}(X) \xrightarrow{p} Gal(k) \to 1$$

The group on the left can be identified with étale fundamental group of the corresponding scheme $\bar{X} = X \times_{\mathrm{Spec}\, k} k^{sep}$ over k^{sep}. If we choose a continuous section to p, we get an action of $Gal(k)$ on $\pi_1^{et}(\bar{X})$, which induces a well defined action on $H^1(X_{et}, A)$. So this is not just a group, but a Galois module. This structure is very important in number theoretic applications.

Chapter 3

Étale topology

3.1 Grothendieck topologies

One of Grothendieck's insights is that by generalizing the definition of a topological space, one gets many other interesting examples such as the étale topology. The starting point is to observe that given a topological space X, the collection of open sets $Open(X)$

- is a partially ordered set,
- and it has is a notion of open coverings $U = \cup U_i$.

We replace $Open(X)$ by a category C with fibre products which plays the role of intersections. A Grothendieck topology on C is a collection of families of morphisms $\{U_i \to U\}$ called coverings satisfying

1. The family consisting of a single isomorphism $\{U \cong U\}$ is a covering.
2. If $\{U_i \to U\}$ and $\{V_{ij} \to U_i\}$ are coverings, then so is the composition $\{V_{ij} \to U\}$ is a covering.
3. If $\{U_i \to U\}$ is a covering and $V \to U$ is a morphism, then $\{U_i \times_U V \to V\}$ is a covering.

A category with a Grothendieck topology is called a site.

Example 3.1.1. *$Open(X)$ can be regarded as a category, where $U \to V$ means $U \subseteq V$. A covering $\{U_i \subset U\}$ is just an open covering in the usual sense.*

Example 3.1.2. *Let X be a scheme. Let $Et(X)$ denote the category have as objects étale morphisms $U \to X$ and as morphisms commuting triangles*

$$\begin{tikzcd} U \arrow[r] \arrow[dr] & V \arrow[d] \\ & X \end{tikzcd}$$

We say that $\{\pi_i : U_i \to U\}$ is a covering if $\cup \pi_i(U_i) = U$. $Et(X)$ with this Grothendieck topology is called the étale site of X. We denote this by X_{et}.

Example 3.1.3. *Let X be a noetherian scheme. Let $Flat(X)$ denote the category having as objects flat morphisms $U \to X$ subject to the finiteness condition that the preimage of any affine open in X is covered by finitely many affine opens in U. Morphisms commuting triangles as above. We say that $\{\pi_i : U_i \to U\}$ is a covering if the images cover U. We call $Flat(X)$ with this Grothendieck topology the flat site[1] of X. We denote this by X_{flat}.*

The for a scheme X, we have a choice of 3 topologies. In order of increasing fineness, we have the Zariski X_{zar}, étale X_{et}, and flat X_{flat} topologies.

3.2 Sheaves

Given a site C, a presheaf of sets, groups... is a contravariant functor F from C to the category of sets, groups.... For any covering $\{U_i \to U\}$, we define the set of patchable sections by

$$\check{H}^0(\{U_i \to U\}, F) = \{(f_i) \in \prod F(U_i) \mid \text{ the image of } f_i = \text{ the image of } f_j \text{ in } F(U_i \times_U U_j)\}$$

There is a canonical map

$$F(U) \to \check{H}^0(\{U_i \to U\}, F)$$

given by sending $f \in F(U)$ to the collection of its images in $F(U_i)$. F is called a sheaf if for any covering $\{U_i \to U\}$

$$F(U) \cong \check{H}^0(\{U_i \to U\}, F)$$

For $Open(X)$ this is the usual condition, so there are plenty of examples. We give a criterion for checking something is a sheaf for the flat site. This can be used for the étale site also since it is coarser.

Proposition 3.2.1. *A presheaf F on X_{flat} or X_{et} is a sheaf if and only*

1. *F is sheaf on the Zariski topology X_{zar}*
2. *For any covering $U' \to U$ of affine schemes in X_{flat} or X_{et}*

$$F(U) \cong \check{H}^0(\{U' \to U\}, F)$$

Proof. See [M1, p 50]. □

Let $X = \operatorname{Spec} k$ where k is a field. The Zariski topology is trivial, but the étale topology is far from it. Recall that elements of X_{et} are given by $\operatorname{Spec} \prod L_i$ where L_i are seperable extensions. Let $K = k^{sep}$ and $G = Gal(K/k)$, then we

[1] This is usually called the fpqc (= fidèlement plat quasi-compact) site

can write this as $\operatorname{Spec}\prod K^{G_i}$ for some open $G_i \subset G$. By a G-module, we mean an abelian group M with discrete topology on which G acts continuously. The last condition is equivalent to requiring that every element of M has a finite orbit. Given M define

$$F_M(\operatorname{Spec}\prod K^{G_i}) = \bigoplus M^{G_i}$$

Then

Theorem 3.2.2. *F_M is a sheaf on X_{et}. The correspondence $M \mapsto F_M$ gives an equivalence between the category of G-modules and sheaves of abelian groups.*

Proof. The conditions of proposition 3.2.1 can be checked for F_M.

Suppose that F is a sheaf on X_{et}. For any Galois extension L/k, $Gal(L/k)$ acts on $F(\operatorname{Spec} L)$. Therefore $M = \varinjlim F(L)$, as L runs over finite Galois extensions of k in K, becomes a G-module. $F \mapsto M$ gives an inverse. See [M1, p 53] for further details.

∎

The above construction can be refined for more general schemes.

Example 3.2.3. *Suppose that X is normal and M is a $\pi_1^{et}(X)$-module. If $U \to X$ is étale and connected, $\pi_1^{et}(U)$ will act on M via the homomorphism $\pi_1^{et}(U) \to \pi_1^{et}(X)$. Then*

$$F_M(U) = M^{\pi_1^{et}(U)}$$

determines a sheaf on X_{et}. Not all sheaves arise this way in general. A sheaf which does is called locally constant. F_M is constant if the $\pi_1^{et}(X)$ action is trivial.

Example 3.2.4. *Let $i : \bar{x} = \operatorname{Spec} k(x)^{sep} \to X$ be a geometric point by which we mean the seperable closure of the residue field of closed point. Given a group M, the skycraper sheaf*

$$i_*M(U) = M^{\#U\times_X \bar{x}}$$

will not be locally constant unless $X = \bar{x}$.

3.3 Faithfully flat descent

An R-algebra S is faithfully flat if it is flat and $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective. This is different from the usual definition. For the equivalence and the following lemma, see [AK, chap V].

Lemma 3.3.1. *If S is faithfully flat then $R \to S$ is injective.*

We need the following weak version of faithfully flat descent in order to construct certain basic examples of sheaves.

Theorem 3.3.2 (Faithfully flat descent). *Let S be a faithfully flat R-algebra. Then*

1. *For any R-module M,*

$$0 \to M \to M \otimes_R S \xrightarrow{d} M \otimes_R \otimes S \otimes_R S$$

is exact where $d(m \otimes s) = m \otimes 1 \otimes s - m \otimes s \otimes 1$

2. *For any R-algebra T,*

$$0 \to Hom_{R\text{-}alg}(T, R) \to Hom_{R\text{-}alg}(T, S) \to Hom_{R\text{-}alg}(T, S \otimes_R S)$$

is exact.

Proof. For 1, see [M1, p 17]. It follows that

$$0 \to Hom_{R\text{-}mod}(T, R) \to Hom_{R\text{-}mod}(T, S) \to Hom_{R\text{-}mod}(T, S \otimes_R S)$$

is exact. To finish the proof of 2, we observe that an $f \in Hom_{R\text{-}mod}(T, R)$ is an algebra homomorphism if its image lies in $Hom_{R\text{-}alg}(T, S)$ because $R \to S$ is injective. $\square$

Corollary 3.3.3. *Let $\mathcal{M}$ be a quasi-coherent sheaf on X_{zar}. Given $\pi : U \to X$ in X_{flat}, let $\mathcal{M}_{flat}(U) = \pi^*\mathcal{M}(U)$. Then $\mathcal{M}_{flat}$ is sheaf on X_{flat}. Therefore $\mathcal{M}_{flat}$ restricts to a sheaf on X_{et}.*

Theorem 3.3.4. *If $Y \to X$ is a morphism of schemes, then*

$$Y(U) = Hom_{Schemes/X}(U, Y)$$

is a sheaf on the flat site, and therefore on the étale site.

Proof. This is more or less proved in [M1, pp 16-18], or a complete proof can be found in [SGA3, IV 6.3.1]. For Y affine, this pretty much follows from prop 3.2.1 and theorem 3.3.2. The general case can be reduced to this. $\square$

Example 3.3.5. *For any set $S = \{s_1, \dots s_n\}$, let $Y = S \times X \cong \coprod_i^n X$. If U is connected, then $Y(U) = S$, otherwise it's bigger. We denote this sheaf by S when there is no confusion. This is by definition a constant sheaf. Note that if S is a group, the corresponding sheaf is a sheaf of groups.*

Example 3.3.6. *Let $Y = \mathbb{A}^1_{\mathcal{X}}$. Then $Y(U) = \mathcal{O}_U(U)$ is the just the structure sheaf in the flat topology. The fact that this is a sheaf also follows from corollary 3.3.3. We usually write this simply as $\mathbb{G}_a$.*

Example 3.3.7. *The multiplicative group scheme $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[x, x^{-1}]$. Let $Y = \mathbb{G}_m \times X$. Then $Y(U) = \mathcal{O}_U^*$ is the sheaf of units. We usually write this simply as $\mathbb{G}_m$.*

Example 3.3.8. *The group scheme of roots of unity $\boldsymbol{\mu}_n = \operatorname{Spec} \mathbb{Z}[x]/(x^n - 1)$. Let $Y = \boldsymbol{\mu}_n \times X$. Then $Y(U) = \{f \in \mathcal{O}(U) \mid f^n = 1\}$. We usually denote this by $\boldsymbol{\mu}_n$. This is locally constant. When X is covered by spectra of rings having n distinct roots of unity, $\boldsymbol{\mu}_n$ is constant and $\boldsymbol{\mu}_n \cong \mathbb{Z}/n\mathbb{Z}$ noncanonically.*

When X is the spectrum of a field k with K, G as above. We can describe the restrictions of these sheaves to X_{et} in terms of G-modules.

1. $S \leftrightarrow S$ with trivial G-action on the right.
2. $\mathbb{G}_a \leftrightarrow K$ with the usual G-action on the right.
3. $\mathbb{G}_m \leftrightarrow K^*$ with the usual G-action on the right.
4. $\boldsymbol{\mu}_n \leftrightarrow \boldsymbol{\mu}_n(K) = \{f \in K \mid f^n = 1\}$ with the usual G-action on the right.

3.4 Schemes modulo equivalence relations

As general as schemes are, it turns out that schemes are sometimes not general enough. It's not that one wants generalizations for their own sake, but rather because one wants certain constructions to make sense. For example, in many situations one would like to form quotients by group actions. However, Hironaka has given an example of a variety X with a fixed point free action by finite group G for which the quotient X/G does not exist as a scheme (see [K]). There are other issues as well. In complex geometry, Grauert showed that one can contract subvarieties with negative normal bundle. While such a result would be desirable in the algebraic world, it not always possible within schemes. However, Artin gave an enlargement of the category of schemes called algebraic spaces, where these kinds of constructions do exist. In rough terms it is a scheme modulo a suitable equivalence relation. The problem is to make the last part precise.

As starting point recall that a functor represented by scheme is sheaf on the étale site of $\operatorname{Spec}\mathbb{Z}$, and this functor determines the scheme. In particular, we can identify X with the corresponding sheaf. This would suggest that all sheaves, not just representable ones, could play the role of generalized schemes. However, these objects can be pretty wild. Artin singled a good subclass of these sheaves. Given a category with products an pair of morphisms $R \rightrightarrows X$ is equivalent to a single morphism $R \to X \times X$. This is called an equivalence relation if for each Y, the map $Hom(Y, R) \to Hom(Y, X) \times Hom(Y, X)$ is injective and the image is an equivalence relation in the usual sense. The quotient, if it exists, is a morphism $X \to X/R$ such that the composition of both projections $R \rightrightarrows X$ with $X \to X/R$ coincide, and such that this is the universal morphism with this property. Quotients by equivalence relations exist in the category of étale sheaves. A sheaf F is an algebraic space if $F = X/R$, where X is a scheme and $R \subset X \times X$ is subscheme which is equivalence relation such that the projections $R \to X$ are étale. So by definition X/G is an algebraic space when G is a finite group acting without fixed points. In particular, Hironaka's example is an algebraic space. A systematic development can be found in [A, K].

Chapter 4

Cohomology

4.1 Exact sequences

Let C be a site, and let $PSh(C)$ denote the category of presheaves of abelian groups. The objects are presheaves, i.e. contravariant functors from C to the category of abelian groups. The morphisms are natural transformations, that is collections of homomorphisms $\eta_U : F(U) \to G(U)$ such that

$$\begin{array}{ccc} F(U) & \xrightarrow{\eta_U} & G(U) \\ \downarrow & & \downarrow \\ F(V) & \xrightarrow{\eta_V} & G(V) \end{array}$$

commutes. The category of $PSh(C)$ is a so called abelian category [Ma] so it comes with a notion of exactness. To make this explicit, a sequence of presheaves

$$\dots F \to G \to H \dots$$

is exact precisely when

$$\dots F(U) \to G(U) \to H(U) \dots$$

is exact for each $U \in C$. Alternatively, define the presheaf kernel and image associated to a morphism η by

$$pker(\eta)(U) = \ker(\eta_U), \quad pim(\eta)(U) = im(\eta_U)$$

The condition for exactness amounts to pim of each morphism in the sequence to coincide with $pker$ of the next morphism.

The category $Sh(C)$ has as its objects sheaves of abelian groups, and morphisms are defined as above. This is also an abelian category, but the notion of *exactness in* $Sh(C)$ *is different from* $PSh(C)$. In other words the inclusion $Sh(C) \subset PSh(C)$ is not an exact functor. The source of the problem is this. If $\eta : F \to G$ is a morphism of sheaves $ker(\eta) = pker(\eta)$ is a sheaf but $pim(\eta)$ usually isn't.

Example 4.1.1. *Let $X = S^1 = \mathbb{R}/\mathbb{Z}$ be the circle with its usual topology. Let $d : C^\infty_X \to C^\infty_X$ take f to its derivative. The constant function $1 \notin d(C^\infty(X))$, even though its restriction to any covering by intervals is the image. So $pim(d)$ is not a sheaf.*

The solution is replace pim by the sheaf image

$$im(\eta)(U) = \{f \in F(U) \mid \exists \text{ a covering } \{U_i\} \text{ such that } f|_{U_i} \in im(\eta_{U_i})\}$$

Then exactness amounts to the condition $ker = im$ as above. Returning to the previous example, we can see that

$$0 \to \mathbb{R}_X \to C^\infty_X \stackrel{d}{\to} C^\infty_X \to 0$$

is an exact sequence of sheaves although not of presheaves. Here $\mathbb{R}_X$ is the sheaf of locally constant real valued functions.

4.2 Stalks in the étale topology

On a topological space X, there is a simple and useful criterion for exactness: A sequence of sheaves

$$\ldots F \to G \to H \ldots$$

is exact if and only if for every $x \in X$, the corresponding sequence of stalks

$$\ldots F_x \to G_x \to H_x \ldots$$

in the usual sense. The bad news/good news is that there is no analogue of this for sites in general, but there is for the étale site.

If X is a scheme, a geometric point consists of a point $x \in X$ together with a choice of seperable closure $k(x)^{sep}$ of the residue field $k(x) = \mathcal{O}_x/m_x$. Equivalently, it is simply a morphism $\bar{x} = \operatorname{Spec} k(x)^{sep} \to X$. An étale neighbourhood of this point is a commutative diagram

$$\begin{tikzcd} \bar{x} \arrow[r] \arrow[dr] & U \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

where π is étale. Given sheaf F on X_{et}, define the stalk at $\bar{x}$ by

$$F_{\bar{x}} = \varinjlim F(U)$$

where U runs over all étale neighbourhoods. This clearly gives a functor $Sh(X_{et}) \to Sets$. The key fact [M1, II 2.15] is

Theorem 4.2.1. *A sequence of sheaves*

$$\ldots F \to G \to H \ldots$$

on X_{et} is exact if only

$$\dots F_{\bar{x}} \to G_{\bar{x}} \to H_{\bar{x}} \dots$$

exact for all geometric points $\bar{x}$.

The stalk $\mathcal{O}_{X,\bar{x}}$, called the strict Henselization of the local ring $\mathcal{O}_{X,x}$, can be described directly in terms of commutative algebra, although it is not something one sees in a basic course.

Theorem 4.2.2. *For any local ring R with residue field k, the strict Henselization R^{sh} is a local ring with residue field k^{sep}. Moreover, R^{sh} satisfies Hensel's lemma which says that given $f(x) \in R^{sh}[x]$ with a simple root $\bar{r} \in k^{sep}$, there exists a root $r \in R^{sh}$ mapping to $\bar{r}$.*

Example 4.2.3. *Let R be local ring of $\mathbb{A}^n_k$ at the origin, i.e. the localization of $k[x_1, \dots x_n]$ at $(x_1, \dots x_n)$. The strict Henselization R^{sh} is isomorphic to the ring of algebraic power series*

$$\{f \in k^{sep}[[x_1, \dots x_n]] \mid \exists p \in k[t] - \{0\}, p(f) = 0\}$$

See [A, p 45].

An extraordinarily useful exact sequence of sheaves in complex manifold theory is the exponential sequence

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \stackrel{e^{2\pi i}}{\to} \mathcal{O}_X^* \to 1$$

Since it uses the exponential function, there is no way to make it algebraic, but there is a good substitute called the Kummer sequence.

Proposition 4.2.4. *Let X be a scheme such that n does not divide the characteristic of any of the residue fields, then the Kummer sequence*

$$1 \to \boldsymbol{\mu}_n \to \mathbb{G}_m \stackrel{f \mapsto f^n}{\longrightarrow} \mathbb{G}_m \to 1$$

is exact on X_{et}.

Proof. We can check exactness on stalks. If $R = \mathcal{O}^{sh}_{X,x}$, then it comes down to verifying

$$1 \to \boldsymbol{\mu}_n(R) \to R^* \stackrel{n}{\to} R^* \to 1$$

is exact. The only nontrivial issue is the surjectivity of the last map. Given $r \in R^*$, $f(x) = x^n - r$ will have a simple root in the separably closed field k^{sep}. By Hensel's lemma this lifts to a root in R. □

4.3 Cohomology

Given a site C, as we mentioned earlier the presheaf kernel is already a sheaf. Therefore

Lemma 4.3.1. *Given a site C, a short exact sequence of sheaves*

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

and an object $X \in C$, there is an exact sequence

$$0 \to F_1(X) \to F_2(X) \to F_3(X)$$

In general, as we saw the last map usually isn't surjective ex 4.1.1. Here's another example

Example 4.3.2. *The Kummer sequence for $n = 2$ on $\mathbb{Q}$ yields*

$$1 \to \{\pm 1\} \to \mathbb{Q}^* \xrightarrow{x \mapsto x^2} \mathbb{Q}^*$$

The last map is not surjective of course.

There is a way to handle this, which is part of a familiar pattern in homological algebra [W]. Suppose we are given a functor $\mathcal{F}$ from one abelian category to another which is additive in the sense it takes sums to sums. Suppose also that $\mathcal{F}$ is also left exact which means that it takes a short exact sequence

$$0 \to A \to B \to C \to 0$$

to an exact sequences

$$0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C)$$

Under some additional assumptions about the abelian categories that every object embeds into an injective we can prolong this to a long exact sequence in a canonical way. These assumptions hold in our case, so to summarize a rather long story:

Theorem 4.3.3. *There exists a sequence of additive functors $H^i(X, -)$, $i = 0, 1, 2, \ldots$ from $Sh(C)$ to abelian groups such that $H^0(X, F) = F(X)$ and a short exact sequence of sheaves*

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

leads to a long exact sequence

$$0 \to H^0(X, F_1) \to H^0(X, F_2) \to H^0(X, F_3) \to H^1(X, F_1) \to H^1(X, F_2) \ldots$$

Of course the details can be found in the references at end, but simply treating everything as a black box is not terribly enlightening. Let us at least try to discover the meaning of some special cases.

4.4 Torsors

Let $\{U_i \to X\}$ be a covering on a site C. Let $U_{ij} = U_i \times_X U_j$ etc. Given a sheaf of not necessarily abelian groups G, by definition a collection $g_i \in G(U_i)$ such that $g_i|_{U_{ij}} = g_j|_{U_{ij}}$ patches to a unique $g \in G(X)$. Let us suppress the restriction symbols and simply write this as $g_i g_j^{-1} = 1$. We refer to this as the 0-cocycle condition. Perhaps this suggests the next step. A collection $g_{ij} \in G(U_{ij})$ satisfies the (Cech) 1-cocycle condition if

$$g_{ij} g_{ik}^{-1} g_{jk} = 1$$

The pattern is perhaps clearer if we write

$$g_{ij\hat{k}} g_{i\hat{j}k}^{-1} g_{\hat{i}jk} = 1$$

By considering degenerate cases such $i = j = k$ we get

$$g_{ii} = 1, \quad g_{ji} = g_{ij}^{-1}$$

Such cocycles come up naturally in the theory of fibre bundles. Let say that C is the site associated to a topological space X, and $\mathcal{G}$ is a topological group, and $G(U)$ is the group of continuous maps from U to $\mathcal{G}$. Given 1-cocycle g_{ij}, and a space F on which G acts we can build a new topological space $\mathcal{F}$ by gluing $F \times U_i$ to $F \times U_j$ by $(f, x) \mapsto (g_{ij}(x) f, x)$. There are some consistency issues, but these are taken care of by the cocycle rule. $\mathcal{F}$ comes with a map to X, and is an example of a locally trivial fibre bundle. In particular the construction applies when $F = \mathcal{G}$ the usual action of $\mathcal{G}$ on itself. We obtain a principal fibre bundle $\mathcal{F} = \mathcal{P}$. A version of this can be carried entirely in setting of sheaves over a site C. Thus to G and a cocycle g_{ij} we obtain a sheaf of sets P with an action $P \times G \to P$ such that the restriction $P(U_i)$ becomes isomorphic to to $G(U_i)$ with usual action of $G(U_i)$ on itself. P is called a torsor or principal homogeneous space. Given a torsor and fixed choice of isomorphisms $P(U_i) \cong G(U_i)$ – such a set of isomorphism is called a trivialization – we can reverse everything and obtain a cocycle.

Let us now assume that G is a sheaf of abelian groups. Let $Z^1(\{U_i \to X\}, G)$ denote the set of 1-cocycles. It is now an abelian group. We have subgroup of 1-coboundaries $B^1(\{U_i \to X\}, G)$ consisting of $g_{ij} = h_i h_j^{-1}$ for $h_i \in G(U_i)$. We define the first Cech cohomology group

$$\check{H}^1(\{U_i \to X\}, G) = Z^1/B^1$$

Taking the direct limit over coverings gives

$$\check{H}^1(XG) = \varinjlim \check{H}^1(\{U_i \to X\}, G)$$

Theorem 4.4.1. $\check{H}^1(X, G) \cong H^1(X, G)$

To see why is reasonable. Consider an exact sequence of sheaves of abelian groups

$$0 \to G \to F \to K \to 0$$

According to the theorem, the obstruction to lifting $k \in K(X)$ to $F(X)$ is given by an element $\delta(k) \in \check{H}^1(X, G)$. By definition of exactness of sheaves, there is a covering $\{U_i \to X\}$ and an elements $f_i \in F(U_i)$ mapping onto $k|_{U_i}$. Then the elements $g_{ij} = f_i f_j^{-1}$ is a 1-cocycle representing $\delta(k)$. If $\delta(k) = 0$ one can show that g_{ij} is a coboundary because the maps in above the direct limit are injective. In this case, we can correct the f_i so that they patch to a section of $F(X)$. Therefore $\delta(k)$ is the obstruction as claimed.

As see saw, elements of $Z^1(\{U_i \to X\}, G)$ correspond to torsors with fixed trvialization over the covering. Dividing by B^1 gets rid of the choice of trivialization, so $\check{H}^1(\{U_i \to X\}, G)$ is the set of bundles which can be trivialized on the given covering. Taking the limit, eliminates the last choice. Thus

Corollary 4.4.2. *The elements of $H^1(X, G)$ correspond to isomorphism classes of torsors for G, with 0 corresponding to the trivial torsor G.*

Here is a basic example. Let X be normal scheme and G be a finite quotient of $\pi_1^{et}(X)$. Then we have an étale cover $Y \to X$ with Galois group G. We wish to show that Y, or more accurately the sheaf $Y(-)$, is a torsor for G on X_{et}. By definition G acts on Y. So it remains to check local triviality. This comes down to

Lemma 4.4.3. *$G \times Y \cong Y \times_X Y$.*

Proof. Define a map $(g, y) \mapsto (gy, y)$. This can be checked to be an isomorphism. $\square$

We can refine this a bit. Suppose G is a finite group with a homomorphism $h : \pi_1^{et}(X) \to G$. Let $H = im(h)$. Then we can form an H-torsor Y as above. Let $P = G \times Y/H$ where H acts by $(g, y) \mapsto (h^{-1}g, yh)$. Then P becomes a G-torsor. Note that P need not be connected. Any G-torsor can be seen to arise this way. When G is abelian, then we have bijections

$$Hom(\pi_1^{et}(X), G) \cong \{G\text{-torsors}\} \cong H^1(X_{et}, G)$$

as sets. This partially justifies our earlier description of cohomology.

One can define higher Cech cohomology for a site by refining the above proceedure. A Cech n-cocycle in a sheaf of abelian groups $\mathcal{F}$ with respect to a covering $\{U_i \to X\}$ is a collection

$$g_{i_0, \ldots g_n} \in \mathcal{F}(U_{i_0, \ldots i_n})$$

satisfying

$$\delta(g) = \sum (-1)^r g_{i_0, \ldots \hat{i}_r, \ldots i_n} = 0$$

One then defines $\check{H}^n(\{U_i \to X\}, \mathcal{F})$ by taking the space of cocycles modulo the image of δ. $\check{H}^n(X, \mathcal{F})$ is defined as the direct limit over all coverings. Unfortunately, this need not be the same as $H^n(X, \mathcal{F})$ even when X is a topological space. Useful criteria for when these are isomorphic can be found in the references. For our purposes, the following is sufficient.

Theorem 4.4.4. *If X is a quasiprojective scheme, then*

$$H^i(X_{et}, \mathcal{F}) \cong \check{H}^i(X_{et}, \mathcal{F})$$

Proof. [M1, III, 2.17]. □

4.5 Galois cohomology

Let $X = \operatorname{Spec} k$, where k is a field, $K = k^{sep}$ and $G = Gal(K/k)$. Then we saw that the category of sheaves on X_{et} is equivalent to the category of G-modules. Under this correspondence

$$F(X) = M^G$$

Thus as, one would expect, the higher cohomology can be described in these terms. Given M, a (Galois) n-cocyle is a continuous map $f : G^n \to M$ satisfying

$$\delta(f)(g_1, \dots g_{n+1}) =$$

$$g_1 f(g_2, \dots g_{n+1}) + \sum (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots g_n) + (-1)^{n+1} f(g_1, \dots g_n) = 0$$

It sometimes convenient to add a normalization condition that f vanishes whenever one of the $g_i = 1$. Let $Z^n(G, M)$ ($Z_{norm}(G, M)$) denote the group of (normalized) cocycles. One checks that $\delta^2 = 0$ so elements in the image of δ, called coboundaries, are in particular cocycles. Let $B^n(G, M)$ ($B_{norm}(G, M)$) denote the group of (normalized) n-coboundaries. Then define the nth Galois cohomology by

$$H^n(G, M) = Z^n(G, M)/B^n(G, M)$$

One can use normalized cocycles above without changing the outcome.

Theorem 4.5.1. *If $F \in X_{et}$ corresponds to M, then*

$$H^i(X_{et}, F) \cong H^i(G, M)$$

This can proved easily once one has set up the general homological machinery [W]. The above functors are isomorphic because they are derived functors of isomorphic functors.

Corollary 4.5.2. *If F is a constant sheaf corresponding to a module M, then $H^1(X_{et}, F) \cong Hom_{cont}(G, M)$.*

Proof. A 1-cocycle satisfies

$$f(g_1g_2) = g_1f(g_2) + f(g_1)$$

in general. In this case, M has trivial action, so it is just a homomorphism. Moreover the coboundaries are trivial. $\square$

It is sometimes convenient to note that Galois cohomology can be expressed as a limit.

Lemma 4.5.3.

$$H^i(G, M) = \varinjlim H^i(Gal(L/k), M^{Gal(K/L)})$$

as L/k runs over finite Galois extensions. (The right hand cohomology is defined as above. Cocycles are automatically continuous.)

Theorem 4.5.4 (Hilbert's theorem 90)**.** $H^1(G, K^*) = 0$

Proof. By the lemma, we have

$$H^1(G, K^*) = \varinjlim H^1(Gal(L/k), L^*)$$

as L/k runs over finite Galois extensions. Fix L/k and set $G = Gal(L/k)$. Given a 1-cocycle $\phi \in Z^1(G, L^*)$ and $\lambda \in L$, set

$$\beta = \sum_{g\in G} \phi(g)g(\lambda)$$

By Galois theory, λ can be chosen so that $\beta \neq 0$ [L, p 303]. Furthermore in this case, one has $\phi(g) = \beta/g(\beta)$ which shows that ϕ is a coboundary. $\square$

There are some important cases, where higher cohomology has a concrete interpretation. The Brauer group $Br(k)$ is the set of isomorphism classess of finite dimensional noncommutative algebras over k which become isomorphic to matrix algebra after extending scalars to some finite Galois extension. The group operations is tensor product. The Brauer group is often highly nontrivial. The determination for $\mathbb{Q}$ is one of the achievements of class field theory. For the reals, $Br(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ with the generator given by the quaternions.

We can express

$$H^2(G, K^*) = \varinjlim H^2(Gal(L/k), L^*)$$

Fix L/k and set $G = Gal(L/k)$. Given normalized 2-cocycle $\phi \in Z^2_{norm}(G, L^*)$, we can define a noncommutative K-algebra $A_\phi = \{\sum a_g * g \mid a_g \in L, g \in G\}$, called the crossed product, with multiplication law $*$ determined by the rules

$$(a * 1) * (b * 1) = (ab) * 1, \quad a, b \in L$$

$$g * a = g(a) * g, \quad a \in L, g \in G$$

$$g * h = \phi(g, h)gh,\ g, h \in G$$

where $g(a)$ is given by the action of G on L. A few facts can be seen readily from these formulas

- The algebra is associative (this uses the cocycle property).
- The centre of this algebra is precisley k
- When $\phi = 1$, we can identify A_ϕ with the matrix algebra $End_k(L)$.

For general ϕ, one can show that after extending scalars $L \otimes_k A_\phi$ becomes a matrix algebra. Therefore A_ϕ determines an element of $Br(k)$.

Theorem 4.5.5. *The map $\phi \mapsto A_\phi$ yields an isomorphism $H^2(G, K^*) \cong Br(k)$*

Chapter 5

Cohomology of Curves

Given a smooth projective curve X of genus g, the usual (singular) cohomology has the following shape

$$H^i(X, A) = \begin{cases} A & \text{if } i = 0 \\ A^{2g} & \text{if } i = 1 \\ A & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

So in particular, this holds when $A = \mathbb{Z}/n\mathbb{Z} \cong \boldsymbol{\mu}_n$. Our goal is to show that étale cohomology gives essentially the same answer.

5.1 Picard group

Fix a smooth projective curve X over an algebraically closed field k A divisor is a finite sum $\sum n_i p_i$ where $n_i \in \mathbb{Z}$ and $p_i \in X(k)$. They form an abelian group $Div(X)$. The degree $deg(\sum n_i p_i = \sum n_i$ defines a surjective homomorphism $deg : Div(X) \to \mathbb{Z}$. Given a rational function $f \in k(X)^*$, define the principal divisor

$$div(f) = \sum ord_p(f)p$$

where ord_p is the discrete valuation measuring the order of the zero of f at p. One has $div(fg) = div(f) + div(g)$, so the the set of principal divisors form a subgroup $Princ(X)$. The Picard group or the divisor class group

$$Pic(X) = Div(X)/Princ(X)$$

The degree of prinicipal divisor is zero, therefore deg factors through $Pic(X)$. Let $Pic^0(X)$ be the kernel.

Theorem 5.1.1. *$Pic(X)$ is isomorphic to the Zariski cohomology group $H^1(X_{zar}, \mathcal{O}_X^*)$.*

Proof. Let $\mathbb{Z}_p$ denote the skyscraper sheaf

$$\mathbb{Z}_p(U) = \begin{cases} \mathbb{Z} & \text{if } p \in U \\ 0 & \text{otherwise} \end{cases}$$

If we write $k(X)^*$ for the constant sheaf $k(X)^*(U) = k(X)^*$, then we have an exact sequence

$$0 \to \mathcal{O}_X^* \to k(X)^* \xrightarrow{div} \bigoplus_p \mathbb{Z}_p \to 0 \tag{5.1}$$

on X_{zar}, where div is defined as above but with the sum over points restricted to the given U. Then we get a long exact sequence

$$H^0(X, k(X)^*) \to H^0(X, \bigoplus \mathbb{Z}_p) \to H^1(X, \mathcal{O}_X^*) \to H^1(X, k(X)^*)$$

The cohomology group on the right vanishes because $k(X)^*$ is flasque [H]. Therefore this sequence becomes

$$k(X)^* \to Div(X) \to H^1(X_{zar}, \mathcal{O}_X^*) \to 0$$

and the theorem follows. □

There is another natural interpretation of this group. An element of $H^1(X_{zar}, \mathcal{O}^*)$ is given by a cocycle g_{ij}. This gives a so-called line bundle, which is a fibre bundle with fibre k. For our purposes, it is more convenient to the sheaf theoretic analogue of this. A line bundle L is for us a rank one locally free sheaf on X, i.e. it is a sheaf of $\mathcal{O}_X$-modules such that $L|_{U_i} \cong \mathcal{O}_{U_i}$ for some Zariski open covering. From the cocycle we can construct L by

$$L(U) = \{f_i \in \prod \mathcal{O}_X(U \cap U_i) \mid f_i = g_{ij} f_j\}$$

We can also construct a line bundle directly from a divisor by

$$\mathcal{O}(\sum n_i p_i) = \prod m_{p_i}^{-n_i}$$

where m_p is the maximal ideal sheaf at p. So to summarize the discussion

Theorem 5.1.2. *There is an isomorphism*

$$Pic(X) \cong H^1(X_{zar}, \mathcal{O}_X^*) \cong \{\textit{line bundles}\}/\cong$$

The group operation on the right is tensor product.

The Picard group has a well known structure of an abelian variety, which is a projective variety which has a structure of an algebraic group [Mu1]. Over $\mathbb{C}$ an abelian variety is a complex torus.

Theorem 5.1.3. *The group $Pic^0(X)$ is the set of rational points of a g dimensional abelian variety called the Jacobian, where $g = \dim H^1(X_{zar}, \mathcal{O}_X)$ is the genus.*

From the structure of abelian varieties

Corollary 5.1.4. *$Pic^0(X)$ is a divisible abelian group. If n is prime to the characteristic of k, then the n-torsion of $Pic^0(X)$, $Pic^0(X)_n$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$.*

5.2 Etale cohomology of $\mathbb{G}_m$

Let X be a smooth projective curve over an algebraically closed field k.

Theorem 5.2.1 (Tsen)**.** *If $K = k(X)$, then $H^i(Gal(K^{sep}/K), K^*) = 0$ for $i > 0$.*

Proof. The vanishing for $i = 1$ is Hilbert's theorem 90. For the more usual form of Tsen's theorem and why the present result follows see [S, chap II, §3]. □

Let $j : \operatorname{Spec} k(X) \to X$ denote the inclusion of the generic point. We can define the so called direct image sheaf $j_*\mathbb{G}_m$ on X_{et} by

$$j_*\mathbb{G}_m(U) = k(U)^*$$

Corollary 5.2.2. *$H^i(X, j_*\mathbb{G}_m) = 0$ for $i > 0$.*

Proof. There are two things going on here. An isomorphism $H^i(X, j_*\mathbb{G}_m) \cong H^i(Gal(K^{sep}/K), K^*)$ and the theorem. The first statement follows from an analysis of the Leray spectral sequence. See [SGA4.5, p 33] for details. □

Note that $j_*\mathbb{G}_m$ restricts to the sheaf $k(X)^*$ on X_{zar}. We have an analogue of (5.1)

$$0 \to \mathbb{G}_m \to j_*\mathbb{G}_m \to \bigoplus_p \mathbb{Z}_p \to 0 \tag{5.2}$$

where $\mathbb{Z}_p$ is the étale version of a skyscraper sheaf (ex 3.2.4).

Lemma 5.2.3. *$H^i(X_{et}, \mathbb{Z}_p) = 0$ for $i > 0$.*

Proof. One checks that $H^i(X_{et}, \mathbb{Z}_p) = H^i(\operatorname{Spec} k(p), \mathbb{Z}_p) = 0$ for $i > 0$. See [SGA4.5, p 34]. □

Putting these results together yields

Theorem 5.2.4.

$$H^i(X_{et}, \mathbb{G}_m) = \begin{cases} k^* & \text{if } i = 0 \\ Pic(X) & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$$

Proof. From (5.2), and corollary 5.2.2, we deduce

$$0 \to H^0(X_{et}, \mathbb{G}_m) \to H^0(X_{et}, j_*\mathbb{G}_m) \to H^0(X_{et}, \bigoplus \mathbb{Z}_p) \to H^1(X_{et}, \mathbb{G}_m) \to 0$$

This can be identified with

$$0 \to H^0(X_{et}, \mathbb{G}_m) \to k(X)^* \to \bigoplus_p \mathbb{Z} \to H^1(X_{et}, \mathbb{G}_m) \to 0$$

Therefore $H^0(X_{et}, \mathbb{G}_m)$ and $H^1(X_{et}, \mathbb{G}_m)$ can be identified with kernel and cokernel of $k(X)^* \to Div(X)$ respectively. The kernel is the set of rational functions on X without poles or zeros, which is known to be exactly k^*. The cokernel of $k(X)^* \to Div(X)$ is $Pic(X)$ as we have seen. The higher cohomologies of $\mathbb{G}_m$ are zero by corollaries 5.2.2 and 5.2.3. □

We can now deduce analogues of the calculations of singular cohomology stated earlier.

Corollary 5.2.5. *Suppose that n is prime to char k. Then*

$$H^i(X_{et}, \boldsymbol{\mu}_n) = \begin{cases} \boldsymbol{\mu}_n(k) & \text{if } i = 0 \\ Pic^0(X)_n & \text{if } i = 1 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = 2 \\ 0 & \text{if } i > 2 \end{cases}$$

Proof. From the Kummer sequence, we obtain

$$0 \to H^0(X_{et}, \boldsymbol{\mu}_n) \to k^* \xrightarrow{n} k^* \to H^1(X_{et}, \boldsymbol{\mu}_n) \to Pic(X) \xrightarrow{n} Pic(X) \to H^2(X_{et}, \boldsymbol{\mu}_n) \to 0$$

The corollary follows by combining this with the exact sequence

$$0 \to Pic^0(X) \to Pic(X) \to \mathbb{Z} \to 0$$

and corollary 5.1.4. □

We deduce from this that there are noncanonical isomorphisms

$$H^i(X_{et}, \mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{if } i = 0 \\ (\mathbb{Z}/n\mathbb{Z})^{2g} & \text{if } i = 1 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = 2 \\ 0 & \text{if } i > 2 \end{cases} \tag{5.3}$$

When X is defined over a subfield $k_0 \subset k$, the Galois group $Gal(k/k_0)$ will act on étale cohomology. The isomorphisms given in the above corollary are better for this purpose, since they respect the Galois action. Discarding this for now, notice the symmetry $rank(H^0(X_{et}, \mathbb{Z}/n\mathbb{Z})) = rank(H^2(X_{et}, \mathbb{Z}/n\mathbb{Z}))$. This is an instance of Poincaré duality. In order to put this in a more canonical footing, we need to explain products. Given sheaves F, G, their tensor product $F \otimes G$ is the sheaf associated to the presheaf $U \mapsto F(U) \otimes G(U)$. There are products

$$\cup : H^n(X_{et}, F) \otimes H^m(X_{et}, G) \to H^{n+m}(X_{et}, F \otimes G)$$

called cup products, which can be described explicitly using Cech cocycles. Given cocycles $f_{i_0,\ldots i_n}$ and $g_{i_0,\ldots i_m}$, their product is given by

$$(f \cup g)_{i_0,\ldots i_{n+m}} = f_{i_0,\ldots i_n} \otimes g_{i_n,\ldots i_{n+m}}$$

Theorem 5.2.6. *Let*

$$tr : H^2(X_{et}, \boldsymbol{\mu}_n) \cong \mathbb{Z}/n\mathbb{Z}$$

denote the canonical isomorphism given above. The pairing

$$tr \circ \cup : H^i(X, \boldsymbol{\mu}_n) \otimes_{\mathbb{Z}/n\mathbb{Z}} H^{2-i}(X, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$$

is perfect i.e. it induces an isomorphism

$$H^i(X, \boldsymbol{\mu}_n) \cong Hom(H^{2-i}(X, \mathbb{Z}/n\mathbb{Z}), \mathbb{Z}/n\mathbb{Z})$$

Proof. The only nontrivial case is $i = 1$. Here the proof hinges on an explicit description of the product [SGA4.5, Dualité §3]. Fix a base point x_0 of X then one has the Abel-Jacobi map $\alpha : X \to Pic^0(X)$ given by $x \mapsto x - x_0$. This map induces an isomorphism on the first étale homology by class field theory. Therefore a $\mathbb{Z}/n\mathbb{Z}$-torsor of X is the pullback of a $\mathbb{Z}/n\mathbb{Z}$-torsor of $Pic^0(X)$ along α. Multiplication by n

$$Pic^0(X) \xrightarrow{n} Pic^0(X)$$

determines a $Pic^0(X)_n$-torsor of $Pic^0(X)$. This determines a class

$$\xi \in H^1(Pic^0(X)_{et}, Pic^0(X)_n)$$

Every $\mathbb{Z}/n\mathbb{Z}$-torsor of $Pic^0(X)$ is the image of ξ under a unique homomorphism $Hom(Pic^0(X)_n, \mathbb{Z}/n\mathbb{Z})$. Therefore, we have an identification

$$H^1(X_{et}, \mathbb{Z}/n\mathbb{Z}) \cong Hom(Pic^0(X)_n, \mathbb{Z}/n\mathbb{Z})$$

The product $H^1(X, \boldsymbol{\mu}_n) \otimes H^1(X, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$ coincides with the evaluation map

$$Pic^0(X)_n \otimes Hom(Pic^0(X)_n, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$$

up to sign, which is perfect by definition. $\square$

5.3 Constructible Sheaves

Let X be as in the last last section. It is convenient to extend the finiteness statements for $H^*(X_{et}, \boldsymbol{\mu}_n)$ to more general coefficients. An étale sheaf F of $\mathbb{Z}/n\mathbb{Z}$-modules on the curve X is called constructible if it has finite stalks and there exists a nonempty Zariski open set $U \subseteq X$ such that $F|_U$ is locally constant. (We can understand the restriction simply as restriction of functors to the subcategory $U_{et} \subset X_{et}$.) Of course, $\boldsymbol{\mu}_n$ and $\mathbb{Z}/n\mathbb{Z}$ are both constructible, as are sky scraper sheaves 3.2.4 i_*A where A is finite.

Theorem 5.3.1. *If F is constructible then $H^i(X_{et}, F)$ is finite for all i and zero when $i > 2$.*

Proof. We make a series of reductions to the case we understand, corollary 5.2.5. This style of proof, called *devissage* or untwisting, is classic Grothendieck.

(1) The theorem is trivially true for skyscraper sheaves.

(2) When F is constant, this follows from corollary 5.2.5.

(3) Suppose that $F|_U$ is constant for some nonempty $j : U \to X$. Let $Z = X - U$ and let $i : Z \to X$ denote the inclusion. We have canonical surjection (or more acurately epimorphism)

$$F \to i_* \bigoplus_{x \in Z} F_x$$

We denote the kernel by $j_! F|_U$. Then the theorem holds for $j_! F|_U$ by usual the long exact sequence and (1) and (2).

(4) Suppose that $F|_U$ is locally constant. Let j be as above. Then there exists an étale cover $\pi : U' \to U$ such that $F_{U'}$ is constant. In fact, we can assume that π is Galois. We can choose smooth compactification X' of U' and extend π to a morphisms $X' \to X$. Let $j' : U' \to U$ denote the inclusion. We obtain a long exact sequence

$$\ldots H^i(X, j_! F|_U) \to H^i(X', j'_! F|_{U'}) \to H^i(X, j'_! F''|_U) \ldots$$

where F'' is again constant. Therefore the theorem holds for $j_! F|_U$.

(5) For the general case. Choose U so that $F|_U$ is locally constant. Use the exact sequence

$$0 \to j_! F_U \to F \to i_* \bigoplus_{x \in Z} F_x \to 0$$

and the previous cases to conclude the theorem.

□

One also has a version of Poincaré duality for constructible sheaves. See [FK, M1, M2, SGA4.5].

Chapter 6

Cohomology of surfaces

The goal of this chapter is apply the machinery of étale cohomology to surfaces. Among other things, we give one of Weil's proofs [Wl] of his bound on the number of points of a curve over a finite field. The proof relies on the geometry of the surface $C \times C$.

6.1 Finiteness of Cohomology

Fix a smooth projective surface (i.e. two dimensional variety) X over an algebraically closed field k. The first key result is the finiteness of the cohomology of X. Let n be coprime to $char\, k$.

Theorem 6.1.1. *The groups $H^i(X_{et}, \boldsymbol{\mu}_n)$ are finitely generated $\mathbb{Z}/n\mathbb{Z}$-modules.*

Sketch. For curves the analagous statement was obtained by direct calculation. For surfaces things are more complicated. The basic idea is to reduce things down to curves by fibering them. Here is a broad outline. The first step is to reduce to the case where X admits a surjection onto $\mathbb{P}^1$. Embed X into projective space. Choose a pencil of hyperplanes $\{H_t\}_{t\in\mathbb{P}^1}$ so that H_0 and H_∞ meet in general position. We can form a new smooth surface

$$X' = \{(x,t) \in X \times \mathbb{P}^1 \mid x \in H_t\}$$

which maps onto both X and $\mathbb{P}^1$. The cohomology of X injects into the cohomology of X', so we may replace X by X'.

Now consider cohomology along the fibres. More formally these are higher direct image sheaves $R^i f_* \boldsymbol{\mu}_n$ which are the sheaves associated to the presheaves

$$U \mapsto H^i((U \times_{\mathbb{P}^1} X)_{et}, \boldsymbol{\mu}_n)$$

The cohomology of X can be computed using the Leray spectral sequence

$$H^i(\mathbb{P}^1_{et}, R^j f_* \boldsymbol{\mu}_n) \Rightarrow H^{i+j}(X_{et}, \boldsymbol{\mu}_n)$$

which we won't explain. The important thing for now is that it reduces the theorem to proving finiteness of $H^i(\mathbb{P}^1_{et}, R^j f_* \boldsymbol{\mu}_n)$ for all i, j. Finiteness will follow from theorem 5.3.1, once we prove

Claim: $R^j f_* \boldsymbol{\mu}_n$ are constructible.

The claim can be proved by a relative version of the arguments used to prove corollary 5.2.5. Let $U \subset \mathbb{P}^1$ be an open set over which f has no singular fibres. Then

$$R^i f_* \boldsymbol{\mu}_n|_U = \begin{cases} \boldsymbol{\mu}_n & \text{if } i = 0 \\ n\text{-torsion of the Picard scheme} & \text{if } i = 1 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

□

There is also a version of Poincaré duality in this setting. This can be proved by refining the previous analysis.

Theorem 6.1.2. *We have an isomorphism*

$$tr : H^4(X, \boldsymbol{\mu}_n^{\otimes 2}) \cong \mathbb{Z}/n\mathbb{Z}$$

where $\boldsymbol{\mu}_n^{\otimes 2} = \boldsymbol{\mu}_n \otimes \boldsymbol{\mu}_n$ *The pairing*

$$tr \circ \cup : H^i(X, \boldsymbol{\mu}_n) \otimes_{\mathbb{Z}/n\mathbb{Z}} H^{4-i}(X, \boldsymbol{\mu}_n) \to \mathbb{Z}/n\mathbb{Z}$$

is perfect.

Proof. [M1, chap VI, §2, §11]. □

To dispel some of the mystery here, we can deduce these theorems from a more elementary result in the case we are mainly interested in. Let C_i be a smooth projective curves over k. Set $X = C_1 \times C_2$. Then

Theorem 6.1.3 (Künneth formula)**.** *The cohomology of X, as a graded group, is the tensor product of the cohomology of C with itself. More explicitly,*

$$H^i(X_{et}, \boldsymbol{\mu}_n^{\otimes(c+d)}) = \bigoplus_{a+b=i} H^a(C_{1,et}, \boldsymbol{\mu}_n^{\otimes c}) \otimes H^b(C_{2,et}, \boldsymbol{\mu}_n^{\otimes d})$$

A tensor $\alpha \otimes \beta$ on the right corresponds to the product $p_1^* \alpha \cup p_2^* \beta$ on the left, where p_i are the projections.

6.2 Divisors on a surface

Let X be as above. A divisor is a finite integral linear combination $\sum n_i C_i$ of irreducible curves on X. As before, these form a group $Div(X)$. The Picard group

$$Pic(X) = Div(X)/Princ(X)$$

where $Princ(X)$ is the group of divisors of the form $div(f) = \sum ord_C(f) C$, $f \in k(X)^*$. Arguments similar to those used earlier show that

Theorem 6.2.1. *$Pic(X) \cong H^1(X_{zar}, \mathcal{O}_X^*) \cong H^1(X_{et}, \mathbb{G}_m)$.*

Suppose that n is coprime to $char\, k$. The Kummer sequence yields a map called the $\mod n$ first Chern class

$$c_1 : Pic(X) \otimes \mathbb{Z}/n\mathbb{Z} \to H^2(X_{et}, \boldsymbol{\mu}_n)$$

For a divisor D, we denote the image by $[D]$. Unlike the case of curves, this map is generally not surjective. Another new feature in the surface case, is the intersection pairing. Given two distinct irreducible curves $C, D \subset X$, defined locally by f, g at $p \in X$, their intersection number at p is

$$(C \cdot D)_p = \dim \mathcal{O}_p/(f, g)$$

Their intersection number is

$$C \cdot D = \sum_p (C \cdot D)_p$$

Theorem 6.2.2. *There exists a symmetric bilinear form on $Pic(X)$ extending $\cdot$ above.*

Proof. See [H, chap V,§1]. ☐

The above product induces a nondegenerate $\mathbb{Z}/n\mathbb{Z}$-valued bilinear form $\langle , \rangle$ on $H^2(X, \boldsymbol{\mu}_n)$.

Theorem 6.2.3. *This pairing is compatible with the intersection pairing on divisors i.e. $\langle [C] \cdot [D] \rangle = (C \cdot D) \mod n$.*

Going mod n involves loosing information. The usual way to recover this is to take the limit. Fix a prime $\ell \neq char\, k$. The ℓ-adic cohomology

$$H^i(X_{et}, \mathbb{Z}_\ell(a)) = \varprojlim_N H^i(X_{et}, \boldsymbol{\mu}_{\ell^N}^{\otimes a})$$

(N.B. This is **not** étale cohomology with coefficients in $\varprojlim \boldsymbol{\mu}_{\ell^N}^{\otimes a}$. If $a < 0$, we should use the $|a|$th power of the dual of $\boldsymbol{\mu}_{\ell^n}$.) The finiteness statements above can be refined to show finite generation of ℓ-adic cohomology as a $\mathbb{Z}_\ell$-module. The maps

$$Pic(X) \to H^2(X_{et}, \boldsymbol{\mu}_{\ell^N})$$

pass to the limit

$$Pic(X) \to H^2({}_{et}X, \mathbb{Z}_\ell(1))$$

and then to

$$Pic(X) \to H^2(X_{et}, \mathbb{Q}_\ell(1)) := H^2(X_{et}, \mathbb{Z}_\ell(1)) \otimes \mathbb{Q}_l$$

The pairing $\langle , \rangle$ also lifts a $\mathbb{Q}_\ell$-valued pairing to the right. Since $\mathbb{Z}$ injects into $\mathbb{Q}_\ell$, we now get $\langle C \cdot D \rangle = (C \cdot D)$ on the nose. The Picard group can be very large.

It is useful to cut it down to something more manageable. A divisor (class) D is numerically equivalent to 0 if $D \cdot D' = 0$ for all D'. The Neron-Severi group $NS(X)$ is the quotient of $Pic(X)$ by the group of divisors numerically equivalent to 0. (Normally one divides by a slightly smaller subgroup, but this is immaterial for our purposes.)

Theorem 6.2.4. *$NS(X)$ is a finitely generated abelian group.*

Proof. The space $H^2(X_{et}, \mathbb{Q}_\ell(1))$ is finite dimensional. Therefore the image of $Pic(X)$ spans a finite dimensional subspace. Let $C_1, \ldots C_N$ be a basis of this subspace. Then the map $NS(X) \to \mathbb{Q}^N$ given by $D \mapsto (C_i \cdot D)$ is necessarily injective. □

By definition the intersection pairing descends to a pairing on $NS(X)$.

6.3 The Lefschetz trace formula

Let C be a smooth projective curve of genus g defined over an algebraically closed field k. Set $X = C \times C$. Then by the Künneth formula

$$H^i(X_{et}, \mathbb{Q}_\ell(c+d)) = \bigoplus_{a+b=i} H^a(C_{et}, \mathbb{Q}_\ell(c)) \otimes H^b(C_{et}, \mathbb{Q}_\ell(d))$$

Corollary 6.3.1. *The dimensions of $H^i(X)$ for $i = 0, 1, 2, 3, 4$ are $1, 2g, 4g + 2, 2g, 1$ respectively.*

Note that $\dim H^*(C_{et}, \mathbb{Q}_\ell(a))$ is independent of the Tate twist parameter a. So it's convenient to disregard the twist for what follows. In fact, we will suppress coefficients completely. Let $f : C \to C$ be an endomorphism. Let $\Gamma_f \subset C$ denote the graph. This is a divisor. For the identity $\Gamma_{id} = \Delta$ is the diagonal. When the graph Γ_f is transverse to Δ, $\Gamma_f \cdot \Delta$ is precisely the number of fixed points of f. In general, we can view $\Gamma_f \cdot \Delta$ as the number of fixed points of f counted correctly. Note that f induces an endomorphism $f^*|H^i(C]$ of cohomology. For $i = 0$ and $i = 2$, this is just the identity and the degree respectively.

Theorem 6.3.2 (Lefschetz trace formula)**.**

$$\Gamma_f \cdot \Delta = \sum (-1)^i trace[f^*|H^i(C_{et}, \mathbb{Q}_\ell)] = 1 - trace[f^*|H^1(C_{et}, \mathbb{Q}_\ell)] + \deg f$$

Proof. The proof uses Poincaré duality along with formal properties of cohomology. By Poincaré duality we have a nondegenerate pairing $\langle \alpha, \beta \rangle = tr(\alpha \cup \beta)$ on $H^*(C)$. This identifies cohomology $H^*(C)$ with its dual. In particular,

$$H^*(X) = H^*(C) \otimes H^*(C) \cong Hom(H^*(C), H^*(C))$$

Under this identification the class of the diagonal $[\Delta]$ corresponds to the identity. To write this explicitly, choose a basis $\alpha_0 \in H^0(C, \mathbb{Q}_\ell)$, $\alpha_1, \ldots \alpha_{2g} \in H^1(C, \mathbb{Q}_\ell)$

$\alpha_{2g+1} \in H^2(C, \mathbb{Q}_\ell)$ for the total cohomology, and let $\alpha^0, \alpha^1, \dots \alpha^{2g+1}$ be the dual basis. We two natural bases of $H^*(X)$ given by $\{\alpha_i \otimes \alpha^j\}$ and $\{\alpha^i \otimes \alpha_j\}$ and these are dual to each other. Then

$$[\Delta] = \sum \alpha_i \otimes \alpha^i$$

Therefore

$$[\Gamma_f] = (f \times id)^* \Delta = \sum f^*(\alpha_i) \otimes \alpha^i$$

We can expand this further by substituing

$$f^* \alpha_i = \sum f_{ij} \alpha_j$$

Note that $f_{00} = 1$ and $f_{2g+2,2g+2} = q$ by the earlier remarks. above. For the next step we need to switch factors. The cup product is graded commutative which means

$$\alpha \cup \beta = (-1)^{\deg \alpha \deg \beta} \beta \cup \alpha$$

Thus we can write

$$[\Delta] = \alpha^0 \otimes \alpha_0 - \sum_1^{2g} \alpha^i \otimes \alpha_i + \alpha^{2g+1} \otimes \alpha_{2g+1}$$

Now

$$\Gamma_f \cdot \Delta = 1 - \sum f_{ii} + q$$

$\square$

Let C be defined over a finite field $\mathbb{F}_q$ with q elements. We write $\bar{C}$ for the corresponding curve over the algebraic closure $\overline{\mathbb{F}_q}$. Then for each integer $n > 0$, let N_n denote the number of points $\#C(\mathbb{F}_{q^n})$. We can assemble these numbers into a generating function called the zeta function

$$Z(t) = \exp(\sum_1^\infty \frac{N_n t^n}{n}) \tag{6.1}$$

The significance of this somewhat peculiar expression can be explained by changing variables. Then after a bit of work, we get an expression [D]

$$\zeta(s) = Z(q^{-s}) = \prod_{x \in C} \frac{1}{1 - 1/N(x)^s}, \quad N(x) = \#k(x)$$

which looks very much like an Euler product for a classical zeta function. Let $F : \bar{C} \to \bar{C}$ denote the qth power Frobenius morphism, which naively is given by raising the coordinates to q. We can identify N_n with the number of fixed points of F^n, and this can be computed by Lefschetz:

$$N_n = 1 - trace[F^{n*}|H^1(\bar{C})] + q^n = 1 + q^n - \sum \lambda_i^n$$

where λ_i are the eigenvalues of $[F^*|H^1(\bar{C})]$. Substituting into (6.1) and simplifying yields

Corollary 6.3.3.

$$Z(t) = \frac{\prod(1-\lambda_i t)}{(1-t)(1-qt)} = \frac{\det(1-Ft|H^1(\bar{C}))}{(1-t)(1-qt)}$$

6.4 Riemann hypothesis for curves

Let X be a smooth projective surface over an algebraically closed field. A divisor is called ample if a positive multiple of it has the same class in $Pic(X)$ as $X \cap H$ for some hyperplane H in a projective space containing X. One can see that an ample divisor has positive self intersection.

Theorem 6.4.1 (Hodge Index theorem). *If H is the class of an ample divisor in $NS(X)$, then $D \cdot H = 0$ implies $D^2 < 0$. In other words, the orthogonal complement of H in $NS(X) \otimes \mathbb{R}$ is negative definite.*

Proof. [H, V, 1.9]. □

Corollary 6.4.2 (Castelnuovo-Severi inequality). *If D is a divisor on a product of curves $X = C_1 \times C_2$,*

$$D^2 \leq 2ab$$

where $a = D \cdot (C_1 \times p_2), b = D \cdot (p_1 \times C_2)$, $p_i \in C_i(k)$.

Proof. Apply the following lemma to the divisors $(C_1 \times p_2), (p_1 \times C_2)$ and D. □

Lemma 6.4.3. *Let $\langle , \rangle$ denote a symmetric bilinear form on $\mathbb{R}^n$ of type $(1, n-1)$. In other words, suppose that it is represented by a matrix with one positive eigenvalue and the rest negative. If $\langle x, x\rangle = \langle y, y\rangle = 0$ and $\langle x, y\rangle = 1$ for some $x, y \in \mathbb{R}^n$. Then for any z, $\langle z, z\rangle \leq 2\langle z, x\rangle\langle z, y\rangle$.*

Proof. As x is in the closure of the cone of vectors with positive square, any vector u orthogonal to x must satisfy $\langle u, u\rangle \leq 0$. Set $u = \langle x, z\rangle y - \langle x, y\rangle z$. Then $\langle u, x\rangle = 0$. Expanding $\langle u, u\rangle \leq 0$ this yields the proof. □

Theorem 6.4.4 (Hasse-Weil). *If C is a smooth projective curve of genus g defined over $\mathbb{F}_q$. Then*

$$|N - 1 - q| \leq 2g\sqrt{q}$$

where $N = \#C(\mathbb{F}_q)$

Proof. Let $\bar{C}$ be the extension of C to the closure. Let Δ and Γ denote the diagonal and the graph of the Frobenius on $X = \bar{C} \times \bar{C}$. Then $N = \Delta \cdot \Gamma$. Note that $\Delta^2 = 2 - 2g$ by the Lefschetz trace formula. Since $\Gamma = (F \times id)^*\Delta$ and $F \times id$ has degree q, we can deduce $\Gamma^2 = q(2 - 2q)$. Let p be a rational point of $\bar{C}$. One checks also that

$$\Delta \cdot (\bar{C} \times p) = \Delta \cdot (p \times \bar{C}) = \Gamma \cdot (\bar{C} \times p) = 1$$

and

$$\Gamma \cdot (p \times \bar{C}) = q$$

Let $D = r\Gamma + s\Delta$ with variable real coefficients. The Castelnuovo-Severi inequality implies that $2(r+s)(rq+s) - D^2 \geq 0$. This simplifies to

$$(2gq)r^2 + 2(1+q-N)rs + 2gs^2 \geq 0$$

for all r, s. This implies that the discriminant of the quadratic form

$$(q+1-N)^2 - 4qg^2 \leq 0$$

This proves the theorem. $\square$

In view of the formula

$$N = 1 + q + \sum \lambda_i$$

obtained earlier, this theorem can be reinterpreted as a bound on the eigenvalues λ_i. From this one can deduce an analogue of the Riemann hypothesis for the zeta function of C defined earlier. This says that the zeros of $\zeta(s)$ lie on the line $s = 1/2$.

The original theorem of Hasse concerns genus one. This was generalized to abitrary genus by Weil [Wl], who gave two proofs including the the one above. This argument really only requires intersection theory for divisors. For the analogous statements in higher dimensions, however, étale cohomology is indispensable.

Chapter 7

Comparison with classical cohomology

In this chapter, we assume that the underlying field k is the field of complex numbers $\mathbb{C}$. The main result, due to Artin, is that étale cohomology of a variety X, with finite coefficients, coincides with singular cohomology or equivalently with cohomology of X equipped with classical Hausdorff topology X_{an}.

7.1 The comparison theorem for curves

In the ensuing discussion it will be convenient to suppress Tate twists, and identify $\mathbb{Z}/n\mathbb{Z} \cong \boldsymbol{\mu}_n$ (noncanonically of course). Given a complex variety X, we let X_{an} denote the usual topology, which on affine opens $U \subset X$ coincides with the Euclidean topology induced from embedding $U \subset \mathbb{C}^N$. We claim that there is a natural map

$$H^i(X_{et}, \mathbb{Z}/n\mathbb{Z}) \to H^i(X_{an}, \mathbb{Z}/n\mathbb{Z}) \tag{7.1}$$

which we refer to as the comparison map. To construct it, we construct a new Grothendieck topology on X, denoted by X_{cl}, where the open sets are local homeomorphisms $Y \to X$. Covers are surjective families. We can see that X_{cl} refines both X_{et} and X_{an}. Consequently, we have maps

$$H^i(X_{et}, \mathbb{Z}/n\mathbb{Z}) \to H^i(X_{cl}, \mathbb{Z}/n\mathbb{Z}) \leftarrow H^i(X_{an}, \mathbb{Z}/n\mathbb{Z})$$

Lemma 7.1.1. *The second arrow is an isomorphism.*

Theorem 7.1.2. *The comparison map (7.1) is an isomorphism, when X is a smooth curve.*

Proof. We will be content to prove that there is an abstract isomorphism between the two cohomology groups in each degree. For $i = 0$, we see that both groups are isomorphic to $\mathbb{Z}/n\mathbb{Z}$ and the map is the identity. For $i = 1$, this is a consequence of theorem 2.2.3 and the subsequent discussion. We have

$$H^1(X_{an}, \mathbb{Z}/n\mathbb{Z}) \cong Hom(\pi_1(X), \mathbb{Z}/n\mathbb{Z})$$

$$H^1(X_{et}, \mathbb{Z}/n\mathbb{Z}) \cong Hom(\widehat{\pi_1(X)}, \mathbb{Z}/n\mathbb{Z})$$

For higher i, the comparison was given in (5.3) when X was smooth and projective. So now assume that X is smooth but not projective. It is known that $H^i(X_{an}, \mathbb{Z}/n\mathbb{Z}) = 0$ for $i > 1$. For etale cohomology, we use the Kummer sequence as in the proof of corollary 5.2.5 to see that

$$Pic(X) \xrightarrow{n} Pic(X) \to H^2(X_{et}, \mathbb{Z}/n\mathbb{Z}) \to 0$$

and

$$H^i(X_{et}, \mathbb{Z}/n\mathbb{Z}) = 0, \text{ for } i > 2$$

In this case, $Pic(X)$ is divisible, so H^2 also vanishes. □

We wish to extend this to singular projective curves. We first define compactly supported etale cohomology of smooth curve U, which is an analogue of the compactly supported cohomology of a manifold defined using differential forms with compact support. Let X denote the unique smooth compactification, and let $j : U \to X$ denote the inclusion, then define

$$H^i_c(U_{et}, \mathbb{Z}/n\mathbb{Z}) = H^i(X_{et}, j_!\mathbb{Z}/n\mathbb{Z})$$

where $j_!$ is defined in section 5.3. $H^i_c(U_{an}, \mathbb{Z}/n\mathbb{Z})$ can be defined in the same way. We note X can be replaced by any compactification:

Lemma 7.1.3. *For any (possibly singular) compactification $k : U \to Y$,*

$$H^i_c(U_{et}, \mathbb{Z}/n\mathbb{Z}) \cong H^i(Y_{et}, k_!\mathbb{Z}/n\mathbb{Z})$$

Poincaré duality (theorem 5.2.6) extends to a smooth curve U in the form

$$H^{2-i}(U_{et}, \mathbb{Z}/n\mathbb{Z}) \cong H^i(U_{et}, \mathbb{Z}/n\mathbb{Z})$$

and similarly for $H^i_c(U_{an}, \mathbb{Z}/n\mathbb{Z})$ Thus as a corollary of the previous theorem, we obtain

Corollary 7.1.4. *If U is a smooth curve, there is a natural isomorphism*

$$H^i_c(U_{et}, \mathbb{Z}/n\mathbb{Z}) \cong H^i_c(U_{an}, \mathbb{Z}/n\mathbb{Z})$$

Theorem 7.1.5. *The comparison map (7.1) is an isomorphism, when X is a projective possibly reducible curve.*

Proof. Let $j : U \to X$ be the complement of the set of singular points Z. Then there is a commutative diagram with exact rows

$$\begin{tikzcd} \dots \arrow[r] & H^i_c(U_{et}) \arrow[r] \arrow[d, "c_U"] & H^i(X_{et}) \arrow[r] \arrow[d, "c_X"] & H^i(Z_{et}) \arrow[r] \arrow[d, "c_Z"] & \dots \\ \dots \arrow[r] & H^i_c(U_{an}) \arrow[r] & H^i(X_{an}) \arrow[r] & H^i(Z_{an}) \arrow[r] & \dots \end{tikzcd}$$

The map labelled c_u is an isomorphism by the previous corollary, and c_z is trivially an isomorphism. Therefore c_X is an isomorphism by the 5-lemma. ☐

7.2 The comparison theorem for surfaces

Theorem 7.2.1. *The comparison map* (7.1) *is an isomorphism when X is a smooth projective surface.*

We will outline the proof. The full details can be found in [SGA4, exp XI]. Here is the first reduction:

Proposition 7.2.2. *Suppose that the following condition holds:* if $i > 0$ and $\eta \in H^i(X_{an})$, then there exists an etale cover $U_j \to X$ such that $\eta|_{U_j} \in H^i(U_{j,an})$ is 0. *Then the theorem holds.*

This follows by analyzing the Leray spectral sequence for the map of sites $X_{cl} \to X_{et}$. The condition above amounts to the vanishing of the direct image associated to this map. Now fix $\eta \in H^i(X_{an})$ as above. We need to construct a cover as above. This is based on the following geometric construction.

Lemma 7.2.3 (Artin)**.** *Given $x \in X$. There exists an open set $x \in V \subset X$ obtained by removing a curve, such that there is a smooth projective map $f : V \to C$ to a smooth affine curve and a divisor $D \subset V$ étale over C with $x \notin D$.*

Let $U = V - D$. This is a Zarski open neighbourhood of x usually called an Artin neighbourhood. By collecting these, we get a Zariski open cover of X. If for each U, we can find an étale $U' \to U$ for which the restriction of η vanishes, then we are done. The map $f : U \to C$ is a fibration, with affine curves as base and fibre. The standard exact sequence for homotopy groups [Sp] implies that the fundamental group is an extension of free groups

$$1 \to \pi_1(f^{-1}(x)) \to \pi_1(U) \to \pi_1(C) \to 1$$

and that the higher homotopy groups vanish. Thus U is a so called $K(\pi, 1)$ space. In particular, standard results in algebraic topology show that

$$H^i(U_{an}, \mathbb{Z}/n\mathbb{Z}) \cong H^i(\pi_1(U), \mathbb{Z}/n\mathbb{Z})$$

where the right side is group cohomology [B]. Given a group Γ, $H^i(\Gamma, \mathbb{Z}/n\mathbb{Z})$ is defined as in section 4.5 but without requiring continuity for cocycles. We can also define the cohomology of the profinite group $H^i(\hat{\Gamma}, \mathbb{Z}/n\mathbb{Z})$ using continuous cocycles. Alternatively

$$H^i(\hat{\Gamma}, \mathbb{Z}/n\mathbb{Z}) = \varinjlim H^i(\Gamma/N, \mathbb{Z}/n\mathbb{Z})$$

as N runs over normal subgroups of finite index. There is a natural map

$$H^i(\hat{\Gamma}, \mathbb{Z}/n\mathbb{Z}) \to H^i(\Gamma, \mathbb{Z}/n\mathbb{Z})$$

Let us call Γ Serre (who calls this good) if this is an isomorphism for all i and n. He shows that extensions of free groups by free groups are Serre [S, pp 13-14]. Thus

$$H^i(\widehat{\pi_1(U)}, \mathbb{Z}/n\mathbb{Z}) \cong H^i(\pi_1(U), \mathbb{Z}/n\mathbb{Z})$$

Now take η, its restriction to U defines an element on the left. But this class must die after restriction $H^i(\pi_1(U)/N)$ for some normal subgroup N of finite index. N determines an étale cover $U' \to U$ such that $\eta|_{U'} = 0$, so the theorem is proved.

7.3 Base change

We can extend the results to other fields using the following special case of the smooth base change theorem ([M1, M2])

Theorem 7.3.1. *Suppose that K/k is an extension of separably closed fields, let n be prime to $char k$. If X is a scheme defined over k, then there is a canonical isomorphism*

$$H^i(X_{et}, \mathbb{Z}/n\mathbb{Z}) \cong H^i((X \times Spec K)_{et}, \mathbb{Z}/n\mathbb{Z})$$

Corollary 7.3.2. *If X is a curve or surface defined over an algebraically closed subfield $K \subset \mathbb{C}$, then the ranks of the etale cohomology of X are precisely the Betti numbers of $X \times Spec\mathbb{C}$.*

Chapter 8

Main Theorems

Here we state some of the main results. Proofs and further elaborations can be found in the references, especially [FK, M1, M2].

For a manifold, compactly supported de Rham cohomology can be defined using differential forms with compact support. It is convenient to have an algebraic version. Fix a field k, and a prime $\ell \neq char\, k$. Let X be a variety k. Then by a theorem of Nagata, there exists a complete variety $\bar{X}$ which contains X as a dense open set. When X is quasiprojective, we can simply take $\bar{X}$ to be the closure of X in a projective space containing it. Let $Z = \bar{X} - X$. and let $j : X \to \bar{X}$ $i : Z \to \bar{X}$ denote the inclusions. We can define $j_! \boldsymbol{\mu}_{\ell^n}$ to be the kernel of the natural restriction map $\boldsymbol{\mu}_{\ell^n} \to i_* \boldsymbol{\mu}_{\ell^n}$.

(1) The space

$$H^i_c(X_{et}, \mathbb{Q}_l(a)) = \varprojlim H^i(\bar{X}_{et}, j_! \boldsymbol{\mu}_{\ell^n}^{\otimes a}) \otimes \mathbb{Q}_\ell$$

depends only on X and not on $\bar{X}$. This is called compactly supported ℓ-adic cohomology. Clearly it is the same as ordinary cohomology if X is complete. As vectors spaces, these are noncanonically the same for different a. However, if X is defined over a subfield of k, the Galois group will act on these spaces. The Galois module structures depend on a.

(2) $H^i_c(X_{et}, \mathbb{Q}_\ell(a))$ is finite dimensional for all i, and it vanishes when $i > 2 \dim X$.

(3) When X is nonsingular of dimension n, Poincaré duality

$$H^i_c(X_{et}, \mathbb{Q}_\ell(a)) \cong H^{2n-i}(X_{et}, \mathbb{Q}_\ell(a-n))^*$$

holds.

(4) The Künneth formula holds. The statement is same as in theorem 6.1.3 but with H_c.

(5) The Lefschetz trace formula holds for H^*_c when X is smooth.

(5) If $k = \mathbb{C}$, then $H^i_c(X_{et}, \mathbb{Q}_\ell(a))$ is isomorphic to compactly supported singular cohomology with coefficents in $\mathbb{Q}_\ell$. A similar statements holds if *char* $k = p > 0$, and X smooth and lifts to characteristic 0.

When X is smooth and complete the Lefschetz formula can be deduced directly from the previous properties by the same method we used earlier. A consequence of these properties are the first few of Weil's conjectures.

Theorem 8.0.3 (Grothendieck). *Suppose that X is a smooth d-dimensional variety over $\overline{\mathbb{F}_q}$ defined over $\mathbb{F}_q$.*

1. *The zeta function*

$$Z(t) = \exp(\sum_1^\infty \frac{N_n t^n}{n}) = \prod [1 - Ft|H^i_c(X_{et}, \mathbb{Q}_\ell)]^{(-1)^i}$$

2. *If X is complete, $Z(t)$ satisfies the functional equation*

$$Z(1/q^d t) = \pm q^{d\chi/2} t^\chi Z(t)$$

where $\chi = \sum (-1)^i \dim H^i(X_{et}, \mathbb{Q}_\ell)$.

Corollary 8.0.4 (Dwork). *$Z(t)$ is a rational function.*

The final Weil conjecture, which is the analogue of the Riemann hypothesis, was proved by Deligne [D].

Theorem 8.0.5 (Deligne). *If X is a smooth complete variety over $\overline{\mathbb{F}_q}$ defined over $\mathbb{F}_q$. The eigenvalues of the qth power Frobenius on $H^i(X_{et}, \mathbb{Q}_\ell)$ are algebraic integers with absolute value $q^{i/2}$.*

A concrete consequence is a refinement of the Hasse-Weil bound.

Corollary 8.0.6. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth degree d hypersurface defined over $\mathbb{F}_q$. Then*

$$|\#X(\mathbb{F}_q) - (1 + q + \ldots + q^n)| \leq b q^{n/2}$$

where b is the nth Betti number (minus 1 if n is even) of a degree d hypersurface in $\mathbb{P}^{n+1}(\mathbb{C})$.

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