AN OVERVIEW OF POLARIZED HODGE MODULES

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Abstract: *We will cover the definition of polarized Hodge modules, and summarize their basic properties. In order to simplify the presentation, we work exclusively with left modules on a nonsingular variety.*

1. Preliminaries

Let X be a complex manifold. The basic example of a Hodge module is provided by a variation of Hodge structure [Sc], which consists of a local system L on X defined over a subring of $\mathbb{R}$, and a filtration $F^\bullet \subset V = \mathcal{O}_X \otimes L$ satisfying Griffiths transversality such that the fibres $(L_x, F_x^\bullet \subset \mathbb{C} \otimes L_x)$ are pure Hodge structures. We have an integrable connection on ∇ on V such that $\ker \nabla \cong \mathbb{C} \otimes L$. V can thus be regarded as a holonomic D-module[1] via $\partial_{x_i} v = \nabla_{\partial_{x_i}} v$. Griffiths transversality ensures that F defines a filtration on the D-module V. This is in fact a good filtration, since F^p is a coherent over $\mathcal{O}_X$. With an eye toward the generalization to Hodge modules, we will find it useful to replace the local system L, by the perverse sheaf $L[\dim X]$. The deeper aspects of the theory require the existence of a polarization, which is flat pairing on L which polarizes the Hodge structures on the fibres in the usual sense. (We will review polarizations in more detail, later on.)

In general, Hodge modules have "singularities". It will be convenient to have a basic prototype. Suppose that $X = \Delta^*$ is a punctured disk with coordinate t and inclusion $j : \Delta^* \to \Delta$ into a disk. Suppose that L is the local system underlying a variation of Hodge structure on Δ^*. The vector bundle $V = \mathcal{O}_{\Delta^*} \otimes L$ is equipped with a connection ∇ such that $\ker \nabla \cong \mathbb{C} \otimes L$. V has several extensions to vector bundles over Δ such that ∇ extends to a logarithmic connection. Let $V^\alpha \subset j_* V$ (respectively $V^{>\alpha}$) be the unique extension where the residues have eigenvalues in $[\alpha, \alpha + 1)$ (respectively $(\alpha, \alpha + 1]$). Then $V^{>\alpha} \subseteq V^\alpha \subseteq V^\beta$ when $\alpha \geq \beta$. We let $\tilde{V} = \bigcup V^\alpha$. Then $\tilde{V}$ is a D-module and V^α is essentially the Malgrange-Kashiwara filtration. In particular, $Gr^\alpha V = V^\alpha / V^{>\alpha}$ is a generalized eigenspace of $t\partial_t$ with eigenvalue α. Under Riemann-Hilbert, $\tilde{V}$ corresponds to the perverse sheaf $\mathbb{R} j_* L[1]$. Our real interest is the intersection cohomology complex or intermediate extension $j_{!*} L[1] = j_* L[1]$. This corresponds to the sub D-module of $\tilde{V}_{min} \subseteq \tilde{V}$ generated by $V^{>-1}$. We filter this by

$$F^p \tilde{V}_{min} = \sum_{i \geq 0} \partial_t^i (j_* F^{p+i} \cap V^{>-1}) \tag{1}$$

We record the following observation, which will basically tell us that $(\tilde{V}_{min}, F, j_{!*} L[1])$ is a Hodge module.

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[1]By default, D-module will mean left module. Regarding D-modules in general, I will follow the conventions and notation of [MS, S] to maintain consistency with Claude Sabbah's lecture.

Lemma 1.1. *When L is polarizable,*

(a) *each $F^p\tilde{V}_{min}$ is coherent over $\mathcal{O}_\Delta$, and $\partial_t F^p\tilde{V}_{min} \subseteq F^{p-1}\tilde{V}_{min}$.*
(b) *if $b > -1$, $tF^p\tilde{V}^b_{min} = F^p\tilde{V}^{b+1}_{min}$.*
(c) *if $b < 0$, $\partial_t F^p Gr^b\tilde{V}_{min} = F^{p-1}Gr^{b-1}\tilde{V}_{min}$.*

Proof. The second formula of (a) is a consequence of Griffiths transversality and (1). The remaining properties can be deduced from the work of Schmid [Sc]. □

In general, a (rational) Hodge module consists of a perverse sheaf $L \in Perv(\mathbb{Q}_X)$ over X with coefficients in $\mathbb{Q}$, and a holonomic D_X-module M equipped with a good filtration F and an isomorphism $\alpha : {}^pDR(L) \cong \mathbb{C} \otimes L$ which is sometimes suppressed. (${}^pDR(M)$ denotes the de Rham complex, shifted so that the result is a perverse sheaf.) Let $MF_h(D_X, \mathbb{Q})$ denote the category of such quadruples (M, F, L, α). Of course, the category $MF_h(D_X)$ is much too big for the purposes of Hodge theory, but it will contain the category of Hodge modules as a full subcategory. We have already described two classes of examples, variations of Hodge structures, and the last example $(\tilde{V}_{min}, F, j_{!*}L[1])$. We note that the category $MF_h(D_X, \mathbb{Q})$ is additive, with kernels and images, although not abelian. The subcategory of Hodge modules will turn out to be abelian, however.

The subcategory of pure Hodge modules is defined by induction. The key inductive axiom is stability under vanishing cycle functors. We recall the basic set up. Given a perverse sheaf L and a holomorphic map $f : X \to \mathbb{C}$, set ${}^p\psi_f L = \mathbb{R}\psi_f L[-1]$ and ${}^p\phi_f L = \mathbb{R}\Phi_f L[-1]$, where the objects on the right are the usual nearby and vanishing cycle sheaves of Deligne [D2]. It is known that ${}^p\psi_f L$ and ${}^p\phi_f L$ are perverse [BBD]. The generator $T \in \pi_1(\mathbb{C}^*)$ acts on these sheaves. In the cases of interest here, we can assume that T acts quasi-unipotently, i.e. that its eigenvalues are roots of unity. We have maps $can : {}^p\psi_f L \to {}^p\phi_f L$ and $Var : {}^p\phi_f L \to {}^p\psi_f L$ whose composition, in either order, is the logarithm[2] N of the unipotent part of T [S1].

On the D-module side, things are more involved. A holonomic D-module M is known to be specializable. In particular, this means that the direct image $\tilde{M} = i_+M$ carries a Malgrange-Kashiwara filtration $V^\bullet\tilde{M}$, where $i : X \to X \times \mathbb{C}$ is the inclusion of the graph of f and $V^\bullet\tilde{M}$ is a decreasing filtration such that $t\partial_t - \alpha$ acts nilpotently on $Gr^\alpha_V\tilde{M} = V^\alpha\tilde{M}/\cup_{\epsilon>0} V^{\alpha+\epsilon}\tilde{M}$. The quasi-unipotency assumption for ${}^pDR(M)$ implies that we can assume that $V^\bullet M$ is indexed by $\mathbb{Q}$. When M is also regular, the associated graded $Gr^b_V\tilde{M}$ corresponds to the $\lambda = \exp(-2\pi i b)$-th eigensheaf $\psi^\lambda_f({}^pDR(M))$, when $-1 < b \leq 0$, and $Gr^{-1}_V\tilde{M}$ to ${}^p\phi^1_f({}^pDR(M))$ [S1, 3.4.12]

We extend the nearby and unipotent vanishing cycle functors to $MF(D_X, \mathbb{Q})$ as follows:

$$\psi_f(M, F, L) := (\bigoplus_{-1<b\leq 0} Gr^b_V\tilde{M}, F, {}^p\psi_f L)$$

$$\phi^1_f(M, F, L) := (Gr^{-1}_V\tilde{M}, F^{\bullet+1}, {}^p\phi_f L)$$

We have maps can and Var induced on the D-module components by $-\partial_t$ and t respectively. We need to impose further conditions on F to get reasonable behaviour. The conditions are essentially the same as those given in lemma 1.1. Given a holonomic D-module with good filtration (M, F), we say that it is strictly specializable with respect to f if the following conditions hold:

[2] The logarithm should be normalized by $\frac{1}{2\pi i}$, but we will ignore this for simplicity.

(a) if $b > -1$, $tF^pV^b\tilde{M} = F^pV^{b+1}\tilde{M}$, where $(\tilde{M}, F) = i_+(M, F)$.
(b) if $b < 0$, $\partial_t F^p Gr_V^b \tilde{M} = F^{p-1}Gr_V^{b-1}\tilde{M}$.

It is called regular with respect to f if in addition

(c) F induces a good filtration on the $Gr^V\tilde{M}$, i.e. $Gr_F^* G_V^* \tilde{M}$ is coherent over $Gr_F^* D_{i(X)}$.

Some explanation for the utility of these conditions is provided by the following two propositions.

Proposition 1.2. *If (M, F) is strictly specializable with respect to f and $\tilde{M}$ is generated by $V^{>-1}\tilde{M}$, then F is determined by the restriction of (M, F) to $U = X - f^{-1}(0)$.*

Proof. See [S1, prop 3.2.2]. We remark that the formula for F on M in terms of $(M, F)|_U$ is essentially the same as (1). □

At the end of the day, at least in the polarizable setting, we want to be able to break up our objects into a sum of simple objects, where the underlying perverse sheaves are intersection cohomology complexes. The following proposition will be a key step.

Proposition 1.3 ([S1, 5.1.4])**.** *If $\mathcal{M} \in MF_h(D_X, \mathbb{Q})$ is regular and quasi-unipotent with respect to f, then the following are equivalent:*

(a) $\phi_f^1(\mathcal{M}) \cong \ker(Var) \oplus \mathrm{im}(can)$
(b) *$\mathcal{M} \cong \mathcal{M}_1 \oplus \mathcal{M}_2$, where $supp\mathcal{M}_2 \subseteq f^{-1}(0)$ and $\mathcal{M}_1$ has no sub or quotient object supported on $f^{-1}(0)$.*

Corollary 1.4. *If the conditions of the proposition hold for $\mathcal{M}$ with respect to every locally defined function f, then $\mathcal{M}$ admits a strict support decomposition which means that*

$$\mathcal{M} = \bigoplus_{Z \subset X \ closed\ analytic} \mathcal{M}_Z$$

where all subquotients of $\mathcal{M}_Z$ have support equal to Z.

2. Hodge Modules

We are now ready to start defining the full subcategory

$$MH(X, n) = MH(X, \mathbb{Q}, n) \subset MF_h(D_X, \mathbb{Q})$$

of pure Hodge modules of weight n. It will be the largest subcategory satisfying axioms (MH1)-(MH3) below.

(MH1) *An object $MH(X, n)$ must be regular and quasi-unipotent and satisfy the conditions of proposition 1.3.*

Thus objects admit strict support decompositions. The remainder of the definition proceeds by induction on the dimension of support. The base case is handled by:

(MH2) *An object (M, F, L) with zero dimensional support lies in $MH(X, n)$ if and only if it is the direct image of a Hodge structure of weight n.*

The rough idea of the inductive axiom is to impose stability under nearby and vanishing functors. However, these constructions usually produce mixed objects, so we need to first take the associated graded with respect to a suitable filtration. This is already evident in the simplest case when L comes from a weight r unipotent polarized variation of Hodge structure on the punctured disk Δ^*. The filtration F induces one on $\psi = \mathbb{R}\psi_t L \otimes \mathbb{C}$, which we can view as just a vector space. This becomes a mixed Hodge structure, called the limit mixed Hodge structure [Sc], when we combine this with the monodromy filtration W shifted by r. This is the unique increasing filtration rational filtration such that

$$NW_i\psi \subset W_{i-2}\psi, \quad N^j : Gr^W_{r+j}\psi \xrightarrow{\sim} Gr^W_{r-j}\psi \tag{2}$$

In general, given an object $(M, F, L) \in MF_h(D_X, \mathbb{Q})$ on $f : X \to \mathbb{C}$, and an integer n which will play the role of weight, let W be the monodromy filtration on ${}^p\psi_f L$, in the category of perverse sheaves, with respect to the nilpotent endomorphism N shifted by $r = n - 1$. We can define a D-module filtration W on $\psi_f M := \oplus Gr^b_V \tilde{M}$ which corresponds to $W \otimes \mathbb{C}$ above under Riemann-Hilbert. The filtration F induces one on the associated graded $Gr^W(\psi_f M)$, resulting in an object

$$Gr^W_i \psi_f(M, F, L) \in MF_h(D_X, \mathbb{Q})$$

Similarly, we have an object

$$Gr^W_i \phi^1_f(M, F, L) \in MF_h(D_X, \mathbb{Q})$$

where W now denotes the monodromy filtration on ${}^p\phi^1_f L$ shifted by n. The final axiom is:

(MH3) *Given $(M, F, L) \in MH(X, n)$ and any holomorphic function f defined on an open subset U,*

$$Gr^W_i \psi_f(M, F, L), Gr^W_i \phi^1_f(M, F, L) \in MH(U, i)$$

These axioms are strong enough to establish:

Theorem 2.1 ([S1, 5.1.14])**.** *$MH(X, n)$ is abelian, and morphisms strictly preserve F.*

This is proved by induction. In order to describe the idea a bit more precisely, we introduce the auxiliary categories. Let $MH_Z(X, n) \subset MH(X, n)$ be the full subcategory of objects with strict support Z, which means that all subquotients of the objects have Z as its support. Let $MHW(X, n)$ be the category of filtered objects of $(\mathcal{M}, W) \in FMF_h(D_X, \mathbb{Q})$ such that $Gr^W_i(\mathcal{M}) \in MH(X, i)$ for all i. (NB: This is strictly bigger than the category of mixed Hodge modules introduced later on in [S2].) Consider the following statements

$A(i)$: $MH_Z(X, n)$ is abelian and the morphisms are strict for $\dim Z \leq i$.

$B(i)$: The subcategory of $MHW(X)$ objects with support having dimension $\leq i$ is abelian and the morphisms are bistrict.

Then by (MH1) and corollary 1.4 it suffices to prove $A(i)$ for all i. $A(0)$ is clear by (MH2). The logic of the rest of the proof is to establish $A(i) \Rightarrow B(i)$ and $B(i) \Rightarrow A(i + 1)$. The first implication $A(i) \Rightarrow B(i)$ uses the orthogonality condition, $Hom(MH_Z(X, i), MH_Z(X, j)) = 0$ when $i > j$ [S1, 5.1.11], and the following general result.

Lemma 2.2 ([S1, 5.1.15])**.** *Let $\mathcal{A}$ be an abelian category. Suppose that $\mathcal{A}_i, i \in \mathbb{Z}$ are full abelian subcategories whose kernels and cokernels coincide with those in $\mathcal{A}$, and suppose also that $Hom(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $i > j$. Then the category $\mathcal{A}W$, of finite filtered objects (A, W) of $\mathcal{A}$ with $Gr_j^W A \in \mathcal{A}_j$, is abelian and morphisms are strict.*

The key point in checking the "bootstrapping" implication $B(i) \Rightarrow A(i+1)$ is that to verify the axioms (MH1) and (MH3) for kernel and cokernel of a map in $MH_Z(X, n)$, we can choose the function f in general position with respect to Z, and thereby cut the dimension of the support.

3. Polarizations

For the deeper results in theory, we need polarizations. Recall that a polarization on a weight n rational Hodge structure H is a pairing $S : H \otimes H \to \mathbb{Q}(-n)$ such that $(2\pi\sqrt{-1})^n S(x, Cy)$ is symmetric and positive definite where the Weil operator C acts by $\sqrt{-1}^{p-q}$ on H^{pq}. It will instructive to recall a basic example. Let X be a d dimensional smooth projective variety with a Lefschetz operator L. On $d - i$ primitive cohomology $P_L H^{d-i}(X) = \ker L^{d+1} : H^{d-i}(X) \to H^{d+i+2}(X)$, $S(-,-) = \pm S_0(-, L^i -)$ gives a polarization where $S_0(\alpha, \beta) = \int_X \alpha \cup \beta$. There is somewhat more structure in this example which is convenient to axiomatize. A Hodge-Lefschetz structure of weight d is a finite sum $\bigoplus H^i$ of Hodge structures of weight $d + i$ together with a collection of morphisms $\ell : H^i \to H^{i+2}(1)$ giving isomorphisms $H^{-i} \cong H^i(i)$. A polarization is a collection of pairings $\langle , \rangle : H^i \times H^{-i} \to \mathbb{Q}(-d)$ such that

$$\langle x, y \rangle = \pm \langle y, x \rangle, \quad \langle \ell x, y \rangle = \pm \langle x, \ell y \rangle \tag{3}$$

and such that $\langle -, \ell^i - \rangle$ polarizes the primitive part $P_\ell H^{-i}$.

There is a bigraded version of the previous notion which is somewhat harder to motivate, but suffice it to say that it will play an essential role in what follows. A bigraded Hodge-Lefschetz structure of weight d consists of a finite sum $\bigoplus H_i^j$ of Hodge structures of weight $d + i + j$ and commuting operators $\ell : H_i^j \to H_i^{j+2}(1)$, $N : H_i^j \to H_{i-2}^j(-1)$ both satisfying hard Lefschetz. A polarization is now a pairing

$$\langle , \rangle : H_{-i}^{-j} \times H_i^j \to \mathbb{Q}(-d)$$

such that the obvious generalization of (3) holds for both operators, and such that $\langle -, N^i \ell^j - \rangle$ polarizes the "bi-primitive" part $\ker N^{i+1} \cap \ell^{j+1}$. Here is a basic example which arises in Steenbrink's work on limit mixed Hodge structures [St, GN]. Given a projective semistable family $f : X \to \Delta$ of relative dimension d, let $Y^{(0)} = X$ and let $Y^{(i)}$ denote the disjoint of i-fold intersections of components of $f^{-1}(0)$ when $i > 0$. Steenbrink constructed a spectral sequence converging to $H^*(\mathbb{R}\psi_f \mathbb{Q})$, where the E_1 term looks like

$$H_i^j = \bigoplus_k H^{j+n-i-2k}(Y^{(2k+i+1)})(-i-k)$$

This is in fact a bigraded Hodge-Lefschetz structure of weight $d + i + j$. The operator $\ell : H_i^j \to H_i^{j+2}$ is simply the Lefschetz operator with respect to a relatively ample class. The remaining operator N is a bit more subtle, so we refer to the original papers [GN, St]. The usual polarizations give a polarization of H_*^* as well. Steenbrink asserted that the filtration on the abutment coming from the spectral sequence coincided with the monodromy filtration, which means that (2) holds. Guillén and Navarro [GN] gave a complete proof based on the following:

Theorem 3.1. *Suppose that* $H = \bigoplus H_i^j$ *is a polarized bigraded Hodge-Lefschetz module with a differential* $d : H_i^j \to H_{i-1}^{j+1}$ *commuting with* ℓ *and* N *and satisfying* $\langle x, dy\rangle = \pm\langle dx, y\rangle$. *Then the cohomology* $\ker d/\operatorname{im} d$ *carries an induced polarized Hodge-Lefschetz structure.*

Proof. See [S1, 4.2.2] or [GN, 4.5]. □

Returning to Hodge modules, let X be a d-dimensional complex manifold. The Verdier dualizing sheaf is just $\mathbb{Q}[2d]$. Given a Hodge module $\mathcal{M} = (M, F, L) \in MH(X, n)$, a polarization is a pairing $S : L \otimes L \to \mathbb{Q}_X[2d](-n)$ satisfying the certain inductive conditions. Since the strict support decomposition is orthogonal with respect to any such pairing, we may as well assume that $\mathcal{M}$ is strictly supported on an irreducible subvariety Z. By adjointness, S corresponds to a map to the Verdier dual $S' : L \to \mathbb{D}L(-n) = \mathbb{R}\mathcal{H}om(L, \mathbb{Q}[2d](-n))$. This induces a map $S' : M \to \mathbb{D}M(-n)$ under Riemann-Hilbert. The operation $\mathbb{D}$ can be lifted to the category of D-modules with good filtration [S1, §2.4]. We say that S is compatible with the filtration if there is a morphism $(M, F) \to \mathbb{D}(M, F)$ coinciding with S' on the first factor. A pairing S compatible with the filtration is called a polarization if it satisfies the following inductive axioms:

(P1) If $\dim Z = 0$, $\mathcal{M}$ is given by a Hodge structure H, S is induced from polarization on H in the usual sense.
(P2) If $\dim Z > 0$ and f is a locally defined function which is not identically 0 on Z, then $S(-, N^i-)$ induces a polarization on the primitive part (with respect to N) of $Gr^W_{n-1+i}{}^p\psi_f L$.

Theorem 3.2. *Any object* $\mathcal{M} \in MH_Z(X, n)$ *is generically a variation of Hodge structure on* Z. *More precisely, there exists an open set* $U \subset Z$ *and a variation of Hodge structure* K *on* U *of weight* $n - \dim Z$ *such* $\mathcal{M}|_{X-(Z-U)}$ *is the direct image of* K. *Moreover, the underlying perverse sheaf is the intersection cohomology complex associated to the local system of* K. *Finally a polarization on* $\mathcal{M}$ *corresponds to a polarization on* K *in the usual sense.*

Proof. [S1, 5.1.10,5.2.12] □

Corollary 3.3. *The full subcategory* $MH(X, n)^p \subset MH(X, n)$ *of polarizable Hodge modules is a semisimple abelian category.*

Proof. This follows from the corresponding result for polarizable variations. □

The theorem admits a converse that any generically defined polarizable variation of Hodge structure extends to a polarizable Hodge module. This was conjectured in the first paper [S1] and proved in the second [S2].

4. Direct images

We come to one of the main results, which involves the behaviour under a projective direct image. Suppose we are given a projective morphism $f : X \to Y$ and Hodge module (M, F, L) on X, what should the ith direct image mean? On the perverse side, we simply use ${}^p\mathcal{H}^i\mathbb{R}f_*L$, which means take the derived direct image and then take the cohomology with respect to the perverse t-structure. On the filtered D-module side, we also have a suitable derived direct image $f_+(M, F)$, but in order to get a well behaved filtered module structure on cohomology, we should make sure that this is (represented by) a complex strictly compatible with F. This is dealt with in the first part of the next theorem. Now we have a candidate for the direct image Hodge module

$$\mathcal{H}^i f_*(M, F, L) := (\mathcal{H}^i f_+(M, L), {}^p\mathcal{H}^i \mathbb{R} f_* L)$$

That this actually is one, is also part of the theorem.

Theorem 4.1. *Let $f : X \to Y$ a be projective morphism of smooth varieties, and let ℓ be the first Chern class of a relatively ample line bundle. Let $(M, F, L) \in MH(X, n)$ have a polarization S. Then the direct image exists as a pure polarized Hodge module and this satisfies hard Lefschetz. More precisely,*

(a) *$f_*(M, F)$ is strict.*
(b) *$\mathcal{H}^i f_*(M, F, L) \in MH(Y, n + i)$.*
(c) *$\ell^i : \mathcal{H}^{-i} f_*(M, F, L) \to \mathcal{H}^i f_*(M, F, L)$ is an isomorphism.*
(d) *$\pm\mathcal{H}^i f_* S(1 \otimes L^i)$ is a polarization of the primitive part (with respect to ℓ) of $\mathcal{H}^i f_*(M, F, L)$.*

We give a broad outline of the proof, focusing on (b) and (c). We refer the reader to [S1, pp 977-988] for the precise details and the remaining parts. Let $\mathcal{M} = (M, F, L) \in MH_Z(X, n)$. The proof is by induction on $d = \dim Z$. There are two cases, where the second case relies on the first with the same value of d. The base case $d = 0$ is trivially true, so we assume $d > 0$.

4.2. **Case 1.** For the first case assume that $\dim f(Z) > 0$. Then choose a (local) function g on Y with $g^{-1}(0)$ in general position and let $h = gf$. Saito establishes an isomorphism

$$\psi_g \mathcal{H}^j f_* \mathcal{M} \cong \mathcal{H}^j f_* \psi_h \mathcal{M}$$

and a similar one for ϕ^1. By induction, theorem 4.1 holds for the primitive parts of the weight graded subquotients of $f_*\psi_h\mathcal{M}$ and $f_*\phi_h^1\mathcal{M}$. The proof of this case of the theorem is completed with the help of the following proposition applied to to $f_*\psi_h\mathcal{M}$ and $f_*\phi_h^1\mathcal{M}$.

Proposition 4.3 ([S1, 5.3.5])**.** *Let $(M', F, L', W) \in MHW(X)$ be equipped with an endomorphism N satisfying* (2) *and a compatible polarization. Suppose that the conclusion of theorem 4.1 holds for the primitive part $P_N Gr^W_{r+i}(M', F, L')$, then there exists a spectral sequence*

$$E_1^{pq} = \mathcal{H}^{p+q} f_* Gr^W_{-p}(M', F, L) \Rightarrow \mathcal{H}^* f_*(M', F, L')$$

This degenerates at E_2, $E_2^{pq} \in MH(Y, q)^p$, and the hard Lefschetz isomorphisms

$$\ell^j : C_i^{-j} \xrightarrow{\sim} C_i^j; \quad N^i : C_{-i}^j \xrightarrow{\sim} C_i^j \tag{4}$$

hold where $C_i^j = E_2^{-i-r, i+j+r}$.

Proof. The spectral sequence is the standard one associated to W on $f_*(M', F, L')$. (In this generality, it goes back to Verdier [S1, 5.1.17]) The assumptions imply that $E_1 \in MH(Y)^p$. The operators N and ℓ respect the strict support decomposition. So we can assume E_1 has strict support Z without loss of generality. E_1 is generically a polarized variation of Hodge structure on Z. So at a general point $z \in Z$, the above assumptions imply that $E_1|_z$ is a bigraded polarized Hodge-Lefschetz structure with a compatible differential. Therefore $E_2|_z$ is also a polarized bigraded Hodge-Lefschetz structure by theorem 3.1. This implies (4). □

4.4. **Case 2.** We now turn to the remaining case where $\dim f(Z) = 0$. We can assume in this case that $Y = pt$ is a point. The theorem implies that the cohomology of L carries a pure polarized Hodge structure and that it satisfies hard Lefschetz. We will be content to prove just this, and refer to [S1] for the remaining properties. The first interesting case essentially goes back to Zucker [Z].

Theorem 4.5. *$H^i(X, L)$ is pure of weight $n + i$ when X is a curve.*

What Zucker actually proved was that intersection cohomology of a curve with coefficients in a polarized variation of Hodge structure has a pure Hodge structure. This was deduced by showing that intersection cohomology coincides with L^2 cohomology with the same coefficients. This is the only point in the proof of theorem 4.1 where analytic methods are needed.

Now suppose that X has arbitrary dimension. We make use of the following easy fact:

Lemma 4.6. *Suppose that V is a pure Hodge structure. Then a direct summand of V in $MF_h(D_{pt}, \mathbb{Q})$ (= the category filtered $\mathbb{C}$-vector spaces with $\mathbb{Q}$-structure) is again a pure Hodge structure of the same weight.*

Let $\pi : \tilde{X} \to X$ be the blow up of the base locus of sufficiently general pencil of hyperplane sections. Let $\tilde{M} = (\tilde{M}, F, \tilde{L}) = \pi^*\mathcal{M}$. Its support is the strict transform of Z, so its dimension is unchanged. We have a surjective map $p : \tilde{X} \to \mathbb{P}^1$. The cohomology $H^i(X, L)$ is summand of $H^i(\tilde{X}, \tilde{L})$ in $MF_h(D_{pt}, \mathbb{Q})$, so suffices to prove that the latter is pure of weight $n + i$. By Case 1, we know that $\mathcal{M}$ satisfies hard Lefschetz relative to p. Therefore Deligne's theorem [D1] shows that the Leray spectral sequence for L with respect to p degenerates at E_2. It follows that

$$H^i(\tilde{X}, \tilde{L}) \cong \bigoplus_{j+k=i} H^j(\mathbb{P}^1, \mathcal{H}^k p_* \tilde{M})$$

Case 1 of the theorem together with theorem 4.5 shows that the latter has a pure Hodge structure of the expected weight.

Let $i : T \to X$ denote the inclusion of a general hyperplane section with respect to a fixed projective embedding. Let $\mathcal{M}' = (M', F, L') = i^*\mathcal{M} \in MH(T, n-1)$. By induction, we can assume that the theorem holds for $T \to pt$. By weak Lefschetz ([BBD, §4.1] and [GM, §7])

$$i^* : H^j(X, L) \to H^{j+1}(T, L')$$

is bijective for $j \leq -2$ and injective for $j = -1$, and

$$i_* : H^{j-1}(T, L') \to H^j(X, L)(1)$$

is bijective for $j \geq 2$ and surjective for $j = 1$. Since we also have $i_* i^* = \ell$, we can conclude, from induction, that we have an isomorphism

$$\ell^j : H^{-j}(X, L) \cong H^j(X, L) \tag{5}$$

when $j \geq 2$. It remains to treat the case where $j = 1$. Since the dimensions coincide, by duality, it is enough to show that (5) is injective. We have equalities

$$H^{-1}(X, L) = \ell H^{-3}(X, L) \oplus P_\ell H^{-1}(X, K)$$

$$H^1(X,L) = \ell^2 H^{-3}(X,L) \oplus \ker \ell$$

The injectivity of

$$\ell : P_\ell H^{-1}(X,K) \to \ker \ell$$

follows from part (c) of the theorem applied to $\mathcal{M}'$.

Corollary 4.7 (Decomposition theorem)**.** *If L is a perverse sheaf underlying a polarizable Hodge module, then $\mathbb{R}f_*L$ is a sum of translated perverse sheaves.*

Proof. By part (b) of the theorem and [D1],

$$\mathbb{R}f_*L \cong \bigoplus {}^p\mathcal{H}^i(\mathbb{R}f_*L)[-i]$$

□

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