

# A quick guide to proofs

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The main goal of this class is to learn algebra, which also means learning how to prove things in algebra. Unfortunately, I realize that many people in class are confused about what a proof actually is. The short answer is that a proof is a sequence of statements, which are either things you know already, or things deduced from previous statements by rules of logic; *the final statement should be what you are trying to prove*. The last item is important to keep in mind. A common mistake is to start with what you want to prove and eventually reduce it to a true statement. To see why this is not valid, let me “prove” that  $1 = 3$ .

*Proof.* Let  $1 = 3$ . Multiply both sides of this by 0 to get  $0 = 0$ , which is true.  $\square$

Yes, but this doesn’t make  $1 = 3$ . To analyze this a bit more, let’s talk about the conditional. This is a compound statement of the form “if (premise) then (conclusion)”. This means that “(premise)” is true, then so is “(conclusion)”. But the reverse may not be valid, e.g. “if pigs can fly then  $0 = 0$ ” is a true statement regardless of whether or not pigs actually do fly.

## 1 Highschool geometry

Probably, the first place where you have actually seen proofs is in a highschool geometry class, and this is a good place to start. We are armed with various facts about lines, angles and triangles, and the goal is derive new theorems, which may or may not be obvious, but nevertheless need to be proved. We summarize a few of these facts:

SSS (Side-Side-Side property) Two triangles with equal sides are congruent, and therefore that the corresponding angles are equal.

SAS (Side-Angle-Side property) Given triangles  $ABC$  and  $A'B'C'$ , they are congruent if we have equalities of the lengths  $AB = A'B'$ ,  $AC = A'C'$  and included angle  $\angle A = \angle A'$ .

**Problem 1.** *Given a triangle  $ABC$  with lengths  $AC = BC$ , prove that the angles  $\angle A = \angle B$ .*

*Proof.* Let  $D$  be the midpoint of the line segment  $AB$ , then the lengths  $AD = \frac{1}{2}AB = BD$ . Form the triangles  $ACD$  and  $BCD$ . We know that  $AC = BC$ ,  $AD = BD$  and that  $DC$  is a common side. Therefore, we may conclude using the SSS property, that  $ACD$  and  $BCD$  are congruent. Therefore we have that  $\angle A = \angle B$ .  $\square$

Try to write down a careful proof of:

**Problem 2.** Given a triangle  $ABC$  with  $\angle A = \angle B$ , prove that  $AC = BC$

## 2 Group theory

The next example is more relevant because it comes from the test.

**Problem 3.** Let  $G$  be a group where  $g^2 = e$  (the identity) for every  $g \in G$ . Prove that  $G$  is a abelian, i.e.  $g_1g_2 = g_2g_1$  for every pair  $g_1, g_2$ .

*Proof.* Since  $g^2 = e$  holds for every  $g$ , it holds for  $g = g_1g_2$ . This means that

$$g_1g_2g_1g_2 = e$$

Multiply both sides of the equation on the left by  $g_1$  to get

$$g_1^2g_2g_1g_2 = g_1e$$

or

$$g_2g_1g_2 = g_1$$

because  $g_1^2 = e$  and  $e$  is the identity. Multiply on the left by  $g_2$  to obtain

$$g_2^2g_1g_2 = g_2g_1$$

This implies that

$$g_1g_2 = g_2g_1$$

because  $g_2^2 = e$ . Therefore  $G$  is abelian.  $\square$

Even if you got that on the test, try this:

**Problem 4.** Suppose that  $G$  is group such that the inverse of  $g_1g_2$  is  $g_1^{-1}g_2^{-1}$ . Prove that  $G$  is abelian.

## 3 Induction

Induction is a powerful tool which is easy to get confused about initially.

**Problem 5.** Prove that

$$2 + 4 + \dots + 2n = n(n + 1)$$

for every natural number  $n$ .

**Problem 6.** *Let say we reverse roles, you're the teacher and I hand in this solution below to you. Does this look OK? If not, how would you fix it?*

*Proof.* When  $n = 0$  both sides are 0, so they are equal. Suppose that

$$2 + 4 + \dots + 2n = n(n + 1)$$

holds for some  $n$ . Since this  $n$  has not been specified, let's call it  $k$ . Now set  $k = n + 1$ . Then

$$2 + 4 + \dots + 2(n + 1) = (n + 1)(n + 1 + 1)$$

must be true as well. So the formula holds by induction. □

## 4 Proof by contradiction

The idea is that is you can prove that something is true by showing that the negation or opposite leads to a contradiction. Here is a really simple example.

**Problem 7.** *If  $n$  is an integer with  $n^2$  even, prove that  $n$  is even.*

*Proof.* Suppose that  $n$  is not even. Then it is odd. Since the product of two odd integers is odd,  $n^2$  would be odd. Which contradicts the assumption. Therefore  $n$  is even. □

**Problem 8.** *Prove that  $\sqrt{2}$  is not a fraction  $a/b$ , with  $a, b \in \mathbb{N}$ .*

*Proof.* Suppose that it is, which is to say that  $\sqrt{2} = a/b$  with  $a, b \in \mathbb{N}$  and  $b \neq 0$ . After canceling common factors, we can assume that  $a, b$  have no common factors. Squaring both sides yields

$$2 = \frac{a^2}{b^2}$$

and therefore

$$2a^2 = b^2$$

This forces  $b$  to be even by the previous problem. Therefore  $b = 2c$ , where  $c \in \mathbb{N}$ . Substitution leads to

$$2a^2 = 4c^2$$

and consequently

$$a^2 = 2c^2$$

Therefore  $a$  is even. Since  $a$  and  $b$  are both even, this contradicts the fact that they have no common factors. □

**Problem 9.** *Prove that  $x^5 + 2x^4 - 3 = 0$  has no solutions which are even integers. (Assume that it does and then derive a contradiction.)*