TRIANGULATED CATEGORIES AND t-STRUCTURES

DONU ARAPURA

Perverse sheaves are objects in a derived category. So it will be necessary to do a certain amount of category theory just to understand the precise definition. I will review the standard construction of the derived category $D^b(\mathcal{A})$ of an abelian category $\mathcal{A}$ in the first part. Then I will explain how to reverse the process. This stuff seems to give a counterexample to the popular claim that category theory has no content.

1. Triangulated categories

Recall that an abelian category $\mathcal{A}$ is an additive category such that morphisms have kernels and cokernels satisfying the expected properties. A very succinct way of expressing this is that for any morphism $f : A \to B$, $coker(\ker(f) \to B)$ (called the coimage) is isomorphic to the image $\ker(B \to coker(f))$ under the canonical map[1]. A typical example is the category of (sheaves of) modules. We can form a new abelian category $C^+(\mathcal{A})$ which is the category of bounded below complexes in $\mathcal{A}$. We also have a subcategory of cohomologically bounded complexes $C^b(\mathcal{A})$. Let $K^b(\mathcal{A})$ ($K^+(\mathcal{A})$) be the associated homotopy category: the objects are the same, but the morphisms are homotopy classes of chain maps. This is no longer abelian because kernels etc. are no longer well defined. Fortunately, there is a partial substitute for short exact sequences. Given a morphism $f : A \to B$ of complexes, the mapping cone

$$cone(f)^n = A^{n+1} \oplus B^n,\ d(a,b) = (-da, db + f(a))$$

fits into a diagram

$$A \to B \to cone(f) \to A[1]$$

or more suggestively

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl] \\ & cone(f) \arrow[ul, rightsquigarrow] & \end{tikzcd}$$

called distinguished triangle. $cone(f)$ plays a role which is combination of kernel and cokernel. However, unlike these earlier constructions, it is homotopy invariant: if f, g are homotopic then $cone(f) \cong cone(g)$. The set of (diagrams isomorphic to) distinguished triangles enjoy the following properties T1-T4, whose proofs range from obvious to challenging. I'll state them a bit imprecisely.

- T1 Every morphism embeds into a distinguished triangle. For the identity, the third vertex is 0.
- T2 The set of distinguished triangles is stable under rotation and translation.

Date: September 9, 2010.

[1]This definition occurs in Grothendieck, Sur quelques points...

T3 Any pair of compatible maps between two vertices of two distinguished

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & {} \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & {} \end{tikzcd}$$

can be extended to a map of distinguished triangles in the obvious sense.

This list is already sufficient for many arguments. But there is one more somewhat technical property called the octahedral axiom because of the way it's sometimes depicted. The previous axioms implies that the third vertex of a triangle extending $f : X \to Y$ is determined up to noncanonical[2] isomorphism. It will be convenient to denote this by Y/X below.

T4 Given distinguished triangles

$$A \to B \to B/A \to$$

$$B \to C \to C/B \to$$

arranged in the upper cap of an octahedral diagram

$$\begin{tikzcd} C/B \arrow[dr, rightsquigarrow] \arrow[dd, rightsquigarrow] & & C \arrow[ll] \\ & B \arrow[ur] \arrow[dl] & \\ B/A \arrow[rr, rightsquigarrow] & & A \arrow[ul] \arrow[uu] \end{tikzcd}$$

(the nondistinguished triangles commute), we can complete this to an octahedral diagram with lower cap

$$\begin{tikzcd} C/B \arrow[dd, rightsquigarrow] & & C \arrow[ll] \arrow[dl] \\ & C/A \arrow[ul] \arrow[dr, rightsquigarrow] & \\ B/A \arrow[ur] \arrow[rr, rightsquigarrow] & & A \arrow[uu] \end{tikzcd}$$

with

$$A \to C \to C/A \to$$

$$B/A \to C/A \to C/B \to$$

distinguished. The last triangle, whose existence is the real point, can be expressed more suggestively as

$$(C/A)/(B/A) \cong C/B$$

[2] The noncanonicity is the source of some headaches and occasional errors

This can be abstracted as follows. A *triangulated category* is an additive category equipped with a endofunctor $A \mapsto A[1]$ called translation, and a set of diagrams, called distinguished triangles, satisfying T1-T4. Aside from homotopy categories, there is one more important class of examples of triangulated categories (for us – there are plenty of others). A map $f : A \to B$ of complexes is called a *quasi-isomorphism* if it induces an isomorphism on all the cohomology "groups" $H^i(A) \cong H^i(B)$.

Theorem 1.1 (Verdier)**.** *Given an abelian category $\mathcal{A}$, there exists a triangulated category $D^b(\mathcal{A})$ (resp. $D^+(\mathcal{A})$) and functor $K^b(\mathcal{A}) \to D^b(A)$ ($K^+(\mathcal{A}) \to D^+(A)$) which takes distinguished triangles to distinguished triangles, and quasi-isomorphisms to isomorphisms. It is the universal such category.*

In outline, the objects of $D^+(\mathcal{A})$ or $D^b(\mathcal{A})$ are still complexes, but the morphisms from $A \to B$ are now equivalence classes of diagrams

$$A \overset{\sim}{\leftarrow} C \to B$$

where the first arrow is a quasi-isomorphism. Another diagram, given as the $AC'B$ path below, is equivalent if it embeds into a commutative diagram

$$\begin{tikzcd}
& C \arrow[dl, "\sim"'] \arrow[dr] & \\
A & C'' \arrow[l, "\sim"'] \arrow[u] \arrow[d] & B \\
& C' \arrow[ul, "\sim"] \arrow[ur] &
\end{tikzcd}$$

When $\mathcal{A}$ has enough injectives, $\mathcal{D}^+(\mathcal{A})$ can also be identified with the homotopy category of complexes of injective objects. It follows that the Hom's in this category have the following interpretation.

Lemma 1.2. *Given $A, B \in \mathcal{A}$,*

$$Hom_{D^+(\mathcal{A})}(A, B[n]) \cong Ext^n(A, B)$$

To give illustration of what we can do with this stuff, we can deduce the long exact sequence for Ext's using only these axioms.. An additive functor F from a triangulated category to an abelian category is *cohomological* if for any triangle

$$A \to B \to C \to$$

there is a long exact sequence

$$\ldots F^n(A) \ldots F^n(B) \to F^n(C) \to F^{n+1}(A) \ldots$$

where $F^n(A) = F(A[n])$. For example, $H(A) = H^0(A)$ is a cohomological functor from $D^+(\mathcal{A}) \to \mathcal{A}$.

Lemma 1.3. *If X is object in a triangulated category, $Hom(X, -)$ is cohomological. $Hom(-, X)$ is cohomological on the opposite category (which is also triangulated).*

Sketch. Suppose that

$$A \xrightarrow{f} B \to C \to$$

is a triangle. Since triangles are stable under rotation and translation (T2), it is enough to check exactness of

$$Hom(X,A) \to Hom(X,B) \to Hom(X,C)$$

Let g be in the first group. By T1 and T3, we have commutative diagram

$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d, "g"] & X \arrow[r] \arrow[d, "f\circ g"] & 0 \arrow[d, dashed] \\ A \arrow[r, "f"] & B \arrow[r] & C \end{tikzcd}$$

So g maps to 0 in $Hom(X,C)$. Suppose $h \in Hom(X,B)$ maps to 0 in $Hom(X,C)$, then from the axioms we can find an arrow m as depicted below

$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d, dashed, "m"] & X \arrow[r] \arrow[d, "h"] & 0 \arrow[d] \\ A \arrow[r, "f"] & B \arrow[r] & C \end{tikzcd}$$

The dual result is similar. □

Suppose that $F : \mathcal{A} \to \mathcal{B}$ is a right exact functor between abelian categories such that $\mathcal{A}$ has enough injectives. By identifying $D^+(\mathcal{A})$ with homotopy category of injectives, we get a well defined extension of F to a triangulated functor $\mathbb{R}F : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ by $\mathbb{R}F(I^\bullet) = F(I^\bullet)$. This is called the right derived functor. There is a dual notion of left derived functor for things like $\otimes, f^*$. This is a bit awkward as the domain is naturally D^- or the unbounded derived category rather than D^+. Fortunately, we won't have to worry about this, since we will be working over a field where modules are automatically flat. So the naive extension will work fine.

2. t-structures

Given $D = D^b(\mathcal{A})$, where $\mathcal{A}$ is abelian, set $D^{\geq n} = D^{\geq n}(\mathcal{A})$ (resp. $D^{\leq n} = D^{\leq n}(\mathcal{A})$) to be full subcategory of complexes such that $H^i(A) = 0$ unless $i \geq n$ (resp. $i \leq n$). This is the prototype of a t-structure. Then

TS1 If $A \in D^{\leq 0}$ and $B \in D^{\geq 1}, Hom(A,B) = 0$.

TS2 $D^{\leq 0} \subset D^{\leq 1}$ and $D^{\geq 0} \supset D^{\geq 1}$.

TS3 For any $A \in D$, there is a distinguished triangle

$$X \to A \to Y \to$$

with $X \in D^{\leq 0}$ and $Y \in D^{\geq 1}$.

To verify TS3, we use the truncation functors

$$X = \tau_{\leq 0}A = \ldots A^{-1} \to \ker d^0 \to 0 \ldots$$

$$Y = \tau_{\geq 1}A = A/\tau_{\leq 0}$$

For TS1, using triangles such as

$$\tau_{\leq -1} \to A \to H^0(A) \to$$

$$H^1(B) \to B \to \tau_{\geq 2}B \to$$

plus induction, we can assume that A and B are sheaves F and G translated to degree ≤ 0 and ≥ 1 respectively. Then

$$Hom(A, B) = Ext^i(F, G) = 0$$

since i will be negative.

A *t-structure* on a triangulated category D is a pair $(D^{\leq 0}, D^{\geq 0})$ satisfying TS1, TS2, TS3, where $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$. Although, we have only one example so far, we will shortly see that there are (non-obvious) perverse t-structures.

Proposition 2.1. *For any t-structure, the inclusion $D^{\leq n} \to D$ (resp $D^{\geq n} \to D$) admits a right (resp. left) adjoint $\tau_{\leq n}$ (resp. $\tau_{\geq n}$). Any object fits into a canonical distinguished triangle*

$$\tau_{\leq 0}A \to A \to \tau_{\geq 1}A \to$$

Proof. In outline, for each $A \in D$ choose a triangle as in TS3. Define $\tau_{\leq 0}A = X$. Observe that by TS1 and TS2

$$Hom(X', A) \cong Hom(X', \tau_{\leq 0}A) \tag{1}$$

for $X' \in D^{\leq 0}$. Thus given $A' \to A$, we get an induced morphism $\tau_{\leq 0}A' \to \tau_{\leq 0}A$, so this is a functor. Equation (1) shows this is the right adjoint to inclusion. The remaining cases are similar. □

Proposition 2.2. *Suppose that $a \leq b$. Then $\tau_{\leq a}\tau_{\leq b} \cong \tau_{\leq a}$, $\tau_{\geq b}\tau_{\geq a} \cong \tau_{\geq b}$, and $\tau_{\geq a}\tau_{\leq b} \cong \tau_{\leq b}\tau_{\geq a}$.*

Proof. The first two isomorphisms are routine, so we prove only the last. The map $\tau_{\leq b}X \to \tau_{\geq a}X$, given as the composition $\tau_{\leq b}X \to X \to \tau_{\geq a}X$, factors through $\tau_{\geq a}\tau_{\leq b}X$. As $\tau_{\geq a}\tau_{\leq b}X \in D^{\leq b}$, we see that $\tau_{\geq a}\tau_{\leq b}X \to \tau_{\geq a}X$ factors through $\tau_{\leq b}\tau_{\geq a}X$. We have to show that this is an isomorphism.

Let Y fit into a distinguished triangle

$$\tau_{<a}X \to \tau_{\leq b}X \to Y \to \tag{2}$$

we can use this along with

$$\tau_{\leq b}X \to X \to \tau_{>b}X \to$$

to generate

$$\tau_{<a}X \to X \to \tau_{\geq a}X \to$$

and

$$Y \to \tau_{\geq a}X \to \tau_{>b}X \to \tag{3}$$

by T4. Since $\tau_{<a}X = \tau_{<a}\tau_{\leq b}X$, (2) implies that $Y \cong \tau_{\geq a}\tau_{\leq b}X$. And since $\tau_{>b}X = \tau_{>b}\tau_{\geq a}X$, we can conclude from (3) that $Y \cong \tau_{\leq b}\tau_{\geq a}X$. □

The heart (“le coeur” in the original) of the t-structure is $D^{\leq 0} \cap D^{\geq 0}$. For the standard t-structure on $D^b(\mathcal{A})$, we can identify the heart with $\mathcal{A}$ itself. Remarkably, the axioms lead to a similar structure in general.

Theorem 2.3 (Beilinson, Bernstein, Deligne)**.** *The heart is abelian. $H^0 = \tau_{\leq 0}\tau_{\geq 0}$ is a cohomological functor from D to the heart.*

Proof. We prove the first statement that the heart $\mathcal{A} = D^{\leq 0} \cap D^{\geq 0}$ is abelian. If $f : A \to B$ is a morphism in $\mathcal{A}$, we need to construct a kernel and cokernel. Extend this to a distinguished triangle

$$A \to B \to S \to$$

Then using

$$B \to S \to A[1] \to$$

we can see that $S \in D^{\leq 0} \cap D^{\geq -1}$. It follows that $C = \tau_{\geq 0} S$ and $K = (\tau_{\leq -1} S)[1]$ are in $\mathcal{A}$. We have a natural map $B = \tau_{\leq 0} B \to C$ which we claim is the cokernel of f. To see this obverse that for any $X \in \mathcal{A}$ we have an exact sequence

$$Hom(A[-1], X) \to Hom(S, X) \to Hom(B, X) \to Hom(A, X)$$

$$Hom(A[-1], X) = 0$$

by the axioms. Also

$$Hom(S, X) = Hom(\tau_{\geq 0} S, X)$$

Thus

$$0 \to Hom(C, X) \to Hom(B, X) \to Hom(A, X)$$

is exact, and this proves the that C is the cokernel. The proof that $K \to A$, induced from $S[-1] \to A$, is the kernel of f is similar.

The final step is to show that the image $im(f) = \ker(B \to C)$ is isomorphic to the coimage $coim(f) = coker(K \to A)$. Using T4, we can use

$$\begin{tikzcd}
C \arrow[dd, squiggly] & & B \arrow[ll] \arrow[dl] \\
& S \arrow[ul] \arrow[dr, squiggly] & \\
K[1] \arrow[ur] \arrow[rr, squiggly] & & A \arrow[uu]
\end{tikzcd}$$

to build

$$\begin{tikzcd}
C \arrow[dd, squiggly] \arrow[dr, squiggly] & & B \arrow[ll] \\
& I \arrow[ur] \arrow[dl] & \\
K[1] \arrow[rr, squiggly] & & A \arrow[uu] \arrow[ul]
\end{tikzcd}$$

Using the upper triangle in the second diagram, we see that $I \cong im(f)$. The bottom triangle shows that $I \cong coim(f)$. $\square$

Remark 2.4. *This does* **not** *say that D is the derived category of its heart. This not always true.*