RIEMANN'S INEQUALITY AND RIEMANN-ROCH

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Fix a compact connected Riemann surface X of genus g. Riemann's inequality gives a sufficient condition to construct meromorphic functions with prescribed singularities. Let $\mathbb{C}(X)$ denote the field of meromorphic functions on X. The most convenient way to specify the zeros and poles of a nonzero element of this field is in terms of divisors. A divisor is a finite formal sum $D = \sum n_p p$ with $p \in X$ and $n_p \in \mathbb{Z}$. We sometimes view this as sum over all points with $n_p = 0$ for all but finitely many p. Divisors form a group under addition. The degree $\deg D = \sum n_p \in \mathbb{Z}$. Given $f \in \mathbb{C}(X)^*$, $div(f) = \sum ord_p(f)p$ is an example of a divisor, where ord_p is the discrete valuation measuring the order at p. D is called effective, written as $D \geq 0$, if all its coefficients are nonnegative. Given a divisor D, let

$$\begin{aligned} L(D) &= \{f \in \mathbb{C}(X)^* \mid div(f) + D \geq 0\} \cup \{0\} \\ &= \{f \in \mathbb{C}(X) \mid \forall p, ord_p(f) \geq -n_p\} \end{aligned}$$

Theorem 0.1 (Riemann's inequality)**.** *If D is a divisor,*

$$\dim L(D) \geq \deg D + 1 - g$$

In particular, $L(D) \neq 0$ as soon as $\deg D \geq g$.

The Riemann-Roch refines this to an equality with a precise "error term". An important consequence which not apriori obvious is:

Corollary 0.2. *$\mathbb{C}(X) \neq \mathbb{C}$, i.e. X carries a nonconstant meromorphic function.*

We give a proof of these results below after a bit of preparation.

1. Harmonic forms

We quickly review some material already discussed in class. We recall that in basic complex analysis there is a close connection between harmonic functions and holomorphic ones. In order to globalize this, we need to introduce the relevant operators. The Hodge star operator is the $\mathbb{C}$-linear operator[1] defined locally by the formulas

$$*dx = dy, \quad *dy = -dx, \quad *1 = dx \wedge dy, \quad *(dx \wedge dy) = 1$$

where x and y are the real and imaginary parts of a local analytic coordinate z. Thus $** = -(1)^p$ on p-forms. The point of the operator is that it gives an inner product

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\bar{\beta}$$

[1]Some authors define it to be antilinear. Also at the risk of complicating the story, $*$ can be defined by the rule $\alpha \wedge *\beta = (\alpha, \bar{\beta})volume$ where $(,)$ is the pointwise inner product associated to a Riemannian metric compatible with the complex structure. This is the right viewpoint in higher dimensions.

on the space of forms. Since $d(\alpha\wedge *\bar\beta) = d\alpha\wedge *\bar\beta \pm \alpha\wedge **d*\bar\beta$. The adjoint of d with respect to this inner product is $d^* = \pm d$ by Stokes' theorem. The Laplacian (aka Hodge Laplacian, or Laplace-Beltrami operator) $\Delta = dd^* + d^*d$. This coincides with minus the classical Laplacian on functions; the reason that this sign is preferred is that one wants the spectrum to be nonnegative. A form α is harmonic if $\Delta\alpha = 0$.

Lemma 1.1. *α is harmonic if and only if $d\alpha = d(*\alpha) = 0$.*

Proof. One direction is clear. The converse is proved by observing that

$$\langle \Delta\alpha, \alpha\rangle = ||d\alpha||^2 + ||*d*\alpha||^2 \tag{1}$$

□

The forms $dz = dx + idy$ and $d\bar z = dx - idy$ are easily seen to be eigenforms of $*$ with eigenvalue $-i$ and $+i$ respectively. A form is of type $(1,0)$ (resp. $(0,1)$) if it is a multiple of dz (resp. $d\bar z$) for every local coordinate z. From the lemma, one quickly deduces that

Corollary 1.2. *A $(1,0)$ (resp. $(0,1)$) form is harmonic if and if it is holomorphic (resp. antiholomorphic or equivalently the conjugate of a holomorphic form). Every harmonic form is a sum of holomorphic form and an antiholomorphic form, and this decomposition is unique.*

Proof. If α is $(1,0)$, then $d\alpha = \bar\partial\alpha$ and $d*\alpha = -i\bar\partial\alpha$. So α is harmonic if and only if it is holomorphic. The $(0,1)$ case is similar.

Any 1-form α can be uniquely decomposed into a sum

$$\alpha = \alpha^{(1,0)} + \alpha^{(0,1)}$$

of a $(1,0)$ and $(0,1)$ form. We can see that

$$\alpha^{(1,0)} = \frac{\alpha + i*\alpha}{2}$$
$$\alpha^{(0,1)} = \frac{\alpha - i*\alpha}{2}$$

Therefore α is harmonic if and only if these components are. □

Theorem 1.3 (Hodge theorem)**.** *The space of C^∞ 1-forms decomposes*

$$\mathcal{E}^1(X) = \{\textit{harmonic } 1\textit{-forms}\} \oplus dC^\infty(X) \oplus *dC^\infty(X)$$

Proof. We give the bare outline of the proof. Details can be found in [D, GH]. It suffices to establish

$$\mathcal{E}^1(X) = \ker\Delta \oplus \operatorname{im}\Delta \tag{2}$$

Equation (1) implies that $\ker\Delta = (\operatorname{im}\Delta)^\perp$. So it is enough to establish

$$\mathcal{E}^1(X) = (\operatorname{im}\Delta)^\perp \oplus \operatorname{im}\Delta$$

For infinite dimensional inner product spaces, such a decomposition is not automatic. However, if we complete $\mathcal{E}^1(X)$ to a Hilbert space $L^2\mathcal{E}^1(X)$, then

$$L^2\mathcal{E}^1(X) = (\operatorname{im}\Delta)^\perp \oplus \overline{\operatorname{im}\Delta}$$

is true by standard functional analysis, where to be clear $()^\perp$ now refers to the orthogonal complement in Hilbert space. But we need to bring things back to the C^∞ world. Given $\alpha, \beta \in L^2\mathcal{E}^1(X)$, we say that

$$\Delta\alpha = \beta \tag{3}$$

holds weakly if it holds in sense of distribution theory or equivalently

$$\langle \alpha, \Delta\gamma \rangle = \langle \beta, \gamma \rangle$$

holds for all C^∞ forms γ. We can see that $(\operatorname{im}\Delta)^\perp$ consists of weak solutions to $\Delta\alpha = 0$. A bit more work shows $\overline{\operatorname{im}\Delta}$ consists of β's for which (3) is weakly solvable. The main analytic result needed at this point is Weyl's "lemma" or the regularity theorem for Δ: if β is C^∞, then a weak solution α to (3) is a true C^∞ solution. It follows that elements of $(\operatorname{im}\Delta)^\perp$ are harmonic forms. We can also see that $\overline{\operatorname{im}\Delta} \cap \mathcal{E}^1(X) = \Delta\mathcal{E}^1(X)$. So that (2) follows. □

Although the following corollary is the most important consequence, we will need the full version of the theorem.

Corollary 1.4. *The first de Rham cohomology of X is isomorphic to the space of harmonic 1-forms*

Proof. By definition

$$H^1(X, \mathbb{C}) = \frac{\ker d}{\operatorname{im} d}$$

on 1-forms. Since $\ker d$ is orthogonal to $im\, d^*$, we have

$$H^1(X, \mathbb{C}) = \frac{\{\text{harmonic 1-forms}\} \oplus dC^\infty(X)}{dC^\infty(X)}$$

□

Corollary 1.5. *The dimension of the space of holomorphic (resp. antiholomorphic) forms is g.*

Proof. By corollaries 1.2 and 1.4

$$2g = \dim H^1(X, \mathbb{C}) = \dim\{\text{holmorphic forms}\} + \dim\{\text{antiholmorphic forms}\}$$

However, the last two spaces are have the same dimension because complex conjugation gives a real isomorphism. □

2. Some sheaf theory

A presheaf (of abelian groups) on X (or any space) is an assignment of an abelian group $U \mapsto \mathcal{A}(U)$ to each open set, and a homomorphism $\rho_{UV} : \mathcal{A}(U) \to \mathcal{A}(V)$ to each inclusion $V \subseteq U$ such that $\rho_{UU} = id$ and $\rho_{VW}\rho_{UV} = \rho_{UW}$. When there is no confusion, we generally write $\alpha|_V$ for $\rho_{UV}(\alpha)$. $\mathcal{A}$ is called a sheaf if for any open cover $\{U_i\}$ of U $\mathcal{A}(U)$ is isomorphic by the set of collections $\alpha_i \in \mathcal{A}(U_i)$ such that $\alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}}$, where $U_{ij} = U_i \cap U_j$. For example we have the sheaf $\mathcal{O}_X$ of holomorphic functions. Here some more examples.

Example 2.1. *Given a divisor $D = \sum n_p p$, let*

$$\mathcal{O}_X(D)(U) = \{f \text{ merom. on } U \mid \forall p \in U, n_p + ord_p(f) \geq 0\}$$

with obvious restrictions.

Example 2.2. *Given $p \in X$, the sky-scraper sheaf*

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} & \text{if } p \in U \\ 0 & \text{otherwise} \end{cases}$$

In fact, $\mathbb{C}$ can be replaced by any abelian group.

A morphism of sheaves $f : \mathcal{A} \to \mathcal{B}$ is a collection of homomorphisms $f_U : \mathcal{A}(U) \to \mathcal{B}(U)$ which commute with restriction i.e. $f_U(\alpha)|_V = f_V(\alpha|_V)$. A pair of morphisms

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

between sheaves is called exact if for every $p \in X$ and $\epsilon > 0$, there exist a neigbourhoods U contained in an ϵ disk around p (with respect to some metric) such that

$$\mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U)$$

is exact in the usual sense. The point is that condition is local about each p. People familiar with stalks will notice that I am translating the usual condition.

Suppose that we are given an exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

Lemma 2.3. *Then there is an exact sequence*

$$0 \to \mathcal{A}(X) \to \mathcal{B}(X) \to \mathcal{C}(X)$$

of groups

However, $\mathcal{B}(X) \to \mathcal{C}(X)$ need not be surjective; the failure of this to be so will be crucial for us to understand. Fix $\gamma \in \mathcal{C}(X)$. Then by definition of exactness, we can find a cover $\{U_i\}$ of X so that $\gamma|_{U_i}$ lifts to $\beta_i \in \mathcal{B}(U_i)$. If $\beta_i|_{U_{ij}} = \beta_j|_{U_{ij}}$, we can patch these to an element $\beta \in B(X)$ and we would be done. The failure of this is measured by $\alpha_{ij} = \beta_i - \beta_j \in \mathcal{A}(U_{ij})$ where suppress writing restrictions to simplify notation. Note that this collection satisfies the 1-cocycle identity:

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0$$

Thus it seems that we have constructed a map $\gamma \mapsto \alpha_{ij}$, but it is not quite well defined because we needed the auxiliary choice of β_i. Any two such choices of β_i differ by an element of $\mathcal{A}(U_i)$. Thus we are led to define the 1st (Cech) cohomology group $\check{H}^1(\{U_i\}, \mathcal{A})$ as the group (which it clearly is) of cocycles as above, modulo the subgroup of coboundaries which are cocycles of the form $\alpha_{ij} = \alpha_i - \alpha_j$, with $\alpha_i \in \mathcal{A}(U_i)$. This depends on the cover, we can get rid of the dependency by taking the direct limit

$$H^1(X, \mathcal{A}) = \varinjlim_{\text{refinement}} \check{H}^1(\{U_i\}, \mathcal{A})$$

but we won't make say much about this last step. We have essentially constructed a homomorphism $\partial : \mathcal{C}(X) \to H^1(X, \mathcal{A})$ by

$$\partial(\gamma) = \text{class of } \alpha_{ij}$$

The notation becomes more uniform if we write $H^0(X, \mathcal{A}) = \mathcal{A}(X)$ etc.

Theorem 2.4. *The previous sequence extends to a 6 term exact sequence*

$$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \to H^1(X, \mathcal{A}) \to \dots$$

Proofs and generalizations can be found in the references [G, GH, H]. The first being the most thorough.

3. Back to Riemann's inequality

We now assume that $D = \sum n_p p \geq 0$, and give a proof in this case. Then $\mathcal{O}_X(D)(U)$ will contain $\mathcal{O}_X(U)$ giving rise to an injective morphism $\mathcal{O}_X \to \mathcal{O}_X(D)$. We want to prolong this to a short exact sequence. To analyze this, let U be a coordinate disk centered at one of the points p of support D and disjoint from the remaining points. Then $\mathcal{O}(D)(U) = \frac{1}{z^{n_p}}\mathcal{O}(U)$. The quotient $\mathcal{O}(D)(U)/\mathcal{O}(U) \cong \mathbb{C}^{n_p}$. So we obtain an exact sequence of sheaves

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \oplus \mathbb{C}_p^{n_p} \to 0$$

giving rise to an exact sequence

$$0 \to \mathcal{O}_X(X) \to \mathcal{O}_X(D)(X) \to \oplus \mathbb{C}^{n_p} \to H^1(X, \mathcal{O}_X)$$

The second space is just $L(D)$. Since X is compact, global holomorphic functions on it are constant by the maximum principle. Therefore we can write the sequence as

$$0 \to \mathbb{C} \to L(D) \to \mathbb{C}^{\deg D} \to H^1(X, \mathcal{O}_X)$$

The rank nullity theorem from linear algebra gives us a lower bound

$$\dim L(D) \geq 1 + \deg D - \dim H^1(X, \mathcal{O})$$

We also get an upper bound

$$\dim L(D) \leq 1 + \deg D$$

which is useful as well. So to finish the proof of Riemann's inequality we just need to prove

Theorem 3.1. *$\dim H^1(X, \mathcal{O}_X) = g$.*

We start by recalling that an element of $H^1(X, \mathcal{O}_X)$ is represented by a cocycle $f_{ij} \in \mathcal{O}(U_{ij})$ on some open cover $\{U_i\}$. Choose a C^∞ partition of unity ψ_i subbordinate to this covering. Set

$$\phi_i = \sum_k \psi_k f_{ik} \tag{4}$$

Then using the cocycle condition and the fact ψ_k is a partition of unity, we have

$$\phi_i - \phi_j = \sum_k \psi_k (f_{ik} - f_{jk}) = f_{ij} \sum_k \psi_k = f_{ij}$$

This says that f_{ij} is a C^∞ coboundary. Applying $\bar{\partial}$ implies that

$$\bar{\partial}(\phi_i - \phi_j) = 0$$

so that $\alpha = \bar{\partial}\phi_i$ is a globally defined $(0,1)$-form on X, which of course depends on the choice of ϕ_i. However, the class

$$[\alpha] \in \{(0,1)\text{-forms}\}/\bar{\partial} C^\infty(X) = H_{\bar{\partial}}^{(0,1)}$$

depend only on the cohomology class associated to f_{ij}

Lemma 3.2. *$H^1(X, \mathcal{O}_X) \cong H_{\bar{\partial}}^{(0,1)}$*

The inverse is given by reversing the process. Given a $(0,1)$ form α, we can write it locally as $\alpha = \bar{\partial}\phi_i$ by variant of Cauchy's formula (cf [GH, p 5]). Then $f_{ij} = \phi_i - \phi_j$ is necessarily a collection holomorphic functions satisfying cocycle identity.

Lemma 3.3. $H^{(0,1)}_{\bar\partial} \cong \{$*antiholomorphic* 1*-forms*$\}$

Given a $(0,1)$-form α, by the Hodge theorem we can decompose it as

$$\alpha = \beta + df + *dg$$

where β is harmonic and f, g are C^∞ functions. If we take $(0,1)$ part of the right side of this equation, it still equals α. The $(0,1)$ part γ of β is still harmonic by corollary 1.2. Therefore

$$\alpha = (\beta + df + *dg)^{(0,1)} = \gamma + \bar\partial f + i\bar\partial g = \gamma + \bar\partial(f + ig)$$

This shows that

$$\{(0,1)\text{-forms}\} = \{\text{antiholomorphic 1-forms}\} + image(\bar\partial)$$

A bit more work shows that the summands are orthogonal, and therefore this a direct sum decomposition. The lemma now follows.

Summarizing, we have an isomorphism

$$H^1(X, \mathcal{O}_X) \cong \{\text{antiholomorphic 1-forms}\}$$

But we know from corollary 1.5 that the last space is g dimensional.

4. Riemann-Roch

So far we have proved Riemann's inequality only for effective divisors. For the general case, it is convenient to first prove a stronger statement, namely the Riemann-Roch theorem.

Theorem 4.1 (Riemann-Roch)**.** *If D is a divisor,*

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) = \deg D + 1 - g$$

Riemann's inequality is an immediate corollary. Of course, this is not the classical statement of Riemann-Roch but the modern version due to Serre, which seems more natural (at least to me). The left side

$$\chi(\mathcal{O}_X(D)) = \dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D))$$

can be understood as an Euler characteristic. This point of view led to the subsequent generalizations due to Hirzebruch and Grothendieck.

Proof. If $D = \sum n_p p$, the proof is by induction on $N = N(D) = \sum |n_p|$. The base case $N = 0$ follows immediately from the facts

$$\dim H^0(X, \mathcal{O}_X) = 1, \quad \dim H^1(X, \mathcal{O}_X) = g$$

proved earlier.

We turn to the inductive step. Let $D' = D - p$. Then $N(D) = N(D') \pm 1$. So it suffices to prove that Riemann-Roch holds for D if and only if it holds for D'. We have an inclusion $\mathcal{O}_X(D') \subset \mathcal{O}_X(D)$, which in a neighbourhood of p is given by $\frac{1}{z^{n_p-1}}\mathcal{O}(U) \subset \frac{1}{z^{n_p}}\mathcal{O}(U)$. The cokernel is one dimensional. This leads to an exact sequence

$$0 \to \mathcal{O}_X(D') \to \mathcal{O}_X(D) \to \mathbb{C}_p \to 0$$

of sheaves. Therefore we have an exact sequence

$$0 \to H^0(X, \mathcal{O}_X(D')) \to H^0(X, \mathcal{O}_X(D)) \to H^0(X, \mathbb{C}_p) \to$$
$$H^1(X, \mathcal{O}_X(D')) \to H^1(X, \mathcal{O}_X(D)) \to H^1(X, \mathbb{C}_p)$$

of vector spaces. We have $H^0(X, \mathbb{C}_p) = \mathbb{C}$, and we claim that the first cohomology of this sheaf is 0. Given an open cover $\{U_i\}$ of X, and a cocycle $f_{ij} \in \mathbb{C}_p(U_{ij})$ we have to show that it is a coboundary. Suppose that $p \in U_1$, then $\psi_1 = 1 \in H^0(X, \mathbb{C}_p)$ and $\psi_i = 0, i > 0$ gives a collection of sections which behaves like a partition of unity. So we can apply formula (4) to show that f_{ij} is a coboundary. This proves the claim. This together with the above exact sequence implies that

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D')) + 1$$

Since the right side of Riemann-Roch behaves the same way:

$$[\deg D + (1 - g)] = [\deg D' + (1 - g)] + 1$$

the proof is complete. □

5. Some consequences

Corollary 5.1. *X can be realized as a branched covering of $\mathbb{CP}^1$.*

Proof. X has a nonconstant meromorphic function. □

Corollary 5.2. *Up to isomorphism, there is unique compact Riemann surface of genus 0, namely $\mathbb{CP}^1$.*

Proof. Choose $p \in X$, and regard it as a degree one divisor. Then $\dim L(p) \geq 2$, so there is nonconstant function in $f \in L(p)$. f has a simple pole at p, and no other singularities. Regarding it as a map $f : X \to \mathbb{CP}^1$, it must have degree one. It is easy to see from this it is an isomorphism. □

Corollary 5.3. *A compact Riemann surface of genus 1 is a projective algebraic curve with affine equation of the form*

$$y^2 = x(x - 1)(x - \lambda)$$

with $\lambda \notin \{0, 1\}$.

Sketch. Choose $p \in X$. A nonconstant function in $L(p)$ would lead to an isomorphism $X \cong \mathbb{CP}^1$ as above. Therefore $L(p) = \mathbb{C}$. Riemann's inequality gives $\dim L(2p) \geq 2$. It follows that there exists $f \in L(2p)$ with a double pole at p. We can regard f as degree 2 map from $X \to \mathbb{CP}^1$. The Riemann-Hurwitz formula gives a total of 4 ramification points including $\infty = f(p)$. We can find an automorphism which sends two of the remaining points to 0 and 1. The fourth point is given by some $\lambda \notin \{0, 1\}$. X can now be described as above. □

References

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