Complex Algebraic Varieties and their Cohomology

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Introduction

In these notes, which originated from various "second courses" in algebraic geometry given at Purdue, I study complex algebraic varieties using a mixture of algebraic, analytic and topological methods. I assumed an understanding of basic algebraic geometry (around the level of [Hs]), but little else beyond standard graduate courses in algebra, analysis and elementary topology. I haven't attempted to prove everything, often I have been content to give a sketch along with a reference to the relevent section of Hartshorne [H] or Griffiths and Harris [GH]. These weighty tomes are, at least for algebraic geometers who came of age when I did, the canonical texts. They are a bit daunting however, and I hope these notes makes some of this material more accessible.

These notes are **pretty rough and somewhat incomplete** at the moment. Hopefully, that will change with time. For updates check

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Chapter 1

Manifolds and Varieties via Sheaves

As a first approximation, a manifold is a space, like the sphere, which looks locally like Euclidean space. We really want to make sure that the function theory of a manifold is locally the same as for Euclidean space. Sheaf theory is a natural language in which to make such a definition, although it's rarely presented this way in introductory texts (e. g. [Spv, Wa]). An algebraic variety can be defined similarly as a space which looks locally like the zero set of a collection of polynomials. The sheaf theoretic approach to varieties was introduced by Serre in the early 1950's, and algebraic geometry has never been the same since.

1.1 Sheaves of functions

In many parts of mathematics, one is interested in some class of functions satisfying some condition. We will be interested in the cases where the condition is local in the sense that it can be checked in a neighbourhood of a point. Formulating this precisely leads almost immediately to the concept of a sheaf.

Let X be a topological space, and Y a set. For each open set $U \subseteq X$, let $Map_Y(U)$ be the set of maps from X to Y.

Definition 1.1.1. *A collection of subsets $P(U) \subset Map_Y(U)$ is called a presheaf of (Y-valued) functions on X, if it is closed under restriction, i. e. $f \in P(U) \Rightarrow f|_U \in P(V)$ when $U \subset V$.*

Definition 1.1.2. *A presheaf of functions P is called a sheaf if $f \in P(U)$ whenever there is an open cover $\{U_i\}$ of U such that $f|_{U_i} \in P(U_i)$.*

Example 1.1.3. *Let $P_Y(U)$ be the set of constant functions to Y. This is a presheaf but not a sheaf in general.*

Example 1.1.4. *A function is locally constant if it is constant in a neighbourhood of a point. The set of locally constant functions, denoted by $Y(U)$ or $Y_X(U)$, is a sheaf. It is called the constant sheaf.*

Example 1.1.5. *Let Y be a topological space, then the set of continuous functions $C_Y(U)$ from U to Y is a sheaf. When Y is discrete, this is just the previous example.*

Example 1.1.6. *Let $X = \mathbb{R}^n$, the sets $C^\infty(U)$ of C^∞ real valued functions forms a sheaf.*

Example 1.1.7. *Let $X = \mathbb{C}$ (or $\mathbb{C}^n$), the sets $\mathcal{O}(U)$ of holomorphic functions on U forms a sheaf.*

Example 1.1.8. *Let L be a linear differential operator on $\mathbb{R}^n$ with C^∞ coefficients (e. g. $\sum \partial^2/\partial x_i^2$). Let $S(U)$ denote the space of C^∞ solutions in U. This is a sheaf.*

Example 1.1.9. *Let $X = \mathbb{R}^n$, the sets $L^1(U)$ of L^1-functions forms a presheaf which is not a sheaf.*

We can always force a presheaf to be become a sheaf by the following construction.

Example 1.1.10. *Given a presheaf P of functions to Y. Define the*

$$P^+(U) = \{f : U \to Y \mid \forall x \in U, \exists \text{ a nbhd } U_x \text{ of } x, \text{ such that } f|_{U_x} \in P(U_x)\}$$

This is a sheaf called the sheafification of P.

When P is a sheaf of constant functions, P^+ is exactly the sheaf of locally constant functions. When this construction is applied to the presheaf L^1, we obtain the sheaf of locally L^1 functions.

1.2 Manifolds

Let k be a field.

Definition 1.2.1. *Let R be sheaf of k-valued functions on X. We call R a sheaf of algebras if each $R(U) \subseteq Map_k(U)$ is a subalgebra.*

Definition 1.2.2. *With the above notation, we call the pair (X, R) a concrete ringed over k, or simply a k-space.*

$(\mathbb{R}^n, C_\mathbb{R})$, $(\mathbb{R}^n, C^\infty)$ and $(\mathbb{C}^n, \mathcal{O})$ are examples of $\mathbb{R}$ and $\mathbb{C}$ spaces. An affine variety over k is a k-space.

Definition 1.2.3. *A morphism of k-spaces $(X, R) \to (Y, S)$ is a continuous map $F : X \to Y$ such that $f \in S(U)$ implies $f \circ F \in R(f^{-1}U)$.*

The collection of k-spaces and morphisms form a category. In any category, one has a notion of an isomorphism. Let's spell it out in this case.

Definition 1.2.4. *An isomorphism of k-spaces $(X, R) \cong (Y, S)$ is a homeomorphism $F : X \to Y$ such that $f \in S(U)$ if and only if $f \circ F \in R(F^{-1}U)$.*

Given a sheaf S on X and and open set $U \subset X$, let $S|_U$ denote the sheaf on U defined by $V \mapsto S(V)$ for each $V \subseteq U$.

Definition 1.2.5. *An n-dimensional C^∞ manifold is an $\mathbb{R}$-space (X, C^∞_X) such that*

1. *The topology of X is given by a metric*[1].
2. *X admits an covering $\{U_i\}$ such that each $(U_i, C^\infty_X|_{U_i})$ is isomorphic to $(B_i, C^\infty|_{B_i})$ for some open ball $B \subset \mathbb{R}^n$.*

The isomorphisms $(U_i, C^\infty|_{U_i}) \cong (B_i, C^\infty|_{B_i})$ correspond to coordinate charts in more conventional treatments. The whole collection of data is called an atlas. There a number of variations on this idea:

1. An n-dimensional topological manifold is defined as above but with $(\mathbb{R}^n, C^\infty)$ replaced by $(\mathbb{R}^n, C_{\mathbb{R}})$.
2. An n-dimensional complex manifold can be defined by replacing $(\mathbb{R}^n, C^\infty)$ by $(\mathbb{C}^n, \mathcal{O})$.

One dimensional complex manifolds are usually called Riemann surfaces.

Definition 1.2.6. *A C^∞ map from one C^∞ manifold to another is just a morphism of $\mathbb{R}$-spaces. A holomorphic map between complex manifolds is defined in the same way.*

C^∞ (respectively complex) manifolds and maps form a category; an isomorphism in this category is called a diffeomorphism (respectively biholomorphism). By definition any point of manifold has neighbourhood diffeomorphic or biholomorphic to a ball. Given a complex manifold $(X, \mathcal{O}_X)$, call $f : X \to \mathbb{R}$ C^∞ if and only if $f \circ g$ is C^∞ for each holomorphic map $g : B \to X$ from a ball in $\mathbb{C}^n$. We can introduce a sheaf of C^∞ functions on any n dimensional complex manifold, so as to make it into a $2n$ dimensional C^∞ manifold.

Let consider some examples of manifolds. Certainly any open subset of $\mathbb{R}^n$ ($\mathbb{C}^n$) is a (complex) manifold. To get less trivial examples, we need one more definition.

[1] It's equivalent and perhaps more standard to require that the topology is Hausdorff and paracompact. (The paracompactness of metric spaces is a theorem of Stone. In the opposite direct use a partition of unity to construct a Riemannian metric, then use the Riemannian distance.)

Definition 1.2.7. *Given an n-dimensional manifold X, a closed subset $Y \subset X$ is called a closed m-dimensional closed submanifold if for any point $x \in Y$, there exists a neighbourhood U of x in X and a diffeomorphism of to a ball $B \subset \mathbb{R}^n$ such that $Y \cap U$ maps to a the interesection of B with an m-dimensional linear space.*

Given a closed submanifold $Y \subset X$, define P_Y to be the presheaf which functions on Y which extend to C^∞ functions on X. More precisely, for each open $U \subset Y$, $f \in P_Y(U)$ in a there exists an open $U \subset V \subset X$ such that and $g \in C^\infty_X(V)$ such that $f = g|_V$. Let $C^\infty_Y = P_Y^+$. In other words, C^∞_Y is the sheaf functions which are locally extendible to C^∞ functions on X.

Lemma 1.2.8. *(Y, C^∞_Y) is a manifold.*

We can make an analogous definition for complex manifolds. One can show, using partitions of unity, that locally extendible C^∞ functions are globally extendible, i.e. $C^\infty_Y = P_Y$. However, the corresponding statement for holomorphic functions on complex manifolds is usually false, as the following example shows.

Example 1.2.9. *Let Let $\mathbb{P}^n_{\mathbb{C}} = \mathbb{CP}^n$ be the set of one dimensional subspaces of $\mathbb{C}^{n+1}$. Let $\pi : k^{n+1} - \{0\} \to \mathbb{P}^n_{\mathbb{C}}$ be the natural projection (in the sequel, we often denote $\pi(x_0, \dots x_n)$ by $[x_0, \dots x_n]$). The topology on this is defined in such a way that $U \subset \mathbb{P}^n$ is open if and only if $\pi^{-1}U$ is open. Define a function $f : U \to k$ to be holomorphic exactly when $f \circ \pi$ is holomorphic. Then the presheaf of holomorphic functions $\mathcal{O}_{\mathbb{P}^n}$ is a sheaf, and the pair $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ is an complex manifold. In fact, if we set*

$$U_i = \{[x_0, \dots x_n] \mid x_i \neq i\}$$

Then the map

$$[x_0, \dots x_n] \mapsto (x_0/x_i, \dots \widehat{x_i/x_i} \dots x_n/x_i)$$

induces an isomomorphism $U_i \cong \mathbb{C}^n$.

Example 1.2.10. *Let $Y \subset \mathbb{P}^1$ be a finite set of at least 2 points $p_1, p_2, \dots p_n$. Then the function which takes the value 1 on p_1 and 0 on $p_2, \dots p_n$ is cannot be extended to global holomorphic function on $\mathbb{P}^1$ since all such functions are constant (this follows from Liouville's theorem).*

With this lemma in hand, it's possible to produce many interesting examples of manifolds starting from $\mathbb{R}^n$.

Exercise 1.2.11.

1. *Let $T = \mathbb{R}^n/Z^n$ be a torus. Let $\pi : \mathbb{R}^n \to T$ be the natural projection. Define $f \in C^\infty(U)$ if and only if the pullback $f \circ \pi$ is C^∞ in the usual sense. Show that (T, C^∞) is a C^∞ manifold.*

2. *Let τ be a nonreal complex number. Let $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ and π denote the projection. Define $f \in \mathcal{O}_E(U)$ if and only if the pullback $f \circ \pi$ is holomorphic. Show that E is a Riemann surface. Such a surface is called an elliptic curve.*

3. *Prove lemma 1.2.8 .*

4. *Check that the quadric defined by $x_1^2 + x_2^2 + \ldots + x_k^2 - x_{k+1}^2 \ldots - x_n^2 = 1$ is a closed $n-1$ dimensional submanifold of $\mathbb{R}^n$.*

1.3 Algebraic varieties

Let k be an algebraically closed field. Write $\mathbb{A}_k^n = k^n$. When $k = \mathbb{C}$, we can use the standard metric space topology on this space, and we will refer to this as the *classical topology.* In general, one has only the Zariski topology, but we may use it even when $k = \mathbb{C}$. This topology can be defined to be the weakest topology which makes the polynomials continuous. The closed sets are precisely the sets of zeros

$$V(S) = \{a \in \mathbb{A}^n \mid f(a) = 0 \, \forall f \in S\}$$

of sets of polynomials $S \subset k[x_0, \ldots x_n]$. Sets of this form are also called algebraic. By Hilbert's nullstellensatz the map $I \mapsto V(I)$ is a bijection between the collection of radical ideals of $k[x_0, \ldots x_n]$ an algebraic subsets of $\mathbb{A}^n$. Will call an algebraic set $X \subset \mathbb{A}^n$ an affine variety if it is irreducible, which means that X is not a union of proper closed subsets, or equivalently if $X = V(I)$ with I prime. The Zariski topology of X has a basis given by affine sets of the form $D(g) = X - V(g)$, $g \in k[x_1, \ldots x_n]$ Call a function $F : D(g) \to k$ regular if it can be expressed as a the rational function with a power of g in the denominator i.e. an element of $k[x_1, \ldots x_n][1/g]$. For a general open set $U \subset X$, we determine regularity of $F : U \to k$ by restricting to the basic open sets. With this notation, then:

Example 1.3.1. *Let $\mathcal{O}_X(U)$ denote the set of regular functions. Then this is a sheaf.*

The irreduciblity guarrantees that $\mathcal{O}(X)$ is an integral domain called the coordinate ring of X. This ring determines X completely. Thus $(X, \mathcal{O}_X)$ is a k-space. In analogy with manifolds, we define:

Definition 1.3.2. *A prevariety over k is a k-space $(X, \mathcal{O}_X)$ such that X is connected and there exists a finite open cover $\{U_i\}$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to an affine variety.*

This is "pre" because we'xre missing a "Hausdorff condition". Before explaining what this means, let's do the example of projective space. Let $\mathbb{P}_k^n$ be the set of one dimensional subspaces of k^{n+1}. Let $\pi : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ be the natural projection. The Zariski topology on this is defined in such a way that $U \subset \mathbb{P}^n$ is open if and only if $\pi^{-1}U$ is open. Equivalently, the closed sets are zeros of sets of homogenous polynomials in $k[x_0, \ldots x_n]$. Define a function $f : U \to k$ to be regular exactly when $f \circ \pi$ is regular. Then the presheaf of regular functions $\mathcal{O}_{\mathbb{P}^n}$ is a sheaf, and the pair $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ is easily seen to be a prevariety with affine open cover U_i as in example 1.2.9.

Now let's make the seperation axiom precise. The Hausdorff condition for a space X is equivalent to the requirement that the diagonal $\Delta = \{(x,x) \,|\, x \in X\}$ is closed in $X \times X$ with its product topology. In the case (pre)varieties, we have to be careful about we mean by products. We certainly expect $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$, but notice that topology on this space is not the product topology. In general, the safest way to define products is terms of a universal property. We define a morphism of prevarieties simply as a morphism of k-spaces. This makes the collection of prevarieties into a category. The following can be found in [M]:

Proposition 1.3.3. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ and be prevarieties. Then the Cartesean product $X \times Y$ carries a topology and a sheaf of functions $\mathcal{O}_{X\times Y}$ such that the projections to X and Y are morphisms. If $(Z, \mathcal{O}_Z)$ is any prevariety which maps via morphisms f and g to X and Y then there the map $f \times g : Z \to X \times Y$ is a morphism.*

In outline, the argument goes as follows. If $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$, then the one checks that the prevariety structure associated to $X \times Y \subset \mathbb{A}^{n+m}$ is the right one. If X and Y have affine coverings $\{U_i\}$ and $\{V_j\}$ respectively, then one constructs $X \times Y$ so that $\{U_i \times V_j\}$ gives an open affine covering for it.

Definition 1.3.4. *A prevariety X is a variety if the diagonal $\Delta \subset X \times X$ is closed.*

Clearly affine varieties are varieties in this sense. To show that $\mathbb{P}^n$ is a variety, one needs to check that the topology on $\mathbb{P}^n \times \mathbb{P}^n$ coincides with the one induced by the Segre embedding $\mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{(n+1)(n+1)-1}$.

Further example s can be obtained by taking open or closed subvarieties. Let $(X, \mathcal{O}_X)$ be an algebraic variety over k. A closed irreducible subset $Y \subset X$ is called a closed subvariety. Imitating the construction for manifolds, given an open set $U \subset Y$ $\mathcal{O}_Y(U)$ to be the set functions which extend to a locally to a regular function on X. Then

Proposition 1.3.5. *$(Y, \mathcal{O}_Y)$ is an algebraic variety.*

It is worth making this explicit for closed subvarieties of projective space. Let $X \subset \mathbb{P}^n_k$ be an irreducible Zariski closed set. The affine cone of X is the affine variety $CX = \pi^{-1}X \cup \{0\}$. Now let π denote the restriction of this map to $CX - \{0\}$. Define a function f on an open set $U \subset X$ to be regular when $f \circ \pi$ is regular.

When $k = \mathbb{C}$, we can use the stronger topology on $\mathbb{P}^n_{\mathbb{C}}$ introduced in 1.2.9, and inhereted by subvarieties will be called the classical topology.

Exercise 1.3.6.

1. *Check that $\mathbb{P}^n$ is an algebraic variety.*
2. *Given an open subset U of an algebraic variety X. Let $\mathcal{O}_U = \mathcal{O}_X|_U$. Prove that $(U, \mathcal{O}_U)$ is a variety.*
3. *Prove proposition 1.3.5.*

1.4 Stalks and tangent spaces

Given two functions defined in possibly different neighbourhoods of a point $x \in X$, we say they have the same germ at x if their restrictions to some common neigbourhood agree. This is is an equivalence relation. The germ at x of a function f defined near X is the equivalence class containing f. We denote this by f_x. The stalk P_x of a presheaf of functions P at x, is the set of germs of functions of contained in some $P(U)$. From a more abstract point of view, P_x is nothing but the direct limit

$$\lim_{\rightarrow} P(U)$$

as U varies over neighbourhoods of x.

When R is a sheaf of algebras of functions, then R_x is a commutative ring. In most of the examples considered earlier, R_x is a local ring, i. e. it has a unique maximal ideal.

Lemma 1.4.1. *R_x is a local ring if and only if the following property holds: If $f \in R(U)$ with $f(x) \neq 0$, then $1/f$ is defined and lies in $R(V)$ for some open set $x \in V \subseteq U$.*

Proof. Let m be the set of germs of functions vanishing at x. Then any $f \in R_x - m$ is invertible which implies that m is the unique maximal ideal. $\square$

We'll say that a k-space is locally ringed if each of the stalks are local rings. Manifolds (in all of the above senses) and algebraic varieties are locally ringed. When $(X, \mathcal{O}_X)$ is an n-dimensional complex manifold, the local ring $\mathcal{O}_{X,x}$ can be identified with ring of convergent power series in n variables. When X is variety, the local ring $\mathcal{O}_{X,x}$ is also well understood. We may replace X by an affine variety with coordinate ring R. Consider the maximal ideal

$$m_x = \{f \in R \mid f(x) = 0\}$$

then

Lemma 1.4.2. *$\mathcal{O}_{X,x}$ is isomorphic to the localization R_{m_x}.*

Proof. Let K be the field of fractions of R. A germ in $\mathcal{O}_{X,x}$ is represented by a by regular function in a neighbourhood of x, but this is fraction $f/g \in K$ with $g \notin m_x$. $\square$

In these two cases the local rings are Noetherian. This is easy to check by a theorem of Krull which says that a local ring R with maximal ideal m is Noetherian if and only if $\cap_n m^n = 0$. By contrast, when (X, C^∞) is a C^∞ manifold, the stalks are non Noetherian local rings, since the intersection $\cap_n m^n$ contains nonzero functions such as

$$\begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(see figure 1.1).

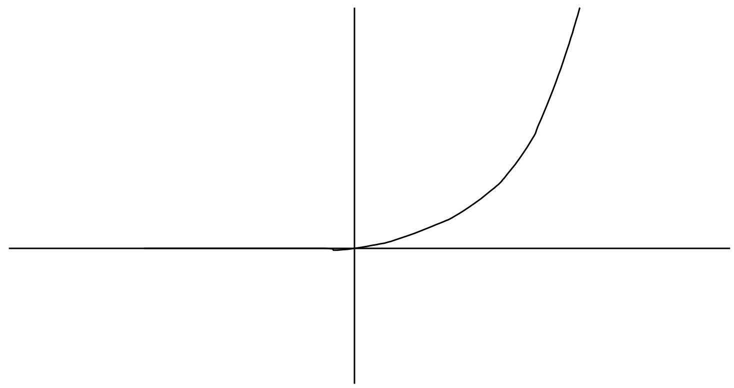


Figure 1.1: function in $\cap_n m^n$

However, the maximal ideals are finitely generated.

Proposition 1.4.3. *If R is the ring of germs at 0 of C^∞ functions on $\mathbb{R}^n$. Then its maximal ideal m is generated by the coordinate functions $x_1, \ldots x_n$.*

If R is local ring with ring with maximal ideal m (which will denote by a pair (R, m)), then R/m is a field called the residue field. The cotangent space of R is the R/m-vector space $m/m^2 \cong m \otimes_R R/m$, and the tangent space is its dual (over R/m). When m is finitely generated, these spaces are finite dimensional. When X is C^∞ or complex manifold or an algebraic variety over k respectively, the local ring $R = \mathcal{O}_{X,x}$ is an algebra over $\mathbb{R}$, $\mathbb{C}$ or k and the map $k \to R \to R/m$ is an isomorphism. In these cases, we denote the tangent space - which is finitef dimensional - by $T_{X,x}$ or simply T_x. When R is the local ring of a manifold or variety X at x, then R/m^2 splits canonically into $k \oplus T_x^*$. Given the germ of a function f, let df be the projection to T_x^*. In other words, $df = f - f(x)$.

Lemma 1.4.4. *$d : R \to T_x^*$ is a k-linear derivation, i. e. it satisfies the Leibnitz rule $d(fg) = f(x)dg + g(x)df$.*

As a corollary, it follows that a tangent vector $v \in T_x = T_x^{**}$ gives rise to a derivation $v \circ d : R \to k$. Conversely, any such derivation corresponds to a tangent vector. In particular,

Lemma 1.4.5. *If (R, m) is the ring of germs at 0 of C^∞ functions on $\mathbb{R}^n$. Then a basis for the tangent space T_0 is given*

$$D_i = \left.\frac{\partial}{\partial x_i}\right|_0 \quad i = 1, \ldots n$$

Exercise 1.4.6.

1. *Prove proposition 1.4.3 (hint: given $f \in m$, let*

$$f_i = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots tx_n)\, dt$$

show that $f = \sum f_i x_i$.)

2. *Prove lemma 1.4.4.*

3. *Let $F : (X, R) \to (Y, S)$ be a morphism of k-spaces. If $x \in X$ and $y = F(x)$, check that the homorphism $F^* : S_y \to R_x$ taking a germ of f to the germ of $f \circ F$ is well defined. When X and S are both locally ringed, show that $F^*(m_y) \subseteq m_x$ where m denotes the maximal ideals.*

4. *When $F : X \to Y$ is a C^∞ map of manifolds, use the previous exercise to construct the induced linear map $dF : T_x \to T_y$. Calculate this for $(X, x) = (\mathbb{R}^n, 0)$ and $(Y, y) = (\mathbb{R}^m, 0)$ and convince yourself that this really is the derivative.*

5. *Check that with the appropriate identification given a C^∞ function on X viewed as a C^∞ map from $f : X \to \mathbb{R}$. df in the sense of 1.4.4 and in the sense of the previous exercise coincide.*

1.5 Nonsingular Varieties

Manifolds are locally quite simple. By contrast algebraic varieties can be locally very complicated. For example, any neighbourhood of the vertex of the cone over a projective variety is as complicated as the variety itself.

We want to say that a point of a variety is nonsingular if it looks like affine space at a microscopic level. The precise definition requires some commutative algebra:

Theorem 1.5.1. *Let $X \subset \mathbb{A}^N_k$ be a closed subvariety defined by the ideal $(f_1, \ldots f_r)$. Choose $x \in X$ and let $R = \mathcal{O}_{X,x}$. Then the following statements are equivalent*

1. *R is a regular local ring i.e. $\dim T_x$ equals the Krull dimension of R.*

2. *The rank of the Jacobian $(\partial f_i / \partial x_j |_x)$ is $N - \dim X$.*

3. *The completion $\hat{R}$ is isomorphic to a ring of formal power series over k.*

If these conditions are fulfilled, x is called a nonsingular point of X. Note that the equivalence of (2) and (3) amounts to a formal implicit function theorem. When $k = \mathbb{C}$, we can apply the holomorphic implicit function theorem to deduce an additional equivalent statement:

4 *There exists a neighbourhood U of $x \in \mathbb{C}^N$ in the usual Euclidean topology, and a biholomorphism (i. e. holomorphic isomorphism) of U to a ball B such that $X \cap U$ maps to the intersection of B and an n-dimensional linear subspace.*

A variety is called nonsingular or smooth if all of its points are nonsingular. Affine and projective spaces are examples of nonsingular varieties. It follows from (4) that a nonsingular affine or projective variety is a complex submanifold of affine or projective space.

1.6 Vector fields

A vector field on a manifold X is a choice $v_x \in T_x$ for each $x \in X$. Of course, we really want this function $x \to v_x$ to be C^∞ in the appropriate sense. There are a few ways to make this precise. For the moment, we will rely on the crutch of coordinates. Choose an atlas

$$F_j : (U_j, C^\infty|_{U_j}) \cong (B_j, C^\infty|_{B_j})$$

then using the construction of the previous exercise, we can push the vector field onto the ball B_j. When we expand this in the basis

$$dF_j v_x = \sum f_i(x) \frac{\partial}{\partial x_i}$$

we require that the components are C^∞ functions. The dual notion is that of 1-form (or covector field). It is a choice of $\omega_x \in T_x^*$ varying in a C^∞ way. This dual notion is perhaps the more fundamental of the two. Given a $\mathbb{C}^i$-function f on X, we can define $df = x \mapsto df_x$. This is a C^∞ 1-form. This allows for a coordinate free formulation of the above notion. A vector field $\{v_x\}$ is C^∞ exactly when the map $x \mapsto v_x \circ df \in C^\infty(X)$ for each $f \in C^\infty(X)$.

Let $\mathcal{T}(X)$ and $\mathcal{E}^1(X)$ denote the space of C^∞ vector fields on X. Then $U \mapsto \mathcal{T}(U)$ and $U \mapsto \mathcal{E}^1(U)$ are easily seen to be sheaves on X denoted by $\mathcal{T}_X$ and $\mathcal{E}^1_X$ respectively. These are prototypes of sheaves of locally free C^∞-modules: Each $\mathcal{T}(U)$ is a $C^\infty(U)$-module, and hence a $C^\infty(V)$-module for any $U \subset V$ and the restriction $\mathcal{T}(V) \to T(U)$ is $C^\infty(V)$-linear. Every point has a neighbourhood U such that are $\mathcal{T}(U)$ and $\mathcal{E}^1(U)$ are free $C^\infty(U)$-modules. More specifically, if U is a coordinate neighbourhood with coordinates $x_1, \ldots x_n$, then $\{\partial/\partial x_1, \ldots \partial/\partial x_n\}$ and $\{dx_1, \ldots dx_n\}$ are bases for $\mathcal{T}(U)$ and $\mathcal{E}^1(U)$ respectively.

These notions are usually phrased in the equivalent language of vector bundles. A rank n (C^∞ real, holomorphic, algebraic) vector bundle is a morphism of C^∞ or complex manifolds or algebraic varieties $\pi : V \to X$ such that there exists an open cover $\{U_i\}$ of X and commutatitve diagrams

$$\begin{tikzcd} \phi_i : \pi^{-1}U_i \arrow[rr, "\cong"] \arrow[dr] & & U_i \times k^n \arrow[dl] \\ & U_i & \end{tikzcd}$$

such that $p_i \circ p_j^{-1}$ are linear on each fiber. Here $k = \mathbb{R}$ or $\mathbb{C}$ in the first two cases. Given a vector bundle $\pi : V \to X$, define the presheaf of sections

$$V(U) = \{s : U \to \pi^{-1}U \mid s \text{ is } C^\infty,\ \pi \circ s = id_U\}$$

This is easily seen to be a sheaf of locally free modules. Conversely, we will see in section 6.3 that every such sheaf arises this way. The vector bundle corresponding to $\mathcal{T}_X$ is called the tangent bundle of X.

Parallel constructions can be carried out for holomorphic (respectively regular) vector fields and forms on complex manifolds and nonsingular algebraic varieties. The corresponding sheaf of forms will be denoted by Ω^1_X.

An explicit example of a nontrivial vector bundle is the tautological bundle. Projective space $\mathbb{P}^n_k$ is the set of lines through 0 in k^{n+1}. Let

$$T = \{(x, l) \in k^{n+1} \times \mathbb{P}^n_k \,|\, x \in l\}$$

Let $P : T \to \mathbb{P}^n_k$ be given by projection onto the second factor. Then T is rank one algebraic vector bundle, or line bundle, over $\mathbb{P}^n_k$. When $k = \mathbb{C}$ this can also be regarded as holomorphic line bundle.

Exercise 1.6.1.

1. *Let $S = S^{n-1} \subset \mathbb{R}^n$ denote the unit sphere. Let*

   $$T_S = \{(v, w) \in \mathbb{R}^n \times S \,|\, v \cdot w = 0\}$$

   where $\cdot$ is the usual dot product. Check that the map $T_S \to S$ given by the second projection makes, T_S into a rank $n-1$ vector bundle. This is an explicit model for the tangent bundle of the sphere.

2. *Check that T really is an algebraic line bundle.*

Chapter 2

Generalities about Sheaves

Up to now, we have been dealing with sheaves primarily as a linguistic device; as sets of functions with some properties. Here we want to do sheaf theory proper.

2.1 The Category of Sheaves

It will be convenient to define sheaves of things other than functions. For instance, one might consider sheaves of equivalence classes of functions. For this more general notion of presheaf, the restrictions maps have to be included as part the data:

Definition 2.1.1. *A presheaf P on a topological space X consists of a set $P(U)$ for each open set U, and maps $\rho_{UV} : P(U) \to P(V)$ for each inclusion $V \subset U$ such that:*

1. $\rho_{UU} = id_{P(U)}$
2. $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$

We will usually write $f|_V = \rho_{UV}(f)$.

Definition 2.1.2. *A presheaf P is a sheaf if for any open covering $\{U_i\}$ of U, given $f_i \in P(U_i)$ satisfying $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, there exists a unique $f \in P(U)$ with $f|_{U_i} = f_i$.*

Definition 2.1.3. *A morphism of presheaves $f_U : P \to P'$ is collection of maps $f_U : P(U) \to P'(U)$ which commute with the restrictions. A morphism of sheaves is defined exactly in the same way.*

A special case of a morphism is the notion of a subsheaf of sheaf. This is a morphism of sheaves where each $f_U : P(U) \subseteq P'(U)$ is an inclusion. For example, the sheaf of C^∞-funtions on $\mathbb{R}^n$ is a subsheaf of the sheaf of continuous functions.

Example 2.1.4. *Given a sheaf of rings of functions R over X, and $f \in R(X)$, the map $R(U) \to R(U)$ given multipication by $f|_U$ is a morphism.*

Example 2.1.5. *Let X be a C^∞ manifold, then $d : C^\infty \to \mathcal{E}^1$ is a morphism of sheaves.*

We now can define the category of presheaves of abelian groups $PAb(X)$ on a topological space X, where we require the maps $f_U : P(U) \to P'(U)$ to be homomorphisms. Let $Ab(X)$ be the full subcategory of sheaves of abelian groups on X. Let Ab denote the category of abelian groups. There are number of functors from $PAb(X)$ to Ab. The global section functor $\Gamma(P) = \Gamma(X, P) = P(X)$. For any $x \in X$, define the stalk P_x of the presheaf P at x, as the direct limit $\lim P(U)$ over neighbourhoods of x. Then $P \to P_x$ determines a functor from $PSh(X) \to Ab$.

There is a functor going backwards from $PAb(X) \to Ab(X)$ called sheafication generalizating the previous construction.

Theorem 2.1.6. *The sheafification functor $P \mapsto P^+$ has the following properties:*

1. *There is a canonical morphism $P \to P^+$.*
2. *If P is a sheaf then this morphism is an isomomorphism*
3. *Any morphism from P to a sheaf factors uniquely through $P \to P^+$*
4. *The map $P \to P^+$ induces an isomorphism on stalks.*

We sketch the construction under a mild (an unnecessary assumption) that $P(X)$ contains at least one element, which we will call 0. The construction will be done in two steps. First, we construct presheaf of functions, then we previous construction to make this sheaf.

Set $Y = \prod P_x$. We define a morphism from P to a sheaf P' of Y-valued functions as follows. There is a canonical map $\sigma_x : P(U) \to P_x$ if $x \in U$; if $x \notin U$ then send everything to 0. Then $f \in P(U)$ defines a function $f'(x) = \sigma_x(f)$. Let $P'(U)$ be the set of f' with $f \in P(U)$. Now we define $P^+ = (P')^+$.

2.2 Exact Sequences

We want to point out, for those who like this sort of thing, that the category $Ab(X)$ is an abelian category [GM, Wl] which means, roughly speaking, that it possesses many of the basic constructions and properties of the category of abelian groups. In particular, there is an intrinsic notion of an exactness in this category. We give a nonintrinsic, but equivalent, formulation of this notion. A sequence of sheaves on X

$$A \to B \to C$$

is called exact if and only if

$$A_x \to B_x \to C_x$$

is exact for every $x \in X$.

Lemma 2.2.1. *Let $f : A \to B$ and $g : B \to C$, then $A \to B \to C$ is exact if and only if for any open $U \subseteq X$*

1. *$g_U \circ f_U = 0$.*
2. *Given $b \in B(U)$ with $g(b) = 0$, there exists an open cover $\{U_i\}$ of U and $a_i \in A(U_i)$ such that $f(a_i) = b|_{U_i}$.*

Proof. We will prove one direction. Suppose that $A \to B \to C$ is exact. Given $a \in A(U)$, $g(f(a)) = 0$, since $g(f(a))_x = g(f(a_x)) = 0$ for all $x \in U$.

For each $x \in U$, b_x is the image of a germ in A at x. Choose a representative for this germ in some $A(U_x)$ where U_x is a neighbourhood of x. □

Corollary 2.2.2. *If $A(U) \to B(U) \to C(U)$ is exact for every open set U, then $A \to B \to C$ is exact.*

The converse is false, but we do have:

Lemma 2.2.3. *If*

$$0 \to A \to B \to C \to 0$$

is an exact sequence of sheaves, then

$$0 \to A(U) \to B(U) \to C(U)$$

is exact for every open set U.

Proof. Let $f : A \to B$ and $g : B \to C$ denote the maps. By lemma 2.2.1,$g \circ f = 0$. Suppose $a \in A(U)$ maps to 0 under f, then $f(a_x) = f(a)_x = 0$ for each $x \in U$. Therefore $a_x = 0$ for each $x \in U$, and this implies that $a = 0$.

Suppose $b \in B(U)$ satisfies $g(b) = 0$. Then by lemma 2.2.1, there exists an open cover $\{U_i\}$ of U and $a_i \in A(U_i)$ such that $f(a_i) = b|_{U_i}$. Then $f(a_i - a_j) = 0$, which implies $a_i - a_j = 0$ by the first part. Therefore $a_i - a_j$ patch together to yield an element of $A(U)$.

□

Example 2.2.4. *Let X denote the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Then*

$$0 \to \mathbb{R}_X \to C^\infty_X \xrightarrow{d} \mathcal{E}^1_X \to 0$$

is exact. However $C^\infty(X) \to \mathcal{E}^1(X)$ is not surjective.

To see the first statement, let $U \subset X$ be an open set diffeomorphic to an open interval. Then the sequence

$$0 \to \mathbb{R} \to C^\infty(U) \xrightarrow{f \to f'} C^\infty(U)dx \to 0$$

is exact by calculus. Thus one gets exactness on stalks. For the second, note that the constant form dx is not the differential of a periodic function.

Example 2.2.5. *Let $(X, \mathcal{O}_Y)$ be C^∞ or complex manifold or algebraic variety and $Y \subset X$ a submanifold or subvariety. Let*

$$\mathcal{I}_Y(U) = \{f \in \mathcal{O}_X(U) \mid f|_Y = 0\}$$

then

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

is exact. Example 1.2.10 shows that $\mathcal{O}_X(X) \to \mathcal{O}_Y(X)$ need not be surjective.

Given a sheaf S and a subsheaf $S' \subseteq S$, we can define a new presheaf with $Q(U) = S(U)/S'(U)$ and restriction maps induced from S. In general, this is not a sheaf.

Exercise 2.2.6.

1. *Finish the proof of lemma 2.2.1.*

2. *Give an example of a subsheaf $S' \subseteq S$, where $Q(U) = S(U)/S'(U)$ fails to be a sheaf. Check that*

   $$0 \to S' \to S \to S/S' \to 0$$

   is an exact sequence of sheaves.

3. *Given a morphism of sheaves $f : S \to S'$, define $ker\, f$ to be the subpresheaf of S with $ker\, f(U) = ker[f_U : S(U) \to S'(U)]$. Check that $ker\, f$ is a sheaf, and check that $(ker\, f)_x \cong ker[S_x \to S'_x]$.*

2.3 The notion of a scheme

A scheme is a massive generalization of the notion of an algebraic variety due to Grothendieck. We will give only the basic flavour of the subject. The canonical reference is [EGA]. Hartshorne's book [H] has become the standard introduction to these ideas for a more most people.

Let R be a commutative ring. Let $Spec\, R$ denote the set of prime ideals of R. For any ideal, $I \subset R$, let

$$V(I) = \{p \in Spec\, R \mid I \subseteq p\}.$$

Lemma 2.3.1. *1. $V(IJ) = V(I) \cup V(J)$.*

2. $V(\sum I_i) = \cap_i V(I_i)$,

As a corollary, it follows that the sets of the form $V(I)$ form the closed sets of a topology on $Spec\, R$ called the Zariski topology. Note that when R is the coordinate ring of an affine variety Y over an algebraically closed field k. The Hilbert Nullstellensatz shows that any maximal ideal of R is of the form $m_y = \{f \in R \mid f(y) = 0\}$ for a unique $y \in Y$. Thus we can embed Y into $Spec\, R$

by sending y to m_y. Under this embedding $V(I)$ pulls back to the algebraic subset

$$\{y \in Y \mid f(y) = 0, \forall f \in I\}$$

Thus this notion of Zariski topologu is an extension of the classical one.

A basis is the Zariski topology on $X = Spec\, R$ is given by $D(f) = X - V(f)$. Thus any open set $U \subset X$ is a union of such sets. Define

$$\mathcal{O}_X(U) = \lim_{\rightarrow} R[\frac{1}{f}]$$

When R is an integral domain with fraction field K, $\mathcal{O}_X(U) \subset K$ consists of the elements r such that for any $p \in U$, $r = g/f$ with $f \notin p$. This remark applies, in particular, to the case where R is the coordinate ring of an algebraic variety Y. In this case, $\mathcal{O}_X(U)$ can be identified with the ring of regular functions on $U \cap Y$ under the above embedding.

Lemma 2.3.2. *$\mathcal{O}_X$ is a sheaf of commutative rings such that $\mathcal{O}_{X,p} \cong R_p$ for any $p \in X$.*

Proof. We give the proof in the special case where R is a domain. This implies that X is irreducible, i. e. any two nonempty open sets intersect, because

$$D(g_i) \cap D(g_j) = D(g_i g_j) \neq \emptyset$$

if $g_i \neq 0$. Consequently the constant presheaf K_X with values in K is already a sheaf. $\mathcal{O}_X$ is a subpresheaf of K_X. Let $U = \cup U_i$ be a union of nonempty open sets, and $f_i \in \mathcal{O}_X(U_i)$. Then $f_i = f_j$ as elements of K. Call the common value f. Since $p \in U$ lies in some U_i, f can be written as a fraction with denominator in $R - p$. Thus $f \in \mathcal{O}_X(U)$, and this shows that $\mathcal{O}_X$ is a sheaf.

One sees readily that the stalk $\mathcal{O}_{X,p}$ is the subring of K of fractions where the denominator can be chosen in $R - p$. Thus $\mathcal{O}_{X,p} \cong R_p$. $\square$

The pair $(X, \mathcal{O}_X)$ is called the *affine scheme* associated to R. A scheme consists of a topological space together with a sheaf of rings which is locally isomorphic to an affine scheme. We have seen how to associate an affine scheme to an affine variety. More generally, given a (pre)variety Y over a field k, there exists a scheme X and embedding $Y \hookrightarrow X$ such that $\mathcal{O}_X$ restricts to the sheaf of regular functions on Y. In particular, all the information about Y can be recovered from X. At this point, one can redefine the varieties as special kinds of schemes. We will take up these ideas again in section 16.1.

2.4 Toric Varieties

Toric varieties are an interesting class of varieties that are explicitly constructed by gluing of affine schemes. The beauty of the subject stems from the interplay between the algebraic geometry and the combinatorics. See [F] for further information (including an explanation of the name).

To simplify our discussion, we will consider only two dimensional examples. Let $\langle , \rangle$ denote the standard inner product on $\mathbb{R}^2$. A cone in $\mathbb{R}^2$ is subset of the form

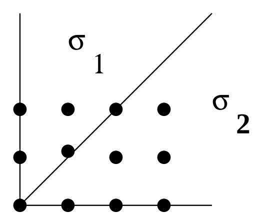
$$\sigma = \{t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \mid t_i \in \mathbb{R}, t_i \geq 0\}$$

It is called rational if the vectors $\mathbf{v}_i$, called generators, can be chosen in $\mathbb{Z}^2$, and strongly convex if the angle between the generators is less than 180^o. If the generators are nonzero and coincide, σ is called a ray. The dual cone can be defined by

$$\sigma^\vee = \{\mathbf{v} \mid \langle \mathbf{v}, \mathbf{w}\rangle \geq 0, \forall \mathbf{w} \in \sigma\}$$

This is rational and spans $\mathbb{R}^2$ if σ is rational and strongly convex. Fix a field k. For each rational strongly convex cone σ, define S_σ to be the subspace of $k[x, x^{-1}, y, y^{-1}]$ spanned by $x^m y^n$ for all $(m, n) \in \sigma^\vee \cap \mathbb{Z}^2$. This easily seen to be a finitely generated subring. The affine toric variety associated to σ is $X(\sigma) = Spec\, S_\sigma$.

A fan Δ in $\mathbb{R}^2$ is a finite collection of nonoverlapping rational strongly convex cones.



Let $\{\sigma_i\}$ be the collection of cones of Δ. Any two cones interest in a ray or in the cone $\{0\}$. The maps $S_{\sigma_i} \to S_{\sigma_i \cap \sigma_j}$ are localizations at single elements. Thus the induced maps

$$\iota_{ij} : X(\sigma_i \cap \sigma_j) \to X(\sigma_i)$$

are inclusions of open sets. We define $X = X(\Delta)$ as a topological space by taking the disjoint union of $X(\sigma_i)$ modulo the equivalence relation generated by $\iota_{ij}(x) \sim \iota_{ji}(x)$ for all $x \in X(\sigma_i \cap \sigma_j)$. A sheaf of rings $\mathcal{O}_X$ can be constructed so that it restricts to $\mathcal{O}_{X(\sigma_i)}$. This turns out to be a variety.

In the example pictured above σ_1 and σ_2 are generated by $(0, 1), (1, 1)$ and $(1, 0), (1, 1)$ respectively. The varieties

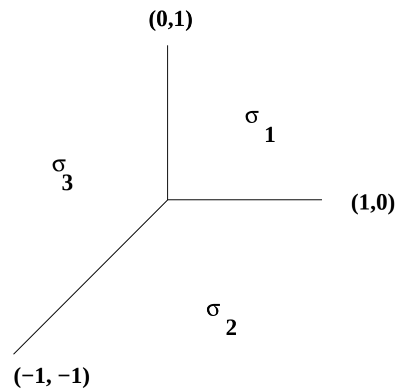
$$X(\sigma_1) = Spec\, k[x, x^{-1}y] = Spec\, k[x, t]$$

$$X(\sigma_2) = Spec\, k[y, xy^{-1}] = Spec\, k[y, s]$$

are both isomorphic to the affine plane. These can be glued by identifying (x, t) in the first plane with $(y, s) = (xt, t^{-1})$ in the second. We will see this example again in a different way. It is the *blow up* of $\mathbb{A}^2$ at $(0, 0)$.

Exercise 2.4.1.

1. *Show that the toric variety corresponding to the fan:*



is $\mathbb{P}^2$

2.5 Sheaves of Modules

Let $\mathcal{R}$ be a sheaf of commutative rings over a space X. The pair $(X, \mathcal{R})$ is called a ringed space (generalizing the notion of section 1.2). A sheaf of $\mathcal{R}$-modules or simply an $\mathcal{R}$-module is a sheaf M such that each $M(U)$ is an $\mathcal{R}(U)$-module and the restrictions $M(U) \to M(V)$ are $M(U)$-linear. The sheaves form a category $\mathcal{R}$-Mod where the morphisms are morphisms of sheaves $A \to B$ such that each map $A(U) \to B(U)$ is $\mathcal{R}(U)$-linear. This is an fact an abelian category. The notion of exactness in this category coincides with the notion introduced in section 2.2.

We have already seen a number of examples in section 1.6. Here are some more:

Example 2.5.1. *Let R be a commutative ring, and M an R-module. Let $X = Spec\, R$. Let $\tilde{M}(U) = M \otimes_R \mathcal{O}_X(U)$. This is an $\mathcal{O}_X$-module. Such a module is called quasi-coherent.*

It will useful to observe:

Lemma 2.5.2. *The functor $M \to \tilde{M}$ is exact.*

Proof. This follows from the fact that $\mathcal{O}_X(U)$ is a flat R-module. □

Example 2.5.3. *The sheaf $\mathcal{I}_Y$ introduced in example 2.2.5 is an $\mathcal{O}_X$-module. It is called an ideal sheaf.*

Most standard linear algebra operations can be carried over to modules.

Definition 2.5.4. *Given a two $\mathcal{R}$-modules M and N, their direct sum is the sheaf $U \mapsto M(U) \oplus N(U)$.*

Definition 2.5.5. *The dual M^* of a $\mathcal{R}$-module M is the sheaf associated to $U \mapsto Hom_{\mathcal{R}(U)}(M(U), \mathcal{R}(U))$.*

For example the sheaf of 1-forms on a manifold is the dual of the tangent sheaf.

Definition 2.5.6. *Given two $\mathcal{R}$-modules M and N, their tensor product is the sheaf associated to $U \mapsto M(U) \otimes_{\mathcal{R}(U)} N(U)$.*

Given a module M over a commutative ring R, recall that the exterior algebra $\wedge^* M$ (respectively symmetric algebra $S^* M$) is the quotient of the free associative algebra with multiplication $\wedge$ (resp. $\cdot$) by the two sided ideal generated $m \wedge m$ (resp. $(m_1 \cdot m_2 - m_2 \cdot m_1)$). $\wedge^k M$ ($S^k M$) is the submodule generated by products of k elements. If V is a finite dimensional vector space, $\wedge^k V^*$ ($S^k V^*$) can be identified with the set of alternating (symmetric) multilinear forms on V in k-variables. After choosing a basis for V, one sees that $S^k V^*$ are degree k polynomials in the coordinates.

Definition 2.5.7. *When M is an $\mathcal{R}$-module, the kth exterior power $\wedge^k M$ is sheaf associated to $U \mapsto \wedge^k M(U)$ When X is manifold the sheaf of k-forms is $\mathcal{E}^k_X = \wedge^k \mathcal{E}^1_X$.*

Definition 2.5.8. *A module M is locally free (of rank n) if for every point has a neighbourhood U, such that $M|_U$ is isomorphic to a finite (n-fold) direct sum $\mathcal{R}|_U \oplus \ldots \oplus \mathcal{R}|_U$*

Given an $\mathcal{R}$-module M over X, the stalk M_x is an $\mathcal{R}_x$-module for any $x \in X$. If M is locally free, then each stalk is free of finite rank. Note that the converse may fail.

As noted in section 1.6, locally free sheaves arise from vector bundles. Let T be the tautological line bundle on projective space $\mathbb{P} = \mathbb{P}^n_k$ over an algebraically closed field k. The sheaf of regular sections is denoted by $\mathcal{O}_{\mathbb{P}}(-1) = \mathcal{O}_{\mathbb{P}^n_k}(-1)$. $\mathcal{O}_{\mathbb{P}}(1)$ is the dual and

$$\mathcal{O}_{\mathbb{P}}(m) = \begin{cases} S^m O(1) = O(1) \otimes \ldots O(1) \ (m \text{ times}) \text{ if } m > 0 \\ \mathcal{O}_{\mathbb{P}} \text{ if } m = 0 \\ S^{-m} O(-1) = O(-m)^* \text{ otherwise} \end{cases}$$

Let $V = k^{n+1}$. By construction $T \subset V \times \mathbb{P}^n$, so $\mathcal{O}_{\mathbb{P}}(-1)$ is a subsheaf of the $n+1$-fold sum $\mathcal{O}_{\mathbb{P}} \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}}$ which can be expressed more canonically as $V \otimes_k \mathcal{O}_{\mathbb{P}}$. Dualizing, gives

$$V^* \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(1) \to 0$$

Taking symmetric powers gives a map, in fact an epimorphism

$$S^m V^* \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(m) \to 0$$

when $m \geq 0$. Taking global sections gives maps

$$S^m V^* \to S^m V^* \otimes \Gamma(\mathcal{O}_{\mathbb{P}}) \to \Gamma(\mathcal{O}_{\mathbb{P}}(m))$$

We will see later that these maps are isomorphisms. Thus the global sections of $\mathcal{O}_{\mathbb{P}}(m)$ are homogenous degree m polynomials in the homogeneous coordinates of m.

Exercise 2.5.9.

1. *Show that the stalk of $\tilde{M}$ at p is precisely the localization M_p.*
2. *Show that direct sums, tensor products, exterior, and symmetric powers of locally free sheaves are locally free.*

2.6 Direct and Inverse images

Given any subset $S \subset X$ of a topological space and a presheaf $\mathcal{F}$, define

$$\mathcal{F}(S) = \lim_{\rightarrow} \mathcal{F}(U)$$

as U ranges over all open neigbourhoods of S. If S is point, this is just the stalk.

Let $F : X \to Y$ be a continous map of topological spaces. Given a sheaf A on X, define the the direct image sheaf F_*A on Y by $F_*A(U) = A(F^{-1}U)$ with obvious restrictions. If B is a sheaf on Y, define $F^{-1}B(V) = B(F(V))$. (The lack of symmetry in the notation will be fixed in a moment.) These operations are clearly functors $f_* : Ab(X) \to Ab(Y)$ and $f^{-1} : Ab(Y) \to Ab(X)$. The relationship is given by the adjointness property:

Lemma 2.6.1. *There is a natural isomorphism*

$$Hom_{Ab(X)}(f^{-1}A, B) \cong Hom_{Ab(Y)}(A, f_*B)$$

Given a morphism F of k-spaces $(X, \mathcal{R}) \to (Y, \mathcal{S})$ (section 1.2), we get a morphism of sheaves rings $\mathcal{S} \to F_*\mathcal{R}$ given by $f \mapsto f \circ F$. More generally, we can define a morphism of ringed spaces to be a continuous map together a morphism of rings $\mathcal{S} \to F_*\mathbb{R}$. Let $f^{-1}S \to R$ be the adjoint map. Given an $\mathcal{R}$-module M, f_*M is naturally an $f_*\mathcal{R}$-module, and hence an $\mathcal{S}$-module by restriction of scalars. Similarly given an $\mathcal{S}$-module N, $f^{-1}N$ is naturally an $f^{-1}S$-module. We define the $\mathcal{R}$-module

$$f^*N = \mathcal{R} \otimes_{f^{-1}S} f^{-1}N$$

The inverse image of a locally sheaf is locally free. This has an an interpretation in the context of vector bundles 1.6. If $\pi : V \to Y$ is a vector bundle, then the vector bundle $f^*V \to X$ can be defined in the context of manifolds or varieties. Set theoretically, it is the projection

$$f^*V = \{(v, x) \,|\, \pi(v) = f(x)\} \to X$$

Then

$$f^*(\text{sheaf of sections of } V) = (\text{sheaf of sections of } f^*V)$$

Exercise 2.6.2.

1. *Check that f_*A and $f^{-1}B$ are really sheaves.*
2. *Prove lemma 2.6.1.*
3. *Generalize lemma 2.2.3 to show that an exact sequence $0 \to A \to B \to C \to 0$ of sheaves gives rise to an exact sequence $0 \to f_*A \to f_*B \to f_*C$.*

Chapter 3

Sheaf Cohomology

In this chapter, we give a rapid introduction to sheaf cohomology. It lies at the heart of everything else in these notes.

3.1 Flabby Sheaves

A sheaf $\mathcal{F}$ on X is called flabby (or flasque) if the restriction maps $\mathcal{F}(X) \to \mathcal{F}(U)$ are surjective for any nonempty open set. Their importance stems from the following:

Lemma 3.1.1. *If $0 \to A \to B \to C$ is an exact sequence of sheaves with A flabby, then $B(X) \to C(X)$ is surjective.*

Proof. We will prove this by the no longer fashionable method of transfinite induction[1]. Let $\gamma \in C(X)$. By assumption, there is an open cover $\{U_i\}_{i\in I}$, such that $\gamma|_{U_i}$ lifts to a section $\beta_i \in B(U_i)$. By the well ordering theorem, we can assume that the index set I is the set of ordinal numbers less than a given ordinal κ. We will define

$$\sigma_i \in B(\cup_{j<i} U_j)$$

inductively, so that it maps to the restriction of γ. Set $\sigma_1 = \beta_0$. If σ_i exists, let α_i be an extension of $\beta_i - \sigma_i$ to $A(X)$. Then set σ_{i+1} to be σ_i on the $U_j, j < i$, and $\beta_i - \alpha_i|_{U_i}$ on U_i. If i is a limit (non-successor) ordinal, then the previous σ's patch to define σ_i. Then σ_κ is a global section of B mapping to γ. $\square$

Corollary 3.1.2. *The sequence $0 \to A(X) \to B(X) \to C(X) \to 0$ is exact if A is flabby.*

Example 3.1.3. *Let X be a space with the property that any open set is connected (e.g. X is irreducible). Then any constant sheaf is flabby.*

[1] For most cases of interest to us, X will have a countable basis, so ordinary induction will suffice

Let $\mathcal{F}$ be a presheaf, define the presheaf $G(\mathcal{F})$ by

$$U \mapsto \prod_{x \in U} \mathcal{F}_x$$

with the obvious restrictions. There is a canonical morphism $\mathcal{F} \to G(\mathcal{F})$.

Lemma 3.1.4. *$G(\mathcal{F})$ is a flabby sheaf, and the morphism is $\mathcal{F} \to G(\mathcal{F})$ is a monomorphism if $\mathcal{F}$ is a sheaf.*

Lemma 3.1.5. *G and $\Gamma \circ G$ are exact functors on the category of sheaves i. e. they preserves exactness.*

Exercise 3.1.6.

1. *Find a proof of lemma 3.1.1 which uses Zorn's lemma.*
2. *Prove that the sheaf of bounded continous real valued functions on $\mathbb{R}$ is flabby*
3. *Prove the same thing for the sheaf of bounded C^∞ functions on $\mathbb{R}$.*
4. *Prove that if $0 \to A \to B \to C$ is exact and A is flabby, then $0 \to f_*A \to f_*B \to f_*C \to 0$ is exact for any continuous map f.*

3.2 Cohomology

Define $C^0(\mathcal{F}) = \mathcal{F}$, $C^1(\mathcal{F}) = coker[\mathcal{F} \to G(\mathcal{F})]$ and $C^{n+1}(\mathcal{F}) = C^1C^n(\mathcal{F})$. Now cohomology can be defined by:

$$\begin{aligned} H^0(X, \mathcal{F}) &= \Gamma(X, \mathcal{F}) \\ H^1(X, \mathcal{F}) &= coker[\Gamma(X, G(\mathcal{F})) \to \Gamma(X, C^1(\mathcal{F}))] \\ H^{n+1}(X, \mathcal{F}) &= H^1(X, C^n(\mathcal{F})) \end{aligned}$$

$H^i(X, -)$ is clearly a functor from $Ab(X) \to Ab$. Another basic property is the following which says in effect that these form a "delta functor" [Gr, H].

Theorem 3.2.1. *Given an exact sequence of sheaves*

$$0 \to A \to B \to C \to 0,$$

there is a long exact sequence

$$0 \to H^0(X, A) \to H^0(X, B) \to H^0(X, C) \to$$

$$H^1(X, A) \to H^1(X, B) \to H^1(X, C) \to \dots$$

First we need:

Lemma 3.2.2. *There is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & G(A) & \to & G(B) & \to & G(C) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C^1(A) & \to & C^1(B) & \to & C^1(C) & & \end{array}$$

Proof. By lemma 3.1.5, there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & G(A) & \to & G(B) & \to & G(C) & \to & 0 \end{array}$$

The snake lemma [AM, GM] (which holds in any abelian category) gives the rest. □

Proof. From the previous lemma and lemmas 2.2.3 and 3.1.5, we get a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \Gamma(G(A)) & \to & \Gamma(G(B)) & \to & \Gamma(G(C)) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Gamma(C^1(A)) & \to & \Gamma(C^1(B)) & \to & \Gamma(C^1(C)) & & \end{array}$$

From the snake lemma, we obtain a 6 term exact sequence

$$0 \to H^0(X, A) \to H^0(X, B) \to H^0(X, C)$$
$$\to H^1(X, A) \to H^1(X, B) \to H^1(X, C)$$

Repeating this with A replaced by $C^1(A), C^2(A) \dots$ allows us to continue this sequence indefinitely. □

Corollary 3.2.3. $B(X) \to C(X)$ *is surjective if* $H^1(X, A) = 0$.

Exercise 3.2.4.

1. *If $\mathcal{F}$ is flabby prove that $H^i(X, \mathcal{F}) = 0$ for $i > 0$. (Prove this for $i = 1$, and that $\mathcal{F}$ flabby implies that $C^1(\mathcal{F})$ is flabby.)*

3.3 Soft sheaves

Up to now, the discussion has been abstract. In this section, we will actually do some computations. We first need to introduce a class of sheaves which are similar to flabby sheaves, but much more plentiful. We assume through out this section that X is a metric space although the results hold under the weaker assumption of paracompactness.

Definition 3.3.1. *A sheaf $\mathcal{F}$ is called soft if the map $\mathcal{F}(X) \to \mathcal{F}(S)$ is surjective for all closed sets.*

Lemma 3.3.2. *If $0 \to A \to B \to C$ is an exact sequence of sheaves with A soft, then $B(X) \to C(X)$ is surjective.*

Proof. The proof is very similar to the proof of 3.1.1. We just indicate the modifications. We can assume that the open cover $\{U_i\}$ consists of open balls. Let $\{V_i\}$ be a new open cover where we shrink the radii of each ball, so that $\bar{V}_i \subset U_i$. Define

$$\sigma_i \in B(\cup_{j<i}\bar{V}_j)$$

inductively as before. □

Corollary 3.3.3. *If A and B are soft then so is C.*

One trivially has:

Lemma 3.3.4. *A flabby sheaf is soft.*

Lemma 3.3.5. *If $\mathcal{F}$ is soft then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

Proof. Lemma 3.3.2 $H^1(\mathcal{F}) = 0$. Lemma 3.3.4 implies that $C^i(\mathcal{F})$ is soft, hence $H^i(\mathcal{F}) = 0$. □

Theorem 3.3.6. *The sheaf $C_{\mathbb{R}}$ of continuous real valued functions on a metric space X is soft.*

Proof. Suppose S is closed subset and $f : U \to \mathbb{R}$ a real valued continuous function defined in a neighbourhood of S. We have to extend the germ of f to X. Let d denote the metric. We extend this to a fuction $d(A, B)$ on pairs of subsets $A, B \subseteq X$ by taking the inf over all $d(a, b)$ with $a \in A$ and $b \in B$. Let $S' = X - U$ and let

$$g(x) = \begin{cases} d(x, S')/\epsilon \text{ if } d(x, S') < \epsilon \\ 1 \text{ otherwise} \end{cases}$$

where $\epsilon = d(S, S')/2$. Then gf extends by 0 to a continous function on X. □

Note that $C_{\mathbb{R}}$ is almost never flabby. We get many more examples of soft sheaves with the following.

Lemma 3.3.7. *Let $\mathcal{R}$ be a soft sheaf of rings, then any $\mathcal{R}$-module is soft.*

Proof. The basic strategy is the same as above. Let f be section of an $\mathcal{R}$-module define in the neighbourhood of a closed set S, and let S' be the complement of this neighbourhood . Since $\mathcal{R}$ is soft, the section which is 1 on S and 0 on S' extends to a global section g. Then gf extends to a global section of the module. □

$U \subset \mathbb{C}$ denote the unit circle, and let $e : \mathbb{R} \to U$ denote the normalized exponential $e(x) = exp(2\pi ix)$. Let us say that X is locally simply connected if every neighbourhood of every point contains a simply connected neighbourhood.

Lemma 3.3.8. *If X is locally simply connected, then the sequence*

$$0 \to \mathbb{Z}_X \to C_{\mathbb{R}} \xrightarrow{e} C_U \to 1$$

is exact.

Lemma 3.3.9. *If X is simply connected and locally simply connected, then $H^1(X, \mathbb{Z}_X) = 0$.*

Proof. Since X is simply connected, any continuous map from X to U can be lifted to a continuous map to its universal cover $\mathbb{R}$. In other words, $C_{\mathbb{R}}(X)$ surjects onto $C_U(X)$. Since $\mathbb{C}_{\mathbb{R}}$ is soft, lemma 3.3.8 implies that $H^1(X, \mathbb{Z}_X) = 0$. $\square$

Corollary 3.3.10. $H^1(\mathbb{R}^n, \mathbb{Z}) = 0$.

Exercise 3.3.11.

1. *Check that the sheaf C^∞ functions on $\mathbb{R}^n$ is soft.*

3.4 Mayer-Vietoris sequence

We will introduce a basic tool for computing cohomology groups which is prelude to Cech cohomology. Let $U \subset X$ be open. For any sheaf, we want to define natural restriction maps $H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F})$. If $i = 0$, this is just the usual restriction. For $i = 1$, we have a commutative square

$$\begin{array}{ccc} \Gamma(X, G(\mathcal{F})) & \to & \Gamma(X, C^1(\mathcal{F})) \\ \downarrow & & \downarrow \\ \Gamma(U, G(\mathcal{F})) & \to & \Gamma(U, C^1(\mathcal{F})) \end{array}$$

which induces a map on the cokernels. In general, we use induction.

Theorem 3.4.1. *Let X be a union to two open sets $U \cup V$, then for any sheaf there is a long exact sequence*

$$\ldots H^i(X, \mathcal{F}) \to H^i(U, \mathcal{F}) \oplus H^j(V, \mathcal{F}) \to H^i(U \cap V, \mathcal{F}) \to H^{i+1}(X, \mathcal{F}) \ldots$$

where the first indicated arrow is the sum of the restrictions, and the second is the difference.

Proof. The proof is very similar to the proof of theorem 3.2.1, so we will just sketch it. Construct a diagram

$$\begin{array}{ccccccccc} 0 & \to & \Gamma(X, G(\mathcal{F})) & \to & \Gamma(U, G(\mathcal{F})) \oplus \Gamma(V, G(\mathcal{F})) & \to & \Gamma(U \cap V, G(\mathcal{F})) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Gamma(X, C^1(\mathcal{F})) & \to & \Gamma(U, C^1(\mathcal{F})) \oplus \Gamma(V, C^1(\mathcal{F})) & \to & \Gamma(U \cap V, C^1(\mathcal{F})) & & \end{array}$$

and apply the snake lemma to get the sequence of the first 6 terms. Then repeat with $C^i(\mathcal{F})$ in place of $\mathcal{F}$. $\square$

Exercise 3.4.2.

1. *Use Mayer-Vietoris to prove that* $H^1(S^1, \mathbb{Z}) \cong \mathbb{Z}$.
2. *Show that* $H^1(S^n, \mathbb{Z}) = 0$ *if* $n \geq 2$.

Chapter 4

De Rham's theorem

In this chapter, we apply the machinery of the last section to the study of C^∞ manifolds.

4.1 Acyclic Resolutions

A complex of abelian groups (or more generally elements in an abelian category) is a possibly infinite sequence

$$\dots F^i \xrightarrow{d^i} F^{i+1} \xrightarrow{d^{i+1}} \dots$$

of groups an homomorphisms satisfying $d^{i+1}d^i = 0$. These condtions guarantee that $image(d^i) \subseteq ker(d^{i+1})$. We denote a complex by $F^\bullet$ and we often suppress the indices on d. The cohomology groups of $F^\bullet$ are defined by

$$H^i(F^\bullet) \cong \frac{ker(d^{i+1})}{image(d^i)}$$

For example, an exact sequence is a complex where the groups are zero.

The connection, between this notion of cohomology and the previous one will be established shortly. A sheaf $\mathcal{F}$ is called acyclic if $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. An acyclic resolution of $\mathcal{F}$ is a exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$$

of sheaves such that each $\mathcal{F}^i$ is acyclic. Given the a complex, the sequence

$$\Gamma(X, \mathcal{F}^0) \to \Gamma(X, \mathcal{F}^1) \to \dots$$

need not be exact, however it is necessarily a complex by functoriallity.

Theorem 4.1.1. *Given an acyclic resolution of $\mathcal{F}$ as above,*

$$H^i(X, \mathcal{F}) \cong H^i(\Gamma(X, \mathcal{F}^\bullet))$$

Proof. Let $K^{-1} = \mathcal{F}$ and $K^i = ker(\mathcal{F}^{i+1} \to \mathcal{F}^{i+2})$. Then there are exact sequences

$$0 \to K^{i-1} \to \mathcal{F}^i \to K^i \to 0$$

Since $\mathcal{F}^i$ are acyclic, theorem 3.2.1 implies that

$$0 \to H^0(K^{i-1}) \to H^0(\mathcal{F}^i) \to H^0(K^i) \to H^1(K^{i-1}) \to 0 \tag{4.1}$$

is exact, and

$$H^j(K^i) \cong H^{j+1}(K^{i-1}) \tag{4.2}$$

for $j > 0$. The sequences (4.1) leads to a commutative diagram

$$\begin{tikzcd}
H^0(K^{i-1}) \arrow[r, dashed] & H^0(\mathcal{F}^i) \arrow[rr] \arrow[dr] & & H^0(\mathcal{F}^{i+1}) \\
& & H^0(K^i) \arrow[ur, dashed] &
\end{tikzcd}$$

where the dashed arrows are injective. Therefore

$$H^0(K^{i-1}) \cong ker[H^0(\mathcal{F}^i) \to H^0(\mathcal{F}^{i+1})]$$

This already implies the first case of the theorem when $i = 0$. This isomorphism together with the sequence (4.1) implies that

$$H^1(K^{i-1}) \cong \frac{ker[H^0(\mathcal{F}^{i+1}) \to H^0(\mathcal{F}^{i+2})]}{image[H^0(\mathcal{F}^i)]}$$

Combining this with the isomorphisms

$$H^{i+1}(K^{-1}) \cong H^i(K^0) \cong \ldots H^1(K^{i-1})$$

of with (4.2) finishes the proof. □

4.2 De Rham's theorem

Let X be a C^∞ manifold and $\mathcal{E}^k = \mathcal{E}^k_X$ the sheaf of k-forms. Note that $\mathcal{E}^0 = C^\infty$.

Theorem 4.2.1. *There exists canonical maps $d : \mathcal{E}^k(X) \to \mathcal{E}^{k+1}(X)$, called exterior derivatives, satisfying the following*

1. *$d : \mathcal{E}^0(X) \to \mathcal{E}^1(X)$ is the operation introduced in section 1.6.*

2. *$d^2 = 0$.*

3. *$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^i \alpha \wedge d\beta$ for all $\alpha \in \mathcal{E}^i(X)$, $\beta \in \mathcal{E}^j(X)$.*

Proof. A complete proof can be found in almost any book on manifolds (e.g. [Wa]). We will only sketch the idea. When X is a ball in $\mathbb{R}^n$ with coordinates x_i, one sees that there is a unique operation satisfying the above rules given by

$$d(f dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots dx_{i_k}$$

This applies to any coordinate chart. By uniqueness, these local d's patch. $\square$

When $X = \mathbb{R}^3$, d can be realized as the $div, grad, curl$ of vector calculus. The theorem tells that $\mathcal{E}^\bullet(X)$ forms a complex. We define the De Rham cohomology groups (actually vector spaces) as

$$H^k_{dR}(X) = H^k(\mathcal{E}^\bullet(X))$$

Notice that the exterior derivative is really a map of sheaves $d : \mathcal{E}^k_X \to \mathcal{E}^{k+1}_X$ satisfying $d^2 = 0$. Thus we have complex. Moreover, $\mathbb{R}_X$ is precisely the kernel of $d : \mathcal{E}^0_X \to \mathcal{E}^1_X$.

Theorem 4.2.2. *The sequence*

$$0 \to \mathbb{R}_X \to \mathcal{E}^0_X \to \mathcal{E}^1_X \ldots$$

is an acyclic resolution of $\mathbb{R}_X$.

Corollary 4.2.3 (De Rham's theorem).

$$H^k_{dR}(X) \cong H^k(X, \mathbb{R})$$

The theorem makes two seperate assertions, first that the complex is exact, then that the sheaves $\mathcal{E}^k$ are acyclic. The exactness follows from:

Theorem 4.2.4 (Poincaré's lemma). *For all* n *and* $k > 0$,

$$H^k(\mathbb{R}^n) = 0.$$

Proof. Assume, by induction, that the theorem holds for $n-1$. Identify $\mathbb{R}^{n-1}$ the hyperplane $x_1 = 0$. Let I be the identity and $R : \mathcal{E}^k(\mathbb{R}^n) \to \mathcal{E}^k(\mathbb{R}^n)$ be restriction to this hyperplane. Note that R commute with d. So if $\alpha \in \mathcal{E}^k(\mathbb{R}^n)$ is closed which means that $d\alpha = 0$. Then $dR\alpha = Rd\alpha = 0$. By the induction assumption, $R\alpha$ is exact which means that it lies in the image of d.

For each k, define a map $h : \mathcal{E}^k(\mathbb{R}^n) \to \mathcal{E}^{k-1}(\mathbb{R}^n)$ by

$$h(f(x_1, \ldots x_n) dx_1 \wedge dx_{i_2} \wedge \ldots) = (\int_0^{x_1} f dx_1) dx_{i_2} \wedge \ldots$$

and

$$h(f dx_{i_1} \wedge dx_{i_2} \wedge \ldots) = 0$$

if $1 \notin \{i_1, i_2, \ldots\}$. Then one checks that $dh + hd = I - R$ (in other words, h is homotopy from I to R). Given $\alpha \in \mathcal{E}^k(\mathbb{R}^n)$ satisfying $d\alpha = 0$. We have $\alpha = dh\alpha + R\alpha$. Which by the above remarks is exact. $\square$

Corollary 4.2.5. *The sequence of theorem 4.2.2 is exact.*

Proof. Any ball is diffeomorphic to Euclidean space, and any point on a manifold has a fundamental system of such neigbourhoods. There the sequence is exact on stalks. □

To prove that the sheaves $\mathcal{E}^k$ are acyclic, it's enough to establish the following.

Lemma 4.2.6. *$\mathcal{E}^0$ is soft.*

In later, we will work with complex valued differential forms. Essentially the same argument shows that $H^*(X, \mathbb{C})$ can be computed using such forms.

Exercise 4.2.7.

1. *We will say that a manifold is of finite type if it has a finite open cover $\{U_i\}$ such that any nonempty intersection of the U_i are diffeomorphic to the ball. Compact manifolds are known to have finite type [Spv, pp 595-596]. Using Mayer-Vietoris and De Rham's theorem, prove that if X is an n-dimensional manifold of finite type, then $H^k(X, \mathbb{R})$ vanishes for $k > n$, and is finite dimensional otherwise.*

4.3 Poincaré duality

Let X be an C^∞ manifold. Let $\mathcal{E}_c^k(X)$ denote the set of C^∞ k-forms with compact support. Since $d\mathcal{E}_c^k(X) \subset \mathcal{E}_c^{k+1}(X)$, we can define compactly supported de Rham cohomology by

$$H^k_{cdR}(X) = H^k(\mathcal{E}_c^\bullet(X)).$$

Lemma 4.3.1. *For all n,*

$$H^k_{cdR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} \text{ if } k = n \\ 0 \text{ otherwise} \end{cases}$$

The computation for Euclidean space suggests that these groups are some opposite to the usual ones. The precise statement, in general, requires the notion of orientation. An orientation on an n dimensional real vector space V is a connected component of $\wedge^n V - \{0\}$. This component serves to tells when an ordered basis $v_1, \ldots v_n$ is positively oriented; it is if $v_1 \wedge \ldots v_n$ lies in this component. An orientation on an n dimensional manifold X is a choice of connected of $\wedge^n T_X$ minus the zero section.

Theorem 4.3.2. *Let X be an oriented n-dimensional manifold. Then*

$$H^k_{cdR}(X) \cong H^{n-k}(X, \mathbb{R})^*$$

With the same notation as above, let $\mathcal{C}^k(U) = \mathcal{E}_c^{n-k}(U)^*$ for any open set $U \subset X$. Given $V \subset U$, $\alpha \in \mathcal{C}^k(U)$, $\beta \in \mathcal{E}^k(V)$, let $\alpha|_V(\beta) = \alpha(\tilde{\beta})$ where $\tilde{\beta}$ is the extension of β by 0. This makes $\mathcal{C}^k$ a presheaf.

Lemma 4.3.3. *$\mathcal{C}^k$ is a sheaf.*

Proof. Let $\{U_i\}$ be an open cover of U, and $\alpha_i \in \mathcal{C}^k(U_i)$. Let $\{\rho_i\}$ be a C^∞ partition of unity subordinate to $\{U_i\}$. Then define $\alpha \in \mathcal{C}^k(U)$ by

$$\alpha(\beta) = \sum_i \alpha_i(\rho_i\beta)$$

Suppose that $\beta \in \mathcal{E}_c^k(U_j)$ is extended by 0 to U. Then $\rho_i\beta$ will be supported in $U_i \cap U_j$. Consequently, $\alpha_i(\rho_i\beta) = \alpha_j(\rho_i\beta)$. Therefore

$$\alpha(\beta) = \alpha_j(\sum_i \rho_i\beta) = \alpha_j(\beta).$$

□

Define a map $\delta : \mathcal{C}^k(U) \to \mathcal{C}^{k+1}(U)$ by $\delta(\alpha)(\beta) = \alpha(d\beta)$. One automatically has $\delta^2 = 0$. Thus one has a complex of sheaves.

The final ingredient is existence of the integral.

Theorem 4.3.4. *Let X be an oriented n-dimensional manifold. There exists a linear map $\int_X : \mathcal{E}_c^n(X) \to \mathbb{R}$ such that $\int_X d\beta = 0$.*

The details can be found in several places such as [Wa]. The last statement is a special case of Stoke's theorem on a manifold. In essence, the construction is similar to the proof of the previous lemma. Using a partition of unity subordinate to an atlas one expresses

$$\int_X \beta = \int_X (\sum_i \rho_i\beta)$$

The right hand itegrals can be expressed in local coordinates in Euclidean space. The orientation is necessary in order to guarantee that these Euclidean integrals are chosen with a consistent sign.

We define a map $\mathbb{R}_X \to \mathcal{C}^0$ induced by the map from the constant presheaf sending $r \to r\int_X$. Then theorem 4.3.2 follows from

Lemma 4.3.5.

$$0 \to \mathbb{R}_X \to \mathcal{C}^0 \to \mathcal{C}^1 \to \dots$$

is an acyclic resolution.

Proof. Lemma 4.3.1 implies that this complex is exact. The sheaves $\mathcal{C}^k$ are soft since they are C^∞-modules. □

We can now use this complex to compute the cohomology of $\mathbb{R}_X$ to get

$$H^i(X,\mathbb{R}) \cong H^i(\mathcal{C}^\bullet(X)) = H^i(\mathcal{E}_c^\bullet(X)^*)$$

and one sees more or less immediately that the right hand space is isomorphic to $H^i_{cdR}(X,\mathbb{R})^*$. This completes the proof of the theorem.

Corollary 4.3.6. *If X is a compact oriented n-dimensional manifold. Then*

$$H^k(X,\mathbb{R}) \cong H^{n-k}(X,\mathbb{R})^*$$

The following is really a corollary of the proof.

Corollary 4.3.7. *If X is a connected oriented n-dimensional manifold. Then the map $\alpha \mapsto \int_X \alpha$ induces an isomorphism (denoted by same symbol)*

$$\int_X : H^n_{cdR}(X,\mathbb{R}) \cong \mathbb{R}$$

From now on, let us suppose that X is compact connected oriented n dimensional manifold. We can make the duality isomorphism much more explicit. The de Rham cohomology is a graded ring with multplication denoted by $\cup$. If α and β are closed (i. e. lie in the kernel of d), then so is $\alpha\wedge\beta$ by theorem 4.2.1. If $[\alpha]$ and $[\beta]$ denote the classes in $H^*_{dR}(X)$ represented by these forms, then define $[\alpha]\cup[\beta] = [\alpha\wedge\beta]$; this is well defined. The following will be proved later on (cor. 7.2.2):

If $f \in H^{n-i}(X,\mathbb{R})^*$, then there exists a unique $\alpha \in H^i(X,\mathbb{R})$ such that $f(\beta) = \int_X \alpha\cup\beta$.

4.4 Fundamental class

Let $Y \subset X$ be a closed connected oriented m dimensional manifold. Denote the inclusion by i. There is a natural restriction map

$$i^* : H^a(X,\mathbb{R}) \to H^a(Y,\mathbb{R})$$

induced by restriction of forms. Using Poincaré duality we get a map going in the opposite direction

$$i_! : H^a(Y,\mathbb{R}) \to H^{a+n-m}(X,\mathbb{R})$$

called the Gysin map. We want to make this more explicit. But first, we need:

Theorem 4.4.1. *There exists an open neigbourhood T, called a tubular neigbourhood, of Y in X and a $\pi : T \to Y$ which makes T a locally trivial rank $(n-m)$ real vector bundle over Y.*

Proof. See [Spv, p. 465]. □

We can factor $i_!$ as a composition

$$H^a(Y) \to H^{a+n-m}_{cdR}(T) \to H^{a+n-m}(X)$$

where the first map is the Gysin map for the inclusion $Y \subset T$ and the second is extension by zero. The first map is an isomorphism since it is dual to

$$H^{m-a}(T) \xrightarrow{\cong} H^{m-a}_{cdR}(Y)$$

Let 1_Y denote constant function 1 on Y. This is the natural generator for $H^0(Y, \mathbb{R})$. The Thom class τ_Y of T is the image of 1_Y under the isomorphism $H^0(Y) \to H^{n-m}_{cdR}(T)$. τ_Y can be reprsented by any closed compactly supported $n-m$ form on T whose integral along any fiber is 1. It is possible to choose a neighbourhood U of a point of Y with local coordinates x_i, such that Y is given by $x_{m+1} = \ldots x_n = 0$ and π is given by $(x_1, \ldots x_n) \mapsto (x_1, \ldots x_m)$. The restriction map

$$H^i_{cdR}(T) \to H^{i-n-m}(U) \otimes H^{n-m}_{cdR}(\mathbb{R}^{n-m})$$

is an isomorphism. Therefore the Thom class can be represented by an expression

$$f(x_{m+1}, \ldots x_n) dx_{m+1} \wedge \ldots dx_n$$

where f is compactly supported in $\mathbb{R}^{n-m}$.

The image of τ_Y in $H^{n-m}(X, \mathbb{R})$ is called the fundamental class $[Y]$ of Y. The basic relation is given by

$$\int_Y i^* \alpha = \int_X [Y] \cup \alpha \tag{4.3}$$

Let $Y, Z \subset X$ be oriented submanifolds such that $dimY + dimZ = n$. Then $[Y] \cup [Z] \in H^n(X, \mathbb{R}) \cong \mathbb{R}$ corresponds to a number $Y \cdot Z$. This has a geometric interpretation. We say that Y and Z are transverse if $Y \cap Z$ is finite and if $T_{Y,p} \oplus T_{Z,p} = T_{X,p}$ for each p in the intersection. Choose ordered bases $v_1(p), \ldots v_m(p) \in T_{Y,p}$ and $v_{m+1}(p) \ldots v_n(p) \in T_{Z,p}$ which are positively oriented with respect to the orientations of Y and Z. We define the intersection number

$$i_p(Y, Z) = \begin{cases} +1 \text{ if } v_1(p) \ldots v_n(p) \text{ is positively oriented} \\ -1 \text{ otherwise} \end{cases}$$

This is easily seen to be independent of the choice of bases.

Proposition 4.4.2. $Y \cdot Z = \sum_p i_p(Y, Z)$

Proof. Choose tubular neighbourhoods T of Y and T' of Z. These can be chosen "small enough" so that $T \cap T'$ is a union of disjoint neighbourhoods around U_p each $p \in Y \cap Z$ diffeomorphic to $\mathbb{R}^n = \mathbb{R}^{\dim Y} \times \mathbb{R}^{\dim Z}$ Then

$$Y \cdot Z = \int_X \tau_Y \wedge \tau_Z = \sum_p \int_{U_p} \tau_Y \wedge \tau_Z$$

Choose coordinates $x_1, \dots x_n$ around p so that Y is given by $x_{m+1} = \dots x_n = 0$ and Z by $x_1 = \dots x_m = 0$. Then as above, the Thom classes of T and T' can be written as

$$\tau_Y = f(x_{m+1}, \dots x_n) dx_{m+1} \wedge \dots dx_n$$

$$\tau_Z = g(x_1, \dots x_m) dx_1 \wedge \dots dx_m$$

Fubini's theorem gives

$$\int_{U_p} \tau_Y \wedge \tau_Z = i_p(Y, Z)$$

$\square$

4.5 Examples

We look as some basic examples to illustrate the previous ideas. Let $T = \mathbb{R}^n/\mathbb{Z}^n$. Let $\{e_i\}$ be the standard basis, and x_i be coordinates on $\mathbb{R}^n$. Then

Proposition 4.5.1. *Every de Rham cohomology class on T contains a unique form with constant coefficients.*

We will postpone the proof until section 7.2.

Corollary 4.5.2. *There is an algebra isomorphism $H^*(T, \mathbb{R}) \cong \wedge^* \mathbb{R}^n$*

Since T is a product of circles, this also follows from repeated application of the Künneth formula:

Theorem 4.5.3. *Let X and Y be C^∞ manifolds, and let $p : X \times Y \to X$ and $q : X \times Y \to Y$ be the the projections. Then the map*

$$\sum \alpha_i \otimes \beta_j \mapsto \sum \alpha_i \wedge \beta_j$$

induces an isomorphism

$$\bigoplus_{i+j=k} H^i(X, \mathbb{R}) \otimes H^j(Y, \mathbb{R}) \cong H^k(X \times Y, \mathbb{R})$$

On the torus, Poincaré duality becomes the standard isomorphism

$$\wedge^k \mathbb{R}^n \cong \wedge^{n-k} \wedge \mathbb{R}^n.$$

If $V_I \subset \mathbb{R}^n$ is the span of $\{e_i \mid i \in I\}$, then $T_I = V/(\mathbb{Z}^n \cap V)$ is a submanifold. Its fundamental class is $dx_{i_1} \wedge \dots dx_{i_d}$, where $i_1 < \dots < i_d$ are the elements of I in increasing order. If J is the complement of I, then $T_I \cdot T_J = \pm 1$.

Next consider, complex projective space $\mathbb{P}^n_{\mathbb{C}}$. Then

$$H^i(\mathbb{P}^n, \mathbb{R}) = \begin{cases} \mathbb{R} \text{ if } 0 \le i \le 2n \text{ is even} \\ 0 \text{ otherwise} \end{cases}$$

This is the basic example for us, and it will be studied further in section 6.2.

Chapter 5

Riemann Surfaces

Recall that Riemann surfaces are the same thing as one dimensional complex manifolds. As such they should be called complex curves and we will later on. For the present, we will stick to the traditional terminology.

5.1 Topological Classification

A Riemann surface can be regarded as a 2 (real) dimensional manifold. It has a canonical orientation: if we identify the real tangent space at any point with the complex tangent space, then for any nonzero vector v, we declare the ordered basis (v, iv) to be positively orientated. Let us now forget the complex structure and consider the purely topological problem of classifying these surfaces up to homeomorphism.

Given two 2 dimensional topological manifolds X and Y with points $x \in X$ and $y \in Y$, we can form new topological manifold $X\#Y$ called the connected sum. To construct this, choose open disks $D_1 \subset X$ and $D_2 \subset Y$ centered around x and y. Then $X\#Y$ is obtained by gluing $X - D_1 \cup S^1 \times [0,1] \cup Y - D_2$ appropriately. Figure (5.1) depicts the connected sum of two tori.

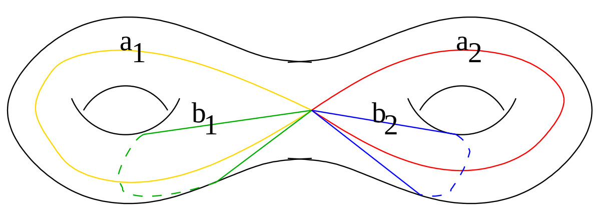


Figure 5.1: Genus 2 Surface

Theorem 5.1.1. *A compact connected orientable 2 dimensional topological manifold is classified, up to homeomorphism, by a nonnegative integer called* *the genus. A genus* 0 *is manifold is homeomorphic to the* 2*-sphere* S^2*. A manifold of genus* $g > 0$ *is homeomorphic to a connected sum of the* 2*-torus and a surface of genus* $g - 1$*.*

There is another standard model for these surfaces which is also quite useful (for instance for computing the fundamental group). Namely that a genus g surface can constructed by gluing the sides of a $2g$-gon. For example, after cutting the genus 2 surface of (5.1) along the indicated curves, it can be opened up to an octagon (5.2).

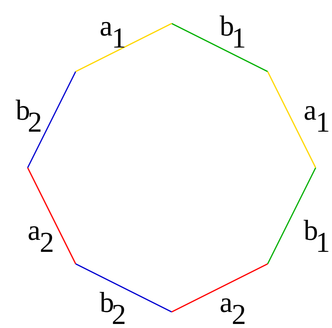


Figure 5.2: Genus 2 surface cut open

The topological Euler characteristic of space X is

$$e(X) = \sum (-1)^i dim\, H^i(X, \mathbb{R}).$$

We need to assume that these dimensions are finite and that all but finitely many of them are zero.

Lemma 5.1.2. *If* X *is a union of two open sets* U *and* V*, then* $e(X) = e(U) + e(V) - e(U \cap V)$*.*

Proof. This follows from the Mayer-Vietoris sequence. □

Corollary 5.1.3. *If* X *is a manifold of genus* g*, then* $e(X) = 2 - 2g$*.*

It is easy to produce Riemann surfaces of every genus. Choose $2g+2$ distinct points in $a_i \in \mathbb{C}$. Let $Y \subset \mathbb{P}^2_{\mathbb{C}}$ be the algebraic curve defined by

$$z^{2g}y^2 - \prod (x - a_i z) = 0$$

where x, y, z are the homogeneous coordinates. This will have a singularity at $[0, 1, 0]$. This can be resolved by normalizing the curve to obtain a smooth projective curve X. Since X is nonsingular, it can be viewed as a Riemann surface. By construction, X comes equipped with a morphism $f : X \to \mathbb{P}^1_{\mathbb{C}}$ which is 2 to 1 except at the branch or ramificationpoints $\{a_i\}$. Curves which can realized as two sheeted ramified coverings or $\mathbb{P}^1_{\mathbb{C}}$ are rather special, and are called hyperelliptic. In the exercises, it will be shown that the genus of X is g. Actually, we're abusing the standard terminology, where the term hyperelliptic is reserved for $g > 1$.

Consider the pairing

$$\alpha \wedge \beta \mapsto \int \alpha \wedge \beta$$

on $H^1(X, \mathbb{C})$. This is skew symmetric and nondegnerate by Poincaré duality (7.2.2). In this case, one can visualize this in terms of intersection numbers of appropriately chosen curves on X. For example, after orientating the curves a_1, a_2, b_1, b_2 in figure (5.1) properly, we get the intersection matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Exercise 5.1.4.

1. *Let S be the toplogical space associated to a finite simplicial complex (jump ahead to the chapter 6 for the definition if necessary). Prove that $e(S)$ is the alternating sum of the number of simplices.*

2. *Check that the genus of the hyperelliptic curve constructed above is g by triangulating in such way that the $\{a_i\}$ are included in the set of vertices.*

5.2 Examples

Many examples of compact Riemann surfaces can be constructed explicitly nonsingular smooth projective curves. We have already done this for hyperelliptic curves.

Example 5.2.1. *Let $f(x, y, z)$ be a homogeneous polynomial of degree d. Suppose that the partials of f have no common zeros in $\mathbb{C}^3$ except $(0, 0, 0)$. Then the $V(f) = \{f(x, y, z) = 0\}$ in $\mathbb{P}^2$ is smooth. We will see later that the genus is $(d-1)(d-2)/2$. In particular, not every genus occurs.*

Example 5.2.2. *Suppose $f(x, y, z)$ is only irreducible, then $V(f)$ may have singularities. After resolving singularities, e. g. by normalizing, we get a Riemman surface. By a generic projection argument one see that every smooth algebraic curve arises this way.*

Example 5.2.3. *Let $f(x, y)$ be a polynomial, nonconstant in both x andy, such that the partials of f have no common zeros in $\mathbb{C}^2$. Projection onto the first factor $V(f) \to \mathbb{C}$ exhibits it as a branched cover. This can be completed to a nonsingular branched cover of $\mathbb{P}^1$. The genus can be calculated by the Riemann-Hurwitz formula.*

From a different point of view, we can construct many examples as quotients of $\mathbb{C}$ or the upper half plane. In fact, the uniformization theorem tells us that all examples other than $\mathbb{P}^1$ arise this way.

Example 5.2.4. *Let $L \subset \mathbb{C}$ be a lattice, i. e. an abelian subgroup generated by two $\mathbb{R}$-linearly independent numbers. The quotient $E = \mathbb{C}/L$ can be made into a Riemann surface (exercise 1.2.11) called an elliptic curve. Since this topologically a torus, the genus is* 1.

This is not an ellipse at all of course. It gets its name because of its relation to elliptic integrals and function. An elliptic function is a meromorphic function on $\mathbb{C}$ which is periodic with respect to the lattice L. A basic example is the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in L,\, \lambda \neq 0} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

This induces a map on the quotient $E \to \mathbb{P}^1$ which is two sheeted and branched at 4 points. One of the branch points will include ∞. We can construct a "hyperelliptic" curve $E' \to \mathbb{P}^1$ with the same branch points. It can be checked that $E \cong E'$, hence E is algebraic.

The group $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\}$ acts on $H = \{z \,|\, im(z) > 0\}$ by fractional linear transformations:

$$z \mapsto \frac{az+b}{cz+d}$$

The action of subgroup $\Gamma \subset PSL_2(\mathbb{R})$ on H is properly discontinuous if every point has a neighbourhood D such that $gD \cap D \neq \emptyset$ for all but finitely many $g \in \Gamma$; it is free and properly discontinuous if $g = I$ is the only such g.

Example 5.2.5. *If Γ acts freely and properly discontinuously on H, the quotient $X = H/\Gamma$ becomes a Riemann surface. If $\pi : H \to X$ denotes the projection, define the structure sheaf $f \in \mathcal{O}_X(U)$ if and only if $f \circ \pi \in \mathcal{O}_H(\pi^{-1}U)$.*

When H/Γ is compact, the quotient has genus $g > 1$. The quickest way to see this is by applying the Gauss-Bonnet theorem to the hyperbolic metric. The fundamental domain for this action will be the interior of a geodesic $2g$-gon. The above construction can be extended to only properly discontinous actions. This is useful since many of the most interesting examples (e.g. $SL_2(\mathbb{Z})$) do have fixed points.

5.3 The $\bar{\partial}$-Poincaré lemma

Let $U \subset \mathbb{C}$ be an open set. Let x and y be real coordinates on $\mathbb{C}$, and $z = x+iy$. Given a complex C^∞ function $f : U \to \mathbb{C}$, let

$$\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}).$$

With this notation, the Cauchy-Riemann equation is simply $\frac{\partial f}{\partial \bar{z}} = 0$.

In order to make it easier to globalize these operators to Riemann surfaces, we reinterpret these in terms of differential forms. In this chapter, $C^\infty(U)$ and $\mathcal{E}^n(U)$ will denote the space of complex valued C^∞ functions and n-forms. The exterior derivative extends to a $\mathbb{C}$-linear operator between these spaces. Define the complex valued 1-forms $dz = dx + idy$ and $d\bar{z} = dx - idy$, and set

$$\partial f = \frac{\partial f}{\partial z} dz$$

$$\bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

We extend this to 1-forms, by

$$\begin{aligned} \partial(f d\bar{z}) &= \frac{\partial f}{\partial z} dz \wedge d\bar{z} \\ \partial(f dz) &= 0 \\ \bar{\partial}(f dz) &= \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz \\ \bar{\partial}(f d\bar{z}) &= 0 \end{aligned}$$

A 1-form α is holomorphic $\alpha = f dz$ with f holomorphic. This is equivalent to $\bar{\partial}\alpha = 0$. The following identities can be easity verified:

$$\begin{aligned} d &= \partial + \bar{\partial} \\ \partial^2 &= \bar{\partial}^2 = 0 \\ \partial\bar{\partial} &+ \bar{\partial}\partial = 0 \end{aligned} \tag{5.1}$$

Theorem 5.3.1. *Let $D \subset \mathbb{C}$ be an open disk. Given $f \in C^\infty(\bar{D})$, there exists $g \in C^\infty(D)$ such that $\frac{\partial g}{\partial \bar{z}} = f$.*

Proof. A solution can be given explicitly as

$$g(\zeta) = \frac{1}{2\pi i} \int_D \frac{f(z)}{z - \zeta} dz \wedge d\bar{z}$$

See [GH, p. 5]. □

5.4 $\bar{\partial}$-cohomology

Let X be a Riemann surface, i. e. a 1-dimensional complex manifold. We write C^∞_X and $\mathcal{E}^i_X$ for the sheaves of complex valued C^∞ functions and i-forms. We define a C^∞-submodule $\mathcal{E}^{(1,0)}_X \subset \mathcal{E}^1_X$ (respectively $\mathcal{E}^{01}_X \subset \mathcal{E}^1_X$), so that for any coordinate neighbourhood U holomorphic cordinate z, $\mathcal{E}_X^{(1,0)}(U) = C^\infty(U)dz$ (resp. $\mathcal{E}_X^{(0,1)}(U) = C^\infty(U)d\bar{z}$). We have a decomposition

$$\mathcal{E}_X^1 = \mathcal{E}_X^{(1,0)} \oplus \mathcal{E}_X^{(0,1)}$$

We set $\mathcal{E}_X^{(1,1)} = \mathcal{E}_X^2$ as this is locally generated by $dz \wedge d\bar{z}$.

Lemma 5.4.1. *There exists $\mathbb{C}$-linear maps $\partial, \bar{\partial}$ on the sheaves $\mathcal{E}_X^\bullet$ which coincide with the previous expressions in local coordinates.*

It follows that the identities (5.1) hold globally, and the kernels of $\bar{\partial}$ are precisely the sheaves of holomorphic functions and forms respectively.

Lemma 5.4.2. *The sequences of sheaves*

$$0 \to \mathcal{O}_X \to C_X^\infty \xrightarrow{\bar{\partial}} \mathcal{E}_X^{(0,1)} \to 0$$

$$0 \to \Omega_X^1 \to \mathcal{E}_X^{(1,0)} \xrightarrow{\bar{\partial}} \mathcal{E}_X^{(1,1)} \to 0$$

are acyclic resolutions.

Proof. Any C^∞-module is soft, hence acyclic, and the exactness follows from theorem5.3.1. □

Corollary 5.4.3.

$$H^1(X, \mathcal{O}_X) = \frac{\mathcal{E}^{(0,1)}(X)}{\bar{\partial} C^\infty(X)}$$

$$H^1(X, \Omega_X^1) = \frac{\mathcal{E}^{(1,1)}(X)}{\bar{\partial}\mathcal{E}_X^{(1,0)}(X)}$$

and

$$H^i(X, \mathcal{O}_X) = H^i(X, \Omega_X^1) = 0$$

if $i > 1$.

Next, we want a holomorphic analogue of the de Rham complex.

Proposition 5.4.4. *There is an exact sequence of sheaves*

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \to 0$$

Proof. The only nontrivial part of the assertion is that $\mathcal{O}_X \to \Omega_X^1 \to 0$ is exact. We can check this by replacing X by a disk D. A holomorphic 1-form on D is automatically closed, therefore exact by the usual Poincaré lemma. If df is holomorphic then $\bar{\partial} f = 0$, so f is holomorphic. □

Corollary 5.4.5. *There is a long exact sequence*

$$0 \to H^0(X, \mathbb{C}) \to H^0(X, \mathcal{O}_X) \to H^0(X, \Omega_X^1) \to \dots$$

Holomorphic 1-forms are closed, and

$$H^0(X, \Omega_X^1) \to H^1(X, \mathbb{C})$$

is the map which sends a holomorphic form to its class in de Rham cohomology.

Lemma 5.4.6. *When X is compact and connected, this map is an injection.*

Proof. It is equivalent to proving that

$$H^0(X, \mathbb{C}) \to H^0(X, \mathcal{O}_X)$$

is surjective i.e. that global holomorphic functions are constant. Let f be a holomorphic function on X. f must attain a maximum at some point, say $x \in X$. Choose a coordinate disk $D \subset X$ centered at x. The we can apply the maximum modulus principle to conclude that f is constant on D. Since $f - f(x)$ has a nonisolated 0, it follows by complex analysis that f is globally constant. $\square$

In section 8.1 we will show that the dimension of $H^0(X, \Omega_X^1)$ is exactly the genus. For now, we prove a weak form of this statement.

Lemma 5.4.7. *If X is compact and connected then dim $H^0(X, \Omega_X^1) \leq g$, where g is the genus.*

Proof. As we have seen, we can view $H^{10} = H^0(X, \Omega_X^1)$ as a subpsace of $H^1(X, \mathbb{C})$. Given two holomorphic 1-forms α, β, $\alpha \wedge \beta = 0$. This implies that H^{10} is an isotropic subpsace under the skew symmetric Poincaré duality pairing. So $dim\, H^{10} \leq \frac{1}{2} dim\, H^1(X, \mathbb{C})$ by linear algebra. $\square$

In the case of the hyperelliptic curve constructed in section 5.1, one can check the opposite inequallity by hand. The expressions $x^i dx/y$ with $0 \leq i < g$ give g linearly elements of $H^0(X\, \Omega_X^1)$ (see exercises).

One additional property whose proof will postponed to section 5.4.3 is that the map

$$H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X)$$

is surjective, or equivalently that $H^1(X, \Omega_X^1)$ injects into $H^2(X, \mathbb{C})$. It follows from this that $dim\, H^1(X, \mathcal{O}_X) = g$. The fact that $H^0(X, \Omega_X) \cong H^1(X, \mathcal{O}_X)$ is usually deduced from the Serre duality theorem.

Exercise 5.4.8.

1. *The differentials $x^i dx/y$ on the hyperelliptic curve are certainly holomorphic on $X - \Sigma$ where $\Sigma = f^{-1}\{a_i\} \cup \{\infty\}$. Check that they holomorphically extend to Σ.*

5.5 Projective embeddings

Fix a compact Riemann surface X. We want to assume a couple of things that stated above but not yet proven, namely that $dim\, H^0(X, \Omega^1_X)$ and $dim\, H^1(X, \mathcal{O}_X)$ both coincide with the genus of X. We introduce some standard shorthand: $h^i = dim\, H^i$, and $\omega_X = \Omega^1_X$.

A divisor D on X is a finite integer linear combination $\sum n_i p_i$ where $p_i \in X$. One says that D is effective if all the coefficients are nonnegative. The degree $deg\, D = \sum n_i$. For every meromphic function defined in a neighbourhood of $p \in X$, let $ord_p(f)$ be the order of vanishing (or minus the order of the pole) of f at p. If D is a divisor define $ord_p(D)$ to be the coefficient of p in D (or 0 if p is absent). Define the sheaf $\mathcal{O}_X(D)$ by

$$\mathcal{O}_X(D)(U) = \{f : U \to \mathbb{C} \cup \{\infty\} \,|\, ord_p(f) + ord_p(D) \geq 0, \forall p \in U\}$$

Lemma 5.5.1. *This is a line bundle.*

Proof. Let U be a coordinate neighbourhood , and let $D = \sum n_i p_i + D'$ where $p_i \in U$ and D' is sum of points not in U. hen we can find functions f_i vanishing at each point $p_i \in U$ to order 1 and nowhere else. It is can be checked that

$$\mathcal{O}_X(D)(U) = \mathcal{O}_X(U)\frac{1}{f_1^{n_1} f_2^{n_2} \dots},$$

which is free of rank one. ∎

Divisors form an abelian group $Div(X)$ in the obvious way.

Lemma 5.5.2. $\mathcal{O}_X(D + D') \cong \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$.

In later terminology, this says that $D \mapsto \mathcal{O}_X(D)$ is a homomorphism from $Div(X) \to Pic(X)$. If D is effective $O(-D)$ is a sheaf of ideals. In particular, $\mathcal{O}_X(-p)$ is exactly the maximal ideal sheaf at p. We have an exact sequence

$$0 \to \mathcal{O}_X(-p) \to \mathcal{O}_X \to \mathbb{C}_p \to 0 \tag{5.2}$$

where

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} \text{ if } p \in U \\ 0 \text{ otherwise} \end{cases}$$

Tensoring (5.2) by $\mathcal{O}_X(D)$ and observing that $\mathbb{C}_p \otimes L \cong \mathbb{C}_p$ for any line bundle, yields

$$0 \to \mathcal{O}_X(D - p) \to \mathcal{O}_X(D) \to \mathbb{C}_p \to 0 \tag{5.3}$$

In the same way, we get a sequence

$$0 \to \omega_X(D - p) \to \omega_X(D) \to \mathbb{C}_p \to 0 \tag{5.4}$$

where $\omega_X(D) = \omega_X \otimes \mathcal{O}_X(D)$.

Lemma 5.5.3. *For all D, $H^i(X, \mathcal{O}_X(D))$ and $H^i(X, \omega_X(D))$ are finite dimensional and 0 if $i > 1$.*

The statement is actually redundant, but we haven't proved that it is.

Proof. Observe that $\mathbb{C}_p$ has no higher cohomology since it is flabby. (5.3) yields

$$0 \to H^0(\mathcal{O}_X(D-p)) \to H^0(\mathcal{O}_X(D)) \to \mathbb{C} \to H^1(\mathcal{O}_X(D-p)) \to H^1(\mathcal{O}_X(D)) \to 0$$

and isomorphisms

$$H^i(\mathcal{O}_X(D-p)) \cong H^i(\mathcal{O}_X(D))\ i > 0$$

By adding or subtracting points, we can reduce this to the case of $\mathcal{O}_X(D) = \mathcal{O}_X$. The argument for $\omega_X(D)$ is the same. $\square$

The residue of a meromorphic 1-form α at p is

$$res_p(\alpha) = \frac{1}{2\pi i}\int_C \alpha$$

where C is any loop "going once counterclockwise" around p and containing no singularities other than p. Alternatively, if $\alpha = f(z)dz$ locally for some local coordinate z at p, $res_p(\alpha)$ is the coefficient of $\frac{1}{z}$ in the Laurant expansion of $f(z)$.

Lemma 5.5.4 (Residue Theorem). *If α is a meromorphic 1-form, it has finitely many singularities, and the sum of the residues is 0.*

Proof. The singularities are isolated hence they form a finite set $\{p_1, \dots p_n\}$. For each i, choose an open disk D_i containing p_i and no other singularity. Then by Stokes' theorem

$$\sum res_{p_i}\alpha = \frac{1}{2\pi i}\int_{X-\cup D_i} d\alpha = 0.$$

$\square$

Theorem 5.5.5. *Suppose that D is a nonzero effective divisor then*

(A) $H^1(\omega_X(D)) = 0$.

(B) $h^0(\omega_X(D)) = deg\, D + g - 1$.

(A) is due to Serre. It is a special case of the Kodaira vanishing theorem. (B) is a weak form of the Riemann-Roch theorem.

Proof. (A) will be proved by induction on the degree of D. Suppose $D = p$. Then $H^0(\omega_X(p))$ consists of the space of meromorphic 1-forms α with at worst a simple pole at p and no other singularities. The residue theorem that such an α must be holomorphic. By the long exact sequence of cohomology groups associated to (5.4), we have

$$0 \to C \to H^1(\omega_X) \to H^1(\omega_X(p)) \to 0$$

Since the space in the middle is one dimensional, this proves (A) in this case. In general, the inductive step follows from a similar application of (5.4).

(B) will again be proved by induction. As already noted when $D = p$, $h^0(\omega_X(D)) = h^0(\omega_X) = g$. In general, (5.4) and (A) shows

$$h^0(\omega_X(D)) = 1 + h^0(\omega_X(D - p)) = deg\, D + g - 1$$

by induction. □

Corollary 5.5.6. *There exists a divisor (called a canonical divisor) such that* $\omega_X \cong \mathcal{O}_X(K)$

Proof. Choose D so that $H^0(\omega_X(D))$ possesses a nonzero section α. Locally $\alpha = f dz$, and we define $ord_p(\alpha) = ord_p(f)$ (this is independent of the coordinate z). Then

$$K = (\alpha) - D = \sum ord_p(\alpha)p - D$$

satisfies the required properties. □

Although, we won't prove that here, the degree of K is known.

Proposition 5.5.7. $deg\, K = 2g - 2$

We say that a line bundle L on X is globally generated if for any point $x \in X$, there exists a section $f \in H^0(X, L)$ such that $f(x) \neq 0$. Suppose that this is the case, Choose a basis $f_0, \ldots f_N$ for $H^0(X, L)$. If fix an isomorphism $\tau : L|_U \cong \mathcal{O}_U$, $\tau(f_i)$ are holomorphic functions on U. Thus we get a holomorphic map $U \to \mathbb{C}^{N+1}$ given by $x \mapsto (\tau(f_i(x)))$. By our assumption, the image lies in the complement of 0, and thus descends to a map to projective space. The image is independent of τ, hence we get a well defined holomorphic map

$$\phi_L : X \to \mathbb{P}^N$$

This map has the property that $\phi_L^* \mathcal{O}_{\mathbb{P}^N}(1) = L$. L is called very ample if ϕ_L is an embedding.

Proposition 5.5.8. *A sufficient condition for L to be globally generated is that $H^1(X, L(-p)) = 0$ for all $p \in X$ A sufficient condition for L to be very ample is that $H^1(X, L(-p - q)) = 0$ for all $p, q \in X$*

Corollary 5.5.9. *$\omega_X(D)$ is very ample if D is nonzero effective with $degD > 2$. In particular, any Riemann surface can be embedded into a projective space.*

5.6 Automorphic forms

Let $\Gamma \subset SL_2(\mathbb{R})$ be a subgroup and k a positive integer, an automorphic form of weight $2k$ is a holomorphic function $f : H \to \mathbb{C}$ on the upper half plane satisfying

$$f(z) = (cz + d)^{-2k} f(\frac{az + b}{cz + d})$$

for each

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

Choose a weight $2k$ automorphic form f. Then $f(z)(dz)^{\otimes k}$ is invariant under the group precisely when f is automorphic of weight $2k$. Since $-I$ acts trivially on H, the action of $SL_2(\mathbb{R})$ factors through $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\}$. Let us suppose that the group $\Gamma/\{\pm I\}/$ acts freely, then the quotient $X = H/\Gamma$ is a Riemann surface, and an automorphic form of weight $2k$ descends to a section of the sheaf $\omega_X^{\otimes k}$. We can apply the previous to the calculate the dimensions of these spaces.

Proposition 5.6.1. *Suppose that $\Gamma/\{\pm I\}$ acts freely on H and that the quotient $X = H/\Gamma$ is compact of genus g. Then the dimension of the space of automorphic functions of weight $2k$ is*

$$\begin{cases} g & \textit{if } k = 1 \\ (g-1)(2k-1) & \textit{if } k > 1 \end{cases}$$

Proof. When $k = 1$, this is clear. When $k > 1$, we have

$$h^0(\omega^{\otimes k}) = h^0(\omega((k-1)K)) = (k-1)(deg\, K) + g - 1 = (2k-1)(g-1)$$

$\square$

Our discussion so far is inadequate since it doesn't deal with examples such as the modular group $SL_2(\mathbb{Z})$. This is a particularly interesting example, since two elliptic curves $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ and $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau'$, with $\tau, \tau' \in H$, are isomorphic if and only if τ and τ' lie in the same orbit of $SL_2(\mathbb{Z})$. The quotient $H/SL_2(\mathbb{Z})$ is noncompact, in fact it is can be identified with $\mathbb{C}$ via the j-function [S4, p. 89]. The natural compactification $\mathbb{P}^1$ can be constructed as a quotient as follows. H has compactification by the circle $\mathbb{R} \cup \{\infty\}$ (this is easier to visualize if we switch to the unit disk model $D \cong H$), however we add only rational points $H^* = D \cup \mathbb{Q} \cup \{\infty\}$ called cusps. Then $SL_2(\mathbb{Z})$ acts on this, and the cusps form a unique orbit corresponding the point at infinity in $\mathbb{P}^1$.

We can apply the same technique to any finite index subgroup of the modular group.

Theorem 5.6.2. *Given a finite index subgroup $\Gamma \subset SL_2(\mathbb{Z})$, H^*/Γ can be made into a compact Riemann surface such that $H^*/\Gamma \to H^*/SL_2(\mathbb{Z}) \cong \mathbb{P}^1$ is holomorphic.*

For example, the nth principle congruence subgroup

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a - 1 \equiv d - 1 \equiv b \equiv c \equiv 0 \mod n \right\}$$

$\Gamma(2)/\{\pm I\}$ acts freely on H, and the quotient is isomorphic to $\mathbb{P}^1 - \{0, 1, \infty\}$. In the unit disk we can choose a fundamental domain for $\Gamma(2)$ as depicted in figure 5.3. The three cusps in the domain correspond to the points $0, 1, \infty$. A pretty consequence of this is Picard's little theorem:

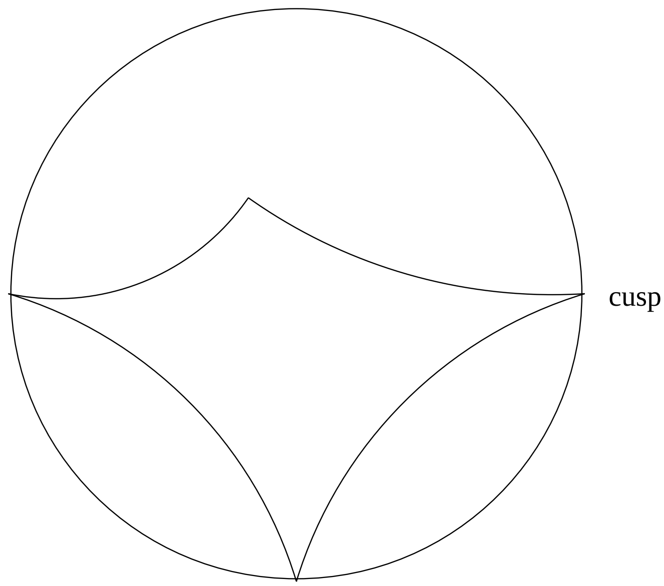


Figure 5.3: Fundamental domain of $\Gamma(2)$

Theorem 5.6.3. *An entire function omitting two or more points must be constant.*

Proof. The universal cover of $\mathbb{P}^1 - \{0, 1, \infty\}$ is H which is isomorphic to the unit disk D. Let f be an entire function omitted two points, which we can assume are 0 and 1. Then f lifts to holomorphic map $\mathbb{C} \to D$ which is bounded and therefore constant by Liouville's theorem. $\square$

Chapter 6

Simplicial Methods

In this chapter, we will develop some tools for actually computing cohomology groups in practice. All of these are based on simplicial methods.

6.1 Simplicial and Singular Cohomology

A systematic development of the ideas in this section can be found in [Sp].

The standard n-simplex is

$$\Delta^n = \{(t_1, \dots t_{n+1}) \in \mathbb{R}^{n+1} \mid \sum t_i = 1,\, t_i \geq 0\}$$

The ith face Δ^n_i is the intersection of Δ^n with the hyperplane $t_i = 0$ (see (6.1). Each face is homeomorphic an $n-1$ simplex by an explicit affine map $\delta_i : \Delta^{n-1} \to \Delta^n_i$. More generally, we refer to the intersection of Δ with the linear space $x_{i_1} = \dots x_{i_k} = 0$ as a face.

Some fairly complicated topological spaces, called polyhedra or triangulable spaces, can be built up by gluing simplices. It is known, although by no means obvious, that manifolds and algebraic varieties (with classical topology) can be triangulated. The combinatorics of the gluing is governed by a simplicial complex. This consists of a set V of vertices, and collection of finite nonempty subsets Σ of V containing all the singletons and closed under taking nonempty subsets. We can construct a space $|(V, \Sigma)|$ out of this roughly as follows. To each maximal element $S \in \Sigma$, choose an n-simplex $\Delta(S)$, where $n+1$ is the cardinality of S. Glue $\Delta(S)$ to $\Delta(S')$ along the face labeled by $S \cap S'$ whenever this is nonempty. (When V is infinite, this gluing process requires some care, see [Sp, chap. 3].)

Let $K = (V, \Sigma)$ be a simplical complex, and assume that V is linearly ordered. We will refer to element of Σ as a n-simplex if it has cardinality $n+1$. We define an n-cochain on a simplicial complex with values in an abelian group A to be function which assigns an element of A to every n-simplex. One can

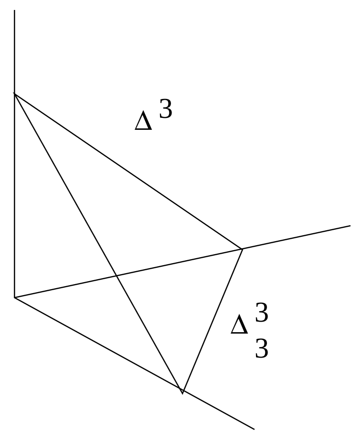


Figure 6.1: 3 simplex

think of a n-cochain some sort of combinatorial analogue of an integral of n-form. As in integration theorem one needs to worry about orientations, and this is where the ordering comes in. An alternative, which is probably more standard, is to use oriented simplices; the complexes one gets this way are bigger, but the resulting cohomology theory is the same. Given an n-cochain F, we define a $(n+1)$-cochain $\partial(F)$. called the coboundary or differential of F. In analogy with Stokes' theorem which allows one to calculate the integral of $d\alpha$ as in integral along the boundary, we define

$$\partial(F)(\{v_0, \ldots v_n\}) = \sum (-1)^i F(\{v_0, \ldots \widehat{v_i} \ldots v_n\}),$$

whenever $v_0 < v_1 \ldots < v_n$. (The notation $\widehat{x}$ means omit x.) Let $C^n(K, A)$ denote the set of n-cochains. This is clearly an abelian group (in fact an A-module if A is ring). We extend $\partial : C^n(K, A) \to C^{n+1}(K, A)$ by linearity. The key relation is

Lemma 6.1.1. $\partial^2 = 0$.

Proof. Let

$$\partial_i(F)(\{v_0, \ldots v_n\}) = F(\{v_0, \ldots \widehat{v_i} \ldots v_n\})$$

so that $\partial = \sum (-1)^i \partial_i$. The lemma follows for the easily verified identity $\partial_j \partial_i = \partial_i \partial_{j-1}$ for $i < j$. □

Thus we have a complex. The simplicial cohomology of K is defined by

$$H^i(K, A) = H^i(C^\bullet(K, A)).$$

Note that when V is finite, these groups are automatically finitely generated and computable. However, triangulations of interesting spaces tend to be quite complicated, so this not a particularly practical method.

When A is replaced by a commutative ring R. There is a product on cohomology analogous to the product in De Rham induced by wedging forms. Given two cochains $\alpha \in C^n(K,R)$, $\beta \in C^m(K,R)$, their cup product $\alpha \cup \beta \in C^{n+m}(K,R)$ is given by

$$\alpha \cup \beta(\{v_0, \ldots v_{n+m}\}) = \alpha(\{v_0, \ldots v_n\})\beta(\{v_n, \ldots v_{n+m}) \tag{6.1}$$

where $v_0 < v_1 < \ldots$. Then:

Lemma 6.1.2. $\partial(\alpha \cup \beta) = \partial(\alpha) \cup \beta + (-1)^n \alpha \cup \partial(\beta)$

Corollary 6.1.3. $\cup$ *induces an operation on cohomology that makes $H^*(K,R)$ into a graded ring.*

Singular cohomology was introduced partly in order to give conceptual proof of the fact that $H^i(K,A)$ depends only on $|K|$, i.e. that simplicial cohomology is independant of the triangulation. It has the advantage, for us, that the relation with sheaf cohomology is easier to establish. A singular n-simplex on a topological space X is simply a continuous map from $f : \Delta^n \to X$. When X is a manifold, we can require the maps to be C^∞. We define a singular n-cochain on a X be a map which assigns an element of A to any n-simplex on X. Let $S^n(X,A)$ ($S^n_\infty(X,A)$) denote the group of (C^∞) n-cochains with values in A. When F is an n-cochain, its coboundary is the $(n+1)$-cochain

$$\partial(F)(f) = \sum (-1)^i F(f \circ \delta_i).$$

The following has more or less the same proof as lemma 6.1.1.

Lemma 6.1.4. $\partial^2 = 0$.

Thus we have a complex. The singular cohomolgy groups of X are

$$H^i_{sing}(X,A) = H^i(S^*(X,A)).$$

A basic property of this cohomology theory is its homotopy invariance. We state this in the form that we will need. A subspace $Y \subset X$ is called a deformation retraction, if there exists a a continuous map $F : [0,1] \times X \to X$ such that $F(0,x) = x$, $F(1,X) = Y$ and $F(1,y) = y$ for $y \in Y$. If Y is a point then X is called contractible.

Proposition 6.1.5. *If $Y \subset X$ is a deformation retraction, then*

$$H^i_{sing}(X,A) \to H^i_{sing}(Y,A)$$

is an isomorphism for any A.

In particular, the higher cohomology vanishes on a contractible space. This is an analogue of Poincaré's lemma. Call space locally contractible if every point has a contractible neighbourhood . Manifolds and varieties with classical topology are examples of such spaces.

Theorem 6.1.6. *If X is paracompact Hausdorff space (e. g. a metric space) which is locally contractible, then $H^i(X, A_X) \cong H^i_{sing}(X, A)$ for any abelian group A.*

A complete proof can be found in [Sp, chap. 6] (note that Spanier uses Čech approach discussed in the next chapter). In the case of manifolds, a proof which is more natural from our point of view can be found in [Wa]. The key step is to consider the sheaves $\mathcal{S}^n$ associated to the presheaves $U \mapsto S^n(U, A)$. These sheaves are soft since they are modules over the sheaf of real valued continuous functions. The local contractability guarantees that

$$0 \to A_X \to \mathcal{S}^0 \to \mathcal{S}^1 \to \dots$$

is a fine resolution. Thus one gets

$$H^i(X, A_X) \cong H^i(\mathcal{S}^*(X))$$

It remains to check that the natural map

$$S^*(X, A) \to \mathcal{S}^*(X)$$

induces an isomorphism on cohomology. We the reader refer to [Wa, pp 196-197] for this.

As a corollary, we obtain the form of De Rham's theorem that most people think of.

Corollary 6.1.7 (De Rham's theorem, version 2). *If X is a manifold,*

$$H^i_{dR}(X, \mathbb{R}) \cong H^i_{sing}(X, \mathbb{R})$$

The theorem holds with C^∞ cochains. The map in the corollary can be defined directly on the level of complexes by

$$\alpha \mapsto (f \mapsto \int_\Delta f^*\alpha)$$

Singular cohomology carries a cup product given by formula (6.1). A stronger form of De Rham's theorem shows that the above map is a ring isomorphism [Wa]. Fundamental classes of oriented submanifolds can be constructed in $H^*(X, \mathbb{Z})$. This explains why the the intersection numbers $Y \cdot Z$ were integers in proposition 4.4.2.

Exercise 6.1.8.

1. *Calculate the simplicial cohomology with $\mathbb{Z}$ coefficients for the "tetrahedron" which is the powerset of $V = \{1, 2, 3, 4\}$ with $\emptyset$ and V removed.*

2. *Let S^n be the n-sphere realized as the unit sphere in $\mathbb{R}^{n+1}$. Let $U_0 = S^n - \{(0, \dots 0, 1)\}$ and $U_1 = S^n - \{(0, \dots 0, -1)\}$. Prove that U_i are contractible, and that $U_0 \cap U_1$ deformation retracts on to the "equatorial" $(n-1)$-sphere.*

3. *Prove that*

$$H^i(S^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } i = 0, n \\ 0 \text{ otherwise} \end{cases}$$

using Mayer-Vietoris.

6.2 $H^*(\mathbb{P}^n, \mathbb{Z})$

Let $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$ with its classical topology.

Theorem 6.2.1.

$$H^i(\mathbb{P}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } 0 \leq i \leq 2n \text{ is even} \\ 0 \text{ otherwise} \end{cases}$$

Before giving the proof, we will need to develop a few more tools. Let X be a space satisfying the assumptions of theorem 6.1.6, and $Y \subset X$ a closed subspace satisfying the same assumptions. We will insert the restriction map

$$H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z})$$

into a long exact sequence. This can be done in a number of ways, by first defining cohomology of the pair (X, Y), using sheaf theory, or using a mapping cone. We will choose the last option. Let C be obtained by first gluing the base of the cylinder $\{1\} \times Y \subset [0, 1] \times Y$ to X along Y, and then collapsing the top to a point P (figure (6.2)).

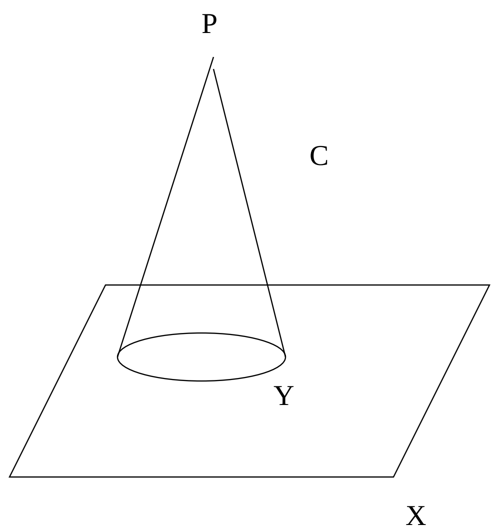


Figure 6.2: Cone

Let $U_1 = C - P$, and $U_2 \subset C$ the open cone $[0,1) \times Y/\{0\} \times Y$ (the notation A/B means collapse B to a point). On sees that U_1 deformation retracts to X, U_2 is contractible, and $U_1 \cap U_2$ deformation retracts to Y. The Mayer-Vietoris sequence, together with these facts, yields a long exact sequence

$$\ldots H^i(C, \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z}) \to H^{i+1}(C, \mathbb{Z}) \ldots$$

when $i > 0$. The make this really useful, note that the map $C \to C/\overline{U_2}$ which collapses the closed cone to a point is a homotopy equivalence. Therefore it induces an isomorphism on cohomology. Since we can identify $C/\overline{U_2}$ with X/Y, we obtain a sequence

$$\ldots H^i(X/Y, \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z}) \to \ldots \tag{6.2}$$

We apply this when $X = \mathbb{P}^n$ and $Y = \mathbb{P}^{n-1}$ embedded as a hyperplane. The complement $X - Y = \mathbb{C}^n$. Collapsing Y to a point amounts to adding a point at infinity to $\mathbb{C}^n$, thus $X/Y = S^{2n}$. Since projective spaces are connected

$$H^0(\mathbb{P}^n, \mathbb{Z}) \cong H^0(\mathbb{P}^{n-1}, \mathbb{Z}) \cong \mathbb{Z}.$$

For $i > 0$, (6.2) and the previous exercise yields isomorphisms

$$H^i(\mathbb{P}^n, \mathbb{Z}) \cong H^i(\mathbb{P}^{n-1}, \mathbb{Z}), \text{ when } i < 2n \tag{6.3}$$

$$H^{2n}(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$$

The theorem follows by induction.

Exercise 6.2.2.

1. *Let $L \subset \mathbb{P}^n$ be a linear subspace of codimension i. Prove that its fundamental class $[L]$ generated $H^{2i}(\mathbb{P}^n, \mathbb{Z})$.*

2. *Let $X \subset \mathbb{P}^n$ be a smooth projective variety. Then $[X] = d[L]$ for some d, where L is a linear subspace of the same dimension. d is called the degree of X. Bertini's theorem, which you can assume, implies that there exists a linear space L' of complementary dimension transverse to X. Check that $X \cdot L' = \#(X \cap L') = d$.*

6.3 Čech cohomology

We return to sheaf theory proper. We will introduce the Čech approach to cohomology which has the advantage of being quite explicit and computable (and the disadvantage of not always giving the "right" answer). Roughly speaking, Čech bears the same relation to sheaf cohomology, as simplicial does to singular cohomology.

One starts with an open covering $\{U_i \,|\, i \in I\}$ of a space X indexed by a totally ordered set I. If $J \subseteq I$, let U_J be the intersection of U_j with $j \in J$. Let $\mathcal{F}$ be a sheaf on X. The group of Cech n-cochains is

$$C^n = C^n(\{U_i\}, \mathcal{F}) = \prod_{i_0 < \ldots < i_n} \mathcal{F}(U_{i_0 \ldots i_n})$$

The coboundary map $\partial : C^n \to C^{n+1}$is defined by

$$\partial(f)_{i_0 \ldots i_{n+1}} = \sum_k (-1)^k f_{i_0 \ldots \hat{i}_k \ldots i_{n+1}}|_{U_{i_0 \cdots i_{n+1}}}$$

where $\hat{i}$ signifies that i should be omitted. This satisfies $\partial^2 = 0$. So we can define the nth Cech cohomology group as

$$\check{H}^n(\{U_i\}, \mathcal{F}) = H^n(C^\bullet) = \frac{ker(\partial : C^n \to C^{n+1})}{im(\partial : C^{n-1} \to C^n)}$$

To get a feeling for this, let us write out the first couple of groups explicitly:

$$\begin{aligned}
\check{H}^0(\{U_i\}, \mathcal{F}) &= \{(f_i) \in \prod \mathcal{F}(U_i) \,|\, f_i = f_j \text{ on } U_{ij} \ \} \\
&= F(X)
\end{aligned}$$

$$\check{H}^1(\{U_i\}, \mathcal{F}) = \frac{\{(f_{ij}) \in \prod \mathcal{F}(U_{ij}) \,|\, f_{ik} = f_{ij} + f_{jk} \text{ on } U_{ijk} \ \}}{\{(f_{ij} \,|\, \exists(\phi_i),\ f_{ij} = \phi_i - \phi_j\}} \tag{6.4}$$

There is a strong similarity with simplicial cohomology. This can be made precise by introducing a simplicial complex called the nerve of the covering. For the set of vertices, we take the index set I. The set of simplices is given by

$$\Sigma = \{\{i_0, \ldots i_n\} \,|\, U_{i_0, \ldots i_n} \neq \emptyset\}$$

If we assume that each $U_{i_0, \ldots i_n}$ is connected, then we see that the Čech complex $C^n(\{U_i\}, A_X)$ coincides with the simplicial complex of the nerve with coefficients in A.

Even though, we are primarily interested in sheaves of *abelian* groups. It will be convenient to extend (6.4) to a sheaf arbitrary groups $\mathcal{G}$.

$$\check{H}^1(\{U_i\}, \mathcal{G}) = \{(g_{ij}) \in \prod_{i<j} \mathcal{G}(U_{ij}) \,|\, g_{ik} = g_{ij} g_{jk} \text{ on } U_{ijk} \ \}/\sim$$

where $(g_{ij}) \sim (\bar{g}_{ij})$ if there exists $(\gamma_i) \in \prod \mathcal{G}(U_i)$ such that $g_{ij} = \gamma_i \bar{g}_{ij} \gamma_j^{-1}$. Note that this is just a set in general. The (g_{ij}) are called 1-cocycles with values in $\mathcal{G}$. It will be useful to drop the requirement that $i < j$ by setting $g_{ji} = g_{ij}^{-1}$ and $g_{ii} = 1$.

As an example of sheaf of nonabelian groups, take $U \mapsto GL_n(\mathcal{R}(U))$, where $(X, \mathcal{R})$ is a ringed space (i.e. space with a sheaf of commutative rings).

Theorem 6.3.1. *Let $(X,\mathcal{R})$ be a manifold or a variety over k, and $\{U_i\}$ and open cover of X. There is a bijection between the following sets:*

1. *The set of isomorphism classes of rank n vector bundles over $(X,\mathcal{R})$ triviallizable over $\{U_i\}$.*

2. *The set of isomorphism classes of locally free $\mathcal{R}$-modules M of rank n such $M|_{U_i}$ is free.*

3. *$\check{H}^1(\{U_i\}, Gl_n(\mathcal{R}))$.*

Proof. We merely describe the correspondences.

$1 \to 2$: Take the sheaf of sections.

$2 \to 3$: Given M as above. Choose isomorphisms $F_i : \mathcal{R}^n_{U_i} \to M|_{U_i}$.Set $g_{ij} = F_i \circ F_j^{-1}$. This determines a well defined element of $\check{H}^1$.

$3 \to 1$: Define an equivalence relation $\sim$ on the disjoint union $W = \coprod U_i \times k^n$ as follows. Given $(x_i, v_i) \in U_i \times k^n$ and $(x_j, v_j) \in U_j \times k^n$, $(x_i, v_i) \sim (x_j, v_j)$ iff $x_i = x_j$ and $v_i = g_{ij}(x)v_j$. Let $V = W/$ with quotient topology. Given an open set $U'/\sim = U \subset V$. Define $f : U \to k$ to be regular, C^∞ or holomorphic (as the case may be) if its pullback to U' has this property. □

Implicit above, is a construction which associates to a 1-cocycle $\gamma = (g_{ij})$, the locally free sheaf

$$M_\gamma(U) = \{(v_i) \in \prod \mathcal{R}(U \cap U_i)^n \mid v_i = g_{ij} v_j\}.$$

Consider the case of projective space $\mathbb{P} = \mathbb{P}^n_k$. Suppose $x_0, \ldots x_n$ are homogeneous coordinates. Let U_i be the complement of the hyperplane $x_i = 0$. Then U_i is isomorphic to $\mathbb{A}^n_k$ by

$$[x_0, \ldots x_n] \to (\frac{x_0}{x_i}, \ldots \widehat{\frac{x_i}{x_i}} \ldots)$$

Define $g_{ij} = x_j/x_i \in O(U_{ij})^*$. This is a 1-cocycle, and $M_{g_{ij}} \cong \mathcal{O}_{\mathbb{P}}(1)$. Likewise $(x_j/x_i)^d$ is the 1-cocyle for $O(d)$.

We get rid of the dependence on coverings by taking direct limits. If $\{V_j\}$ is refinement of $\{U_i\}$, there is a natural restriction map

$$\check{H}^i(\{U_i\}, \mathcal{F}) \to \check{H}^i(\{V_j\}, \mathcal{F})$$

We define

$$\check{H}^i(X, \mathcal{F}) = \lim_{\to} \check{H}^i(\{U_i\}, \mathcal{F})$$

Corollary 6.3.2. *There is a bijection between the following sets:*

1. *The set of isomorphism classes of rank n vector bundles over $(X,\mathcal{R})$.*

2. *The set of isomorphism classes of locally free $\mathcal{R}$-modules M of rank n.*

3. *$\check{H}^1(X, Gl_n(\mathcal{R}))$.*

A line bundle is a rank one vector bundle. We won't distinguish between lines bundles and rank one locally free sheaves. The set of isomorphism classes of line bundles carries the stucture of a group namely $\check{H}^1(X, \mathcal{R}^*)$. This group is called the Picard group, and is denoted by $Pic(X)$.

Exercise 6.3.3.

1. *Check the description of* $\mathcal{O}_{\mathbb{P}}(1)$ *given above.*

2. *Show that multiplication in* $Pic(X)$ *can be interpreted as tensor product of line bundles.*

6.4 Čech versus sheaf cohomology

Definition 6.4.1. *An open covering* $\{U_i\}$ *is called a Leray covering for a sheaf* $\mathcal{F}$ *if* $H^n(U_J, \mathcal{F}) = 0$ *for all nonempty finite sets* J *and* $n > 0$.

Lemma 6.4.2. *Suppose that* $H^1(U_J, A) = 0$ *for all nonempty finite sets* J. *Then given an exact sequence*

$$0 \to A \to B \to C \to 0$$

of sheaves, there is a long exact sequence:

$$0 \to \check{H}^0(\{U_i\}, A) \to \check{H}^0(\{U_i\}, B) \to \check{H}^0(\{U_i\}, C) \to \check{H}^1(\{U_i\}, A) \ldots$$

Proof. The hypothesis guarantees that there is a short exact sequence of complexes:

$$0 \to C^\bullet(\{U_i\}, A) \to C^\bullet(\{U_i\}, B) \to C^\bullet(\{U_i\}, C) \to 0.$$

The long exact now follows from a standard result in homological algebra... □

Lemma 6.4.3. *Suppose that* $\mathcal{F}$ *is flabby, then* $\check{H}^n(\mathcal{F}) = 0$ *for all* $n > 0$.

Theorem 6.4.4. *If* $\{U_i\}$ *is a Leray with respect to* $\mathcal{F}$ *then*

$$\check{H}^n(\{U_i\}, \mathcal{F}) \cong H^n(X, \mathcal{F})$$

Proof. This is clearly true for $n = 0$. Next consider the case where $n = 1$. Lemmas 6.4.2 and 6.4.3 imply that

$$\check{H}^1(\mathcal{F}) = coker[\Gamma(X, G(\mathcal{F})) \to \Gamma(X, C^1(\mathcal{F}))] = H^1(X, \mathcal{F}).$$

The remaining cases follow by dimension shifting, e.g.

$$\check{H}^2(\mathcal{F}) = \check{H}^1(C^1(\mathcal{F})) = H^2(X, \mathcal{F})$$

□

We state few more general results.

Proposition 6.4.5. *For any sheaf $\mathcal{F}$*

$$\check{H}^1(X,\mathcal{F}) \cong H^1(X,\mathcal{F})$$

Proof. See [G, Cor. 5.9.1] □

Theorem 6.4.6. *If X is a paracompact space then for any sheaf and all i,*

$$\check{H}^i(X,\mathcal{F}) \cong H^i(X,\mathcal{F})$$

Proof. See [G, Cor. 5.10.1] □

6.5 First Chern class

Let $(X,\mathcal{O}_X)$ be a complex manifold or algebraic variety over $\mathbb{C}$. Then we have isomorphisms

$$Pic(X) \cong \check{H}^1(X,\mathcal{O}_X^*) \cong H^1(X,\mathcal{O}_X^*)$$

The exponential sequence is

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \xrightarrow{e^{2\pi i}} \mathcal{O}_X^* \to 1 \tag{6.5}$$

Definition 6.5.1. *Given a line bundle L, its first Chern class $c_1(L) \in H^2(X,\mathbb{Z})$ is the image of L under the connecting map $Pic(X) \to H^2(X,\mathbb{Z})$*

This can be carried out for C^∞ manifolds as well. Provided one interprets $\mathcal{O}_X$ as the sheaf of complex valued C^∞ functions, and $Pic(X)$ as group of C^∞ complex line bundles. In this case, c_1 is an isomorphism. It is clear that the construction is functorial:

Lemma 6.5.2. *If $f : X \to Y$ is C^∞ map between manifolds, $c_1(f^*L) = f^*c_1(L)$.*

We want to calculate this explicitly for $\mathbb{P} = \mathbb{P}^1_{\mathbb{C}}$. We can use the standard covering $U_i = \{x_i \neq 0\}$. We identify U_1 with $\mathbb{C}$ with the coordinate $z = x_0/x_1$. The 1-cocycle (section 6.3) of $O(1)$ is $g_{01} = z^{-1}$. The logarithmic derivative $dlog\, g_{01} = -dz/z$ is 1-cocycle with values in $\mathcal{E}^1$. Since this sheaf is soft, this cocycle is coboundary, i.e. there exists forms $\alpha_i \in \mathcal{E}^1(U_i)$ such that $-dz/z = \alpha_1 - \alpha_0$ on the intersection. There $d\alpha_i$ patch to yield a global 2-form $\beta \in \mathcal{E}^2(\mathbb{P})$. $\beta/2\pi i$ gives an explicit representative of the image of $C_1(O(1))$ in $H^2(X,\mathbb{C})$. We have an isomorphism of this space with $\mathbb{C}$ given by integration. In order to evaluate the integral $\int_{\mathbb{P}} \beta/2\pi i$, divide the sphere into two hemispheres $H_0 = \{|z| \leq 1\}$ and $H_1 = \{\||z| \geq 1\}$. Let C be the curve $|z| = 1$ with positive orientation. Note the boundary of H_1 is $-C$. Then with the help of Stokes' theorem, we get

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\mathbb{P}} \beta &= \frac{1}{2\pi i}\left(\int_{H_0} d\alpha_0 + \int_{H_1} d\alpha_1\right) \\
&= \frac{1}{2\pi i}\left(\int_C \alpha_0 - \int_C \alpha_1\right) \\
&= \frac{1}{2\pi i}\int_C \frac{dz}{z} = 1
\end{aligned}$$

Thus $c_1(O(1))$ is the fundamental class of $H^2(\mathbb{P}^1, \mathbb{Z})$. By the same kind of argument, we obtain:

Lemma 6.5.3. *If D is a divisor on a compact Riemann surface X, $c_1(\mathcal{O}(D)) = \deg(D)[X]$*

Since c_1 is compatible with restriction, (6.3) implies that the same holds for $\mathbb{P}^n$:

Lemma 6.5.4. *$c_1(\mathcal{O}_{\mathbb{P}^n}(1)) = [H]$ where $H \subset \mathbb{P}^n$ is a hyperplane.*

Chapter 7

The Hodge theorem

Thus far, our approach has been pretty much algebraic or topological. We are going to need a basic analytic result, namely the Hodge theorem which says that every de Rham cohomology class has a unique "smallest" element. Standard acccounts of basic Hodge theory can be found in the books of Griffiths-Harris [GH], Warner [Wa] and Wells [W]. However, we will depart slightly from these treatments by outling the heat equation method of Milgram and Rosenbloom [MR]. This is an elegant and comparatively elementary approach to the Hodge theorem. As a warm up, we will do a combinatorial version which requires nothing more than linear algebra.

7.1 Hodge theory on a simplicial complex

In order to motivate the general Hodge theorem, we work this out for a finite simplicial complex. This will require nothing beyond linear algebra

Let $K = (V, \Sigma)$ be a finite simplicial complex. Choose inner products on the spaces of cochains $C^*(K, \mathbb{R})$. For each simplex S, let

$$\delta_S(S') = \begin{cases} 1 \text{ if } S = S' \\ 0 \text{ otherwise} \end{cases}$$

These form a basis. A particularly natural choice of inner product is determined by making this basis orthonormal. Let $\partial^* : C^i(K, \mathbb{R}) \to C^{i-1}(K, \mathbb{R})$ be the adjoint to ∂, and let $\Delta = \partial\partial^* + \partial^*\partial$. Δ is the discrete Laplacian.

Lemma 7.1.1. *Let α be a cochain. The following are equivalent:*

1. $\alpha \in (im\partial)^{\perp} \cap ker\partial$.
2. $\partial\alpha = \partial^*\alpha = 0$.
3. $\Delta\alpha = 0$.

The cochains satisfying the above conditions are called harmonic.

Lemma 7.1.2. *Every simplical cohomology class has a unique harmonic representative. These are precise the elements of smallest norm.*

Proof. One sees that the obvious map

$$(im\partial)^{\perp} \cap ker\partial \to H^n(K, \mathbb{R})$$

is an isomorphism. Given an element α in the lefthand space

$$||\alpha + \partial\beta||^2 = ||\alpha||^2 + ||\partial\beta||^2 > ||\alpha||^2$$

unless $\partial\beta = 0$. □

Exercise 7.1.3.

1. *Prove lemma 7.1.1.*
2. *Prove that Δ is a positive semidefinite symmetric operator.*
3. *Use this to prove that limit of the "heat kernel"*

$$H = \lim_{t\to\infty} e^{-t\Delta}$$

is the orthogonal projection to the space of harmonic cochains.

7.2 Harmonic forms

Let X be an n dimensional compact oriented manifold. We want to prove an analogue of lemma 7.1.2 for de Rham cohomology. In order to formulate this, we need inner products. A Riemannian metric $(,)$, i.e. a family of inner products on the tangent spaces which vary in a C^∞ fashion. The existence is standard partition of unity argument [Wa]. A metric determines inner products on exterior powers of the cotangent bundle which will also be denoted by $(,)$. X possesses an differential form $dvol \in \mathcal{E}^n(X)$ called the volume form which is roughly the square root of the determinant of the metric (this requires an orientation). The Hodge star operator is a $C^\infty(X)$-linear operator $* : \mathcal{E}^p(X) \to \mathcal{E}^{n-p}(X)$, determined by

$$\alpha \wedge *\beta = (\alpha, \beta)dvol.$$

One can choose a local orthonormal basis or frame e_i for $\mathcal{E}^1_X$ in a neighbourhood of any point. Then $*$ is easy to calculate in this frame, e.g. $*e_1 \wedge \ldots e_k = e_{k+1} \wedge \ldots e_n$. From this one can check that $** = \pm 1$. The spaces $\mathcal{E}^p(X)$ carry inner products:

$$< \alpha, \beta > = \int_X (\alpha, \beta)dvol = \int_X \alpha \wedge *\beta$$

and hence norms.

Then basic result of Hodge (and Weyl) is:

Theorem 7.2.1. *Every de Rham cohomology class has a unique representative which minimizes norm. This is called the harmonic representative.*

As an application of this theorem, we have get a new proof of a Poincaré duality in strengthened form.

Corollary 7.2.2. *(Poincaré duality, version 2). The pairing*

$$H^i(X, \mathbb{R}) \times H^{n-i}(X, \mathbb{R}) \to \mathbb{R}$$

induced by $(\alpha, \beta) \mapsto \int \alpha \wedge \beta$ *is a perfect pairing*

Proof. $*$ induces an isomorphism between the space of harmonic i-forms and $n-i$-forms. This proves directly that $H^i(X, \mathbb{R})$ and $H^{n-i}(X, \mathbb{R})$ are isomorphic.

Consider the map

$$\lambda : H^i(X, \mathbb{R}) \to H^{n-i}(X, \mathbb{R})^*$$

given by $\lambda(\alpha) = \beta \mapsto \int \alpha \wedge \beta$. We need to prove that λ is an isomorphism. Since these spaces have same dimension, it is enough to prove that $ker(\lambda) = 0$. But this clear since $\lambda(\alpha)(*\alpha) \neq 0$ whenver α is a nonzero harmonic form. $\square$

To understand the meaning of this condition, we can find the Euler-Lagrange equation. Let α be a harmonic p-form. Then for any for any $(p-1)$-form β, we would have to have

$$\frac{d}{dt}||\alpha + td\beta||^2|_{t=0} = 2 < \alpha, d\beta >= 2 < d^*\alpha, \beta >= 0,$$

which forces $d^*\alpha = 0$, where d^* is adjoint to d. A straight forward integration by parts shows that $d^* = \pm * d *$. Thus harmonicity can be expressed as a pair of differential equations $d\alpha = 0$ and $d^*\alpha = 0$. It's sometimes more convenient to combine these into a single equation $\Delta\alpha = 0$ where $\Delta = d^*d + dd^*$ is the Hodge Laplacian. The equivalence follows from the identity $< \Delta\alpha, \alpha >= ||d\alpha||^2 + ||d^*\alpha||^2$.

The hard work is contained in the following:

Theorem 7.2.3. *There are linear operators H (harmonic projection) and G (Green's operator) taking C^∞ forms to C^∞ forms, which are characterized by the following properties*

1. *$H(\alpha)$ is harmonic,*
2. *$G(\alpha)$ is orthogonal to the space of harmonic forms,*
3. *$\alpha = H(\alpha) + \Delta G(\alpha)$,*

for any C^∞ form α.

Let's see how to prove the existence part of theorem 7.2.1 given this result. Any α can be written as $\alpha = \beta + dd^*\gamma + d^*d\gamma$ with β harmonic. All these terms are in fact orthogonal to each other, so

$$||d^*d\gamma||^2 =< d^*d\gamma, \alpha >=< d\gamma, d\alpha >,$$

and this vanishes if α is closed. Therefore α is cohomologous to the harmonic form β. The uniqueness is straightforward: if β' is another harmonic representative, then $\beta - \beta'$ is both exact and harmonic. Denoting this by $d\gamma$, we then have $||d\gamma||^2 =< d^*d\gamma, \gamma >= 0$.

It will be useful to record part of this argument as a corollary.

Corollary 7.2.4. *There is an orthogonal direct sum*

$$\mathcal{E}^i(X) = (\textit{harmonic forms}) \oplus d\mathcal{E}^{i-1}(X) \oplus \mathcal{E}^{i+1}(X)$$

Before doing the general case, let's work out the easy, but instructive, example of the torus $X = \mathbb{R}^n/\mathbb{Z}^n$, with the Euclidean metric. A differential form α can be expanded in a Fourier series

$$\alpha = \sum_{\lambda\in\mathbb{Z}^n} \sum_{i_1<\ldots<i_p} a_{\lambda,i_1\ldots i_p} e^{2\pi i\lambda\cdot\mathbf{x}} dx_{i_1} \wedge \ldots \wedge dx_{i_p} \tag{7.1}$$

By direct calculation, one finds the Laplacian

$$\Delta = -\sum \frac{\partial^2}{\partial x_i^2},$$

the harmonic projection

$$H(\alpha) = \sum a_{0,i_1\ldots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

and Green's operator

$$G(\alpha) = \sum_{\lambda\in\mathbb{Z}^n-\{0\}} \sum \frac{a_{\lambda,i_1\ldots i_p}}{4\pi^2|\lambda|^2} e^{2\pi i\lambda\cdot\mathbf{x}} dx_{i_1} \wedge \ldots \wedge dx_{i_p}.$$

Since the image of H are forms with constant coefficients, this proves propostion 4.5.1.

7.3 Heat Equation

We will sketch an approach to 7.2.3 using the heat equation due Milgram and Rosenbloom ([MR] [Ch]). The heuristic behind it is that if α is thought of as an initial temperature, then the temperature should approach a harmonic steady state as the manifold cools. So our task is to solve the heat equation:

$$\frac{\partial A(t)}{\partial t} = -\Delta A(t) \tag{7.2}$$

$$A(0) = \alpha \tag{7.3}$$

for all $t > 0$, and study the behaviour as $t \to \infty$. For simplicial complexes, this was the content of the previous exercise. Before describing this, we make a few general remarks. If $A(t)$ is a C^∞ solution of 7.2, upon differentiating $||A(t)||^2$ we obtain:

$$2 < \frac{\partial A}{\partial t}, A >= -2 < \Delta A, A >= -2(||dA||^2 + ||d^*A||^2) \leq 0.$$

This implies that $||A||^2$ is nonincreasing. Therefore if $A(0) = 0$, it remains 0. This proves the uniqueness because the equation is linear. For existence, it is enough to prove that there exists ϵ independent of α, such that solutions exist for all $0 < t < \epsilon$. For if T is the supremum of the t_0 for which the above system has a solution for $t < t_0$. Then T must be ∞, since otherwise picking $T - \epsilon/2 < t_0 < T$ and using $A(t_0)$ as the new initial condition, we see that A can be extended to the interval $t < T + \epsilon/2$.

The starting point for the proof of existence is the observation that when X is replaced by Euclidean space and α is a compactly supported function, then one has an explicit solution

$$A(t) = \int_{\mathbb{R}^n} K(x, y, t)\alpha(y)dy$$

where the heat kernel

$$K(x, y, t) = (4\pi t)^{-n/2} e^{-||x-y||^2/4t}$$

This can be modified to make sense for a p-form on X. When working with forms on $X \times X$, we will abuse notation a bit and write them as local expressions such as $\eta(x, y)$. Then $d_x\eta(x, y)$ etcetera will indicate that the operations $d \ldots$ are preformed with y treated as a constant (or more correctly, these operations are preformed fiberwise along the second projection). Let $dist(x, y)^2$ be a nonnegative C^∞ function which agrees with the square of the Riemannian distance function in neighbourhood of the diagonal. Let $\beta = -\frac{1}{2}d_x d_y dist(x, y)^2$ and set

$$K(x, y, t) = (4\pi t)^{-n/2} e^{-dist(x,y)^2/4t} \beta^p$$

This is only an approximation to the true heat kernel on X, however one can still get some useful information from it. For small t, it behaves like the Euclidean heat kernel, in particular in the limit it acts like a δ-function along the diagonal:

$$\lim_{t \to 0} < A(y, \tau), K(x, y, t) >_y = A(x, \tau)$$

This together with the self adjointness of Δ and integration by parts shows that for any p-form A on $X \times [0, \infty)$

$$\begin{aligned} A(x, t) \quad &= \quad < A(x, 0), K(x, y, t) >_y + \int_0^t < K_1(x, y, t), A(y, t) >_y dt \\ &\quad - \int_0^t < K(x, y, t), (\Delta_y + \partial/\partial t)A(y, t) >_y dt \end{aligned}$$

where

$$K_1(x,y,t) = (\Delta_y - \partial/\partial t)K(x,y,t).$$

In particular, if A is a solution to equations 7.2 and 7.3, then it satisfies the integral equation

$$A =< \alpha(x), K(x,y,t) >_y + \int_0^t < K_1(x,y,t), A(y,t) >_y dt \tag{7.4}$$

and conversely a solution to this equation satisfies the heat equation. Let $R_t(A)$ denotes the right hand side of the above equation which we can regard as an operator acting on forms on $X \times [0,t]$. $||K_1(x,y,t)||$ is known to be bounded by a constant M independent of y and t [MR]. This implies that for $t < \epsilon = 1/M$, R_t is a contraction operator. Thus there is a fixed point, and it is C^∞ because K is. So by the above remarks, we have a C^∞ solution for all $t > 0$.

Since we have established existence and uniqueness of 7.2 and 7.3, the operator $T(\alpha) = A(t)$ is well defined. One has the semigroup property $T_{t_1+t_2} = T_{t_1}T_{t_2}$ because $A(t_1+t_2)$ can be obtained by solving the heat equation with initial condition $A(t_2)$ and then evaluating it at $t = t_1$. We claim, furthermore that T_t is formally selfadjoint. To see this, calculate

$$\frac{\partial}{\partial t} < T_t\eta, T_\tau\xi >=< \frac{\partial}{\partial t}T_t\eta, T_\tau\xi >=< \Delta T_t\eta, \xi >$$

$$=< \eta, T_\tau\Delta\xi >=< T_t\eta, \frac{\partial}{\partial \tau}T_\tau\xi >= \frac{\partial}{\partial \tau} < T_t\eta, T_\tau\xi >$$

which implies that $< T_t\eta, T_\tau\xi >$ can be written as a function of $t+\tau$, say $g(t+\tau)$. Therefore $< T_t\eta, \xi >= g(t) =< \eta, \xi >$. These properties imply that for $h \geq 0$ we have

$$\begin{aligned}
||T_{t+2h}\alpha - T_t\alpha||^2 &= ||T_{t+2h}\alpha||^2 + ||T_t\alpha||^2 - 2 < T_{t+2h}\alpha, T_t\alpha > \\
&= ||T_{t+2h}\alpha||^2 + ||T_t\alpha||^2 - 2||T_{t+h}\alpha||^2 \\
&= (||T_{t+2h}\alpha|| - ||T_t\alpha||)^2 - 2(||T_{t+h}\alpha||^2 - ||T_{t+2h}\alpha|| \cdot ||T_t\alpha||)
\end{aligned}$$

$||T_t\alpha||^2$ converges because it is nonincreasing. Therefore the above expression can be made arbitrarily small for large enough t. This implies that $T_t\alpha$ converges in the L^2 sense to an L^2 form $H(\alpha)$. In fact, a little more work shows uniform convergence.

We'll outline the remaining steps. Equation 7.4 and the semigroup law implies the relation $T_{t-\tau}\alpha = R_\tau T_t\alpha$. In the limit as $t \to \infty$, we get $H(\alpha) = R_\tau H(\alpha)$ which implies that $H\alpha$ is C^∞ and harmonic. $||T_t\alpha - H\alpha||$ can be shown to decay rapidly, so the integral

$$G(\alpha) = \int_0^\infty (T_t\alpha - H\alpha)dt$$

is well defined. We'll verify formally that this is Green's operatorr:

$$\Delta G(\alpha) = \int_0^\infty \Delta T_t\alpha dt = -\int_0^\infty \frac{\partial T_t\alpha}{\partial t}dt = \alpha - H(\alpha),$$

and for β harmonic

$$< G(\alpha), \beta >= \int_0^\infty < (T_t - H)\alpha, \beta > dt = \int_0^\infty < \alpha, (T_t - H)\beta > dt = 0$$

as required.

Let's return to the example of a torus $X = \mathbb{R}^n/\mathbb{Z}^n$, where things can be calculated explicitly. Given α as in (7.1), the solution to the heat equation with initial value α is given by

$$T_t\alpha = \sum_{\lambda\in\mathbb{Z}^n} \sum a_{\lambda,i_1\ldots i_p} e^{(2\pi i\lambda\cdot\mathbf{x} - 4\pi^2|\lambda|^2 t)} dx_{i_1} \wedge \ldots \wedge dx_{i_p},$$

and this converges to the harmonic projection

$$H(\alpha) = \sum a_{0,i_1\ldots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

$T_t\alpha - H\alpha$ can be integrated term by term to obtain Green's operator

$$G(\alpha) = \sum_{\lambda\in\mathbb{Z}^n-\{0\}} \sum \frac{a_{\lambda,i_1\ldots i_p}}{4\pi^2|\lambda|^2} e^{2\pi i\lambda\cdot\mathbf{x}} dx_{i_1} \wedge \ldots \wedge dx_{i_p}.$$

Chapter 8

Toward Hodge theory for Complex Manifolds

In this chapter, we take the first few steps toward Hodge theory in the complex setting. This is really a warm up for the next chapter.

8.1 Riemann Surfaces Revisited

This section is a warm up for things to come. We also tie up a few loose ends from chapter 5. Fix a compact Riemann surface X. In order to apply the techniques from the previous chapter, we need a metric which is a C^∞ family of inner products. But we need to impose a compatability condition. As one learns in first course in complex analysis, conformal maps are angle preserving, and this means that we have a well defined notion of the angle between two tangent vectors. Among other things, compatability will mean that the angles determined by the metric agree with the ones above. To say this more precisely, view X as two dimension real C^∞ manifold. Choosing an analytic local coordinate z in a neighbourhood of U, the vectors $v_1 = \partial/\partial x$ and $v_2 = \partial/\partial y$ give a basis (or frame) of the real tangent sheaf $T_X^{\mathbb{R}}$ of X restricted to U. The automorphism $J_p : T_X^{\mathbb{R}}|_U \to T_X^{\mathbb{R}}|_U$ represented by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in the basis v_1, v_2 is independent of this basis, and hence globally well defined. A Riemannian metric $(,)$ is compatible with the complex structure, or hermitean, if the transformations J_p are orthogonal. In terms of the basis v_1, v_2 this forces the matrix of the bilinear $(,)$ to be a positive multiple of I by some function h. Standard partition of unity arguments show that hermitean metrics exist. Fix one. In coordinates, the metric would be represented by a tensor $h(x, y)(dx \otimes dx + dy \otimes dy)$. The volume form (which is globally well defined) is represented by $hdx \wedge dy$. It follows that $*dx = dy$ and $*dy = -dx$. In other words, $*$ is the transpose of J which is independent of h. Once we have $*$, we can define all the previous operators and talk about harmonicity. After having made such a fuss about metrics, it turns out that for our purposes one is as good as any other (for the statements though not the proofs).

Lemma 8.1.1. *A 1-form is harmonic if and only if its $(1,0)$ and $(0,1)$ parts are respectively holomorphic and antiholomorphic.*

Proof. A 1-form α is harmonic if and only if $d\alpha = d * \alpha = 0$. Given a local coordinate z, it can be checked that $*dz = -idz$ and $*d\bar{z} = id\bar{z}$. Therefore the $(1,0)$ and $(0,1)$ parts of a harmonic 1-form α are closed. If α is a $(1,0)$-form, then $d\alpha = \bar{\partial}\alpha$. Thus α is closed if and only if it is holomorphic. Conjugation yields the analogous statement for $(0,1)$-forms. □

Corollary 8.1.2. *$dim\, H^0(X, \Omega^1_X)$ equals the genus of X.*

Proof. The first Betti number $dim\, H^1(X, \mathbb{C})$ is the dimension of the spaces of holomorphic and antiholomorphic forms, and both these spaces have the same dimension. □

Lemma 8.1.3. *The image of Δ and $\partial\bar{\partial}$ on $\mathcal{E}^2(X)$ coincide.*

Proof. Computing in local coordinates yields

$$\partial\bar{\partial}f = -\frac{i}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy$$

and

$$d * df = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy.$$

This finishes the proof because $\Delta = -d * d*$. □

Proposition 8.1.4. *The map $H^1(X, \mathcal{O}_X) \to H^1(X, \Omega^1_X)$ induced by d vanishes.*

Proof. We use the use the $\bar{\partial}$-cohomology descriptions of these spaces 5.4.3. Given $\alpha \in \mathcal{E}^{01}(X)$, let $\beta = d\alpha$. We have to show that β lies in the image of $\bar{\partial}$. Applying theorem 7.2.3 we can write $\beta = H(\beta) + \Delta G(\beta)$. Since β is exact, we can conclude that $H(\beta) = 0$ by corollary 7.2.4. Therefore β lies in the image of $\partial\bar{\partial} = -\bar{\partial}\partial$. □

Corollary 8.1.5. *The map $H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X)$ is surjective.*

8.2 Dolbeault's theorem

We now extend the results from Riemann surfaces to higher dimensions. Given an n dimensional complex manifold X, let $\mathcal{O}_X$ denote the sheaf of holomorphic functions. We can regard X as a $2n$ dimensional (real) C^∞ manifold as explained in section 1.2. As with Riemann surfaces, from now on C^∞ and $\mathcal{E}^k_X$ will denote the sheaves of complex valued $\mathbb{C}^i$ functions and forms. We use the notation $C^\infty_{X,\mathbb{R}}$ and $\mathcal{E}^k_{X,\mathbb{R}}$ to denote the sheaf of real valued C^∞ functions and k-forms. More formally,

$$\mathcal{E}^k_X(U) = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{E}^k_{X,\mathbb{R}}(U).$$

This implies that this space has complex conjugation given by $\overline{a \otimes \alpha} = \bar{a} \otimes \alpha$. Recall that Ω^p_X is the sheaf of holomorphic k-forms. This is a subsheaf of $\mathcal{E}^p_X$.

Definition 8.2.1. *Let $\mathcal{E}^{(p,0)}_X(U)$ denote the $C^\infty(U)$ submodule of $\mathcal{E}^p(U)$ generated by $\Omega^p_X(U)$. Let $\mathcal{E}^{(0,p)}_X(U) = \overline{\mathcal{E}^{(p,0)}_X(U)}$, and $\mathcal{E}^{(p,q)}_X(U) = \mathcal{E}^{(p,0)}_X(U) \wedge \mathcal{E}^{(0,q)}_X(U)$.*

Suppose $U \subset \mathbb{C}^n$ is an open subset. Let $z_1, \ldots z_n$ be the coordinates on $\mathbb{C}^n$. Then $\mathcal{E}^{(p,q)}_X(U)$ is a free $C^\infty(U)$-module with a basis

$$\{dz_{i_1} \wedge \ldots dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \ldots d\bar{z}_{j_q} \,|\, i_1 < \ldots < i_p,\, j_1 < \ldots < j_q\}$$

To simplify formulas, we'll write this the above forms as $dz_I \wedge d\bar{z}_J$.

All of the operations of sections 5.3 and 5.4 can be extended to the higher dimensional case. The operators

$$\partial : \mathcal{E}^{(p,q)}_X \to \mathcal{E}^{(p+1,q)}_X$$

and

$$\bar{\partial} : \mathcal{E}^{(p,q)}_X \to \mathcal{E}^{(p,q+1)}_X$$

are given locally by

$$\partial(\sum_{I,J} f_{I,J} dz_I \wedge d\bar{z}_J) = \sum_{I,J} \sum_{i=1}^n \frac{\partial f_{I,J}}{\partial z_i} dz_I \wedge d\bar{z}_J$$

$$\bar{\partial}(\sum_{I,J} f_{I,J} dz_I \wedge d\bar{z}_J) = \sum_{I,J} \sum_{j=1}^n \frac{\partial f_{I,J}}{\partial \bar{z}_j} dz_I \wedge d\bar{z}_J$$

The identities

$$\begin{aligned} d &= \partial + \bar{\partial} \\ \partial^2 &= \bar{\partial}^2 = 0 \\ \partial\bar{\partial} + \bar{\partial}\partial &= 0 \end{aligned} \tag{8.1}$$

hold.

The analogue of theorem 5.3.1 is

Theorem 8.2.2. *Let $D \subset \mathbb{C}^n$ be an open polydisk (i.e. a product of disks). Given $\alpha \in \mathcal{E}^{(p,q)}(\bar{D})$ with $\bar{\partial}\alpha = 0$, there exists $\beta \in C^\infty(D)$ such that $\alpha = \bar{\partial}\beta$*

See [GH]. Let Ω^p_X be sheaf of holomorphic p-forms which are given locally as sums of $f dz_I$ with f holomorphic. One checks immediately that this is the kernel of the $\bar{\partial}$ operator on $\mathcal{E}^{(p,0)}_X$.

Corollary 8.2.3 (Dolbeault's Theorem).

$$0 \to \Omega^p_X \to \mathcal{E}^{(p,0)}_X \xrightarrow{\bar{\partial}} \mathcal{E}^{(p,1)}_X \ldots$$

is a fine resolution.

8.3 Complex Tori

A complex torus is a quotient $X = V/L$ of a finite dimensional complex vector space by a lattice (i.e. a discrete subgroup of maximal rank). Thus it is both a complex manifold and a torus. Let us identify V with $\mathbb{C}^n$. Let $z_1, \ldots z_n$ be the standard complex coordinates on $\mathbb{C}^n$, and let $x_i = Re(z_i)$, $y_i = Im(z_i)$.

We have already seen that harmonic forms with respect to the flat metric are the forms with constant coefficients. The space of harmonic forms can be decomposed into (p, q) type.

$$H^{(p,q)} = \bigoplus_{\#I=p,\#J=q} \mathbb{C} dz_I \wedge d\bar{z}_J$$

These forms are certainly $\bar{\partial}$-closed. Thus we get a map to $\bar{\partial}$-cohomology.

Proposition 8.3.1. *This map induces an isomorphism $H^{(p,q)} \cong H^q(X, \Omega^p_X)$.*

The isomorphism $V \cong L \otimes \mathbb{R}$ induces a natural real structure on V. Therefore it makes sense to speak of antilinear maps from V to C; let $\bar{V}^*$ denote the space of these.

Corollary 8.3.2. $H^q(X, \Omega^p_X) \cong \wedge^q V \otimes \wedge^p \bar{V}^*$

It's possible to give a fairly elementary proof of this proposition [MA]. However, we will indicate the that generalizes well.

Let ∂^* and $\bar{\partial}^*$ denote the adjoints to ∂ and $\bar{\partial}$ respectively (note that this convention is reversed in [GH]). These operators can be calculated explicitly. Let i_k and $\bar{i}_k$ denote contraction with the vector fields $2\partial/\partial z_k$ and $2\partial/\partial \bar{z}_k$. These are the adjoints to $dz_k\wedge$ and $d\bar{z}_k\wedge$. Then

$$\partial^* \alpha = -\sum \frac{\partial}{\partial z_k} i_k \alpha$$

$$\bar{\partial}^* \alpha = -\sum \frac{\bar{\partial}}{\bar{\partial} z_k} i_k \alpha$$

We can define the ∂ and $\bar{\partial}$-Laplacians by

$$\Delta_\partial = \partial^* \partial + \partial \partial^*,$$

$$\Delta_{\bar{\partial}} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$$

Theorem 8.3.3. $\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_\partial$.

Proof of proposition 8.3.1. By Dolbeault's theorem

$$H^q(X, \Omega^p_X) \cong \frac{ker[\bar{\partial} : \mathcal{E}^{(p,q)}(X) \to \mathcal{E}(p, q+1)]}{im[\bar{\partial} : \mathcal{E}^{(p,q-1)}(X) \to \mathcal{E}(p, q)]}$$

Let α be a $\bar{\partial}$-closed (p, q)-form. Decompose

$$\alpha = \beta + \Delta\gamma = \beta + 2\Delta_{\bar{\partial}}\gamma = \beta + \bar{\partial}\gamma_1 + \bar{\partial}^*\gamma_2$$

with β harmonic by theorem7.2.3.

$$||\bar{\partial}^*\gamma_2||^2 =< \gamma_2, \bar{\partial}\bar{\partial}^*\gamma_2 >=< \gamma_2, \bar{\partial}\alpha >= 0.$$

Therefore the $\bar{\partial}$-class of α is represented by β. The uniqueness can be proved by arguments identical to those of section 7.2. Briefly, suppose β' is another harmonic form representing this class. Then $\beta - \beta' = \bar{\partial}\eta$ must be orthogonal to itself. □

Introduce the operators

$$\omega = \frac{\sqrt{-1}}{2}\sum dz_k \wedge d\bar{z}_k = \sum dx_k \wedge dy_k$$

$$L\alpha = \omega \wedge \alpha$$

and

$$\Lambda = -\frac{\sqrt{-1}}{2}\sum \bar{i}_k i_k$$

A straight forward computation yields the first order Kähler identities:

Proposition 8.3.4. *If* $[A, B] = AB - BA$ *then*

1. $[\Lambda, \bar{\partial}] = -i\partial^*$
2. $[\Lambda, \partial] = i\bar{\partial}^*$

Proof. [GH, p. 114]. □

Upon substituting these into the definitions of the various laplacians some remarkable cancelations take place, and we obtain:

Proof of theorem 9.2.1. We first establish $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$,

$$i(\partial\bar{\partial}^* + \bar{\partial}^*\partial) =$$

$$= \partial(\Lambda\partial - \partial\Lambda) - (\Lambda\partial - \partial\Lambda)\partial$$

$$= \partial\Lambda\partial - \partial\Lambda\partial = 0$$

Similarly, $\partial^*\bar{\partial} + \bar{\partial}\partial^* = 0$.

Next expand Δ,

$$\begin{aligned}\Delta &= (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}) \\ &= (\partial\partial^* + \partial^*\partial) + (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}) + (\partial\bar{\partial}^* + \bar{\partial}^*\partial) + (\partial^*\bar{\partial} + \bar{\partial}\partial^*) \\ &= \Delta_\partial + \Delta_{\bar{\partial}}\end{aligned}$$

Finally, we check $\Delta_\partial = \Delta_{\bar{\partial}}$,

$$\begin{aligned}-i\Delta_\partial &= \partial(\Lambda\bar{\partial} - \bar{\partial}\Lambda) + (\Lambda\bar{\partial} - \bar{\partial}\Lambda)\partial \\ &= \partial\Lambda\bar{\partial} - \partial\bar{\partial}\Lambda + \Lambda\bar{\partial}\partial - \bar{\partial}\Lambda\partial \\ &= (\partial\Lambda - \Lambda\partial)\bar{\partial} + \bar{\partial}(\partial\Lambda - \Lambda\partial) = -i\Delta_{\bar{\partial}}\end{aligned}$$

□

Chapter 9

Kähler manifolds

In this chapter, we extend the results of the previous section to an important class of manifolds called Kähler manifolds.

9.1 Kähler metrics

Let X be a compact complex manifold with complex dimension n. Fix a hermitean metric H, which is a choice of hermitean inner product on the complex tangent spaces which vary in C^∞ fashion. More precisely, H would be given by a section of the $\mathcal{E}_X^{(1,0)} \otimes \mathcal{E}_X^{(0,1)}$, sthat in some (any) locally coordinate system around each point is given by

$$H = \sum h_{ij} dz_i \otimes d\bar{z}_j$$

with h_{ij} positive definite hermitean. By standard linear algebra, the real and imaginary parts of h_{ij} are respectively symmetric positive definite and nondegenerate skew symmetric matrices. Geometrically, the real part is just a Riemannian structure, while the (suitably normalized) imaginary part of gives a $(1,1)$-form ω called the Kähler form; in local coodinates

$$\omega = \frac{\sqrt{-1}}{2} \sum h_{ij} dz_i \wedge d\bar{z}_j.$$

X has a canonical orientation, so the Hodge star operator associated to the Riemannian structure can be defined. $*$ will be extended to $\mathbb{C}$-linear operator on $\mathcal{E}_X$, and set $\bar{*}(\alpha) = *\bar{\alpha}$. $*$ is compatible with the natural bigrading on forms in the sense that

$$*\mathcal{E}^{(p,q)}(X) \subseteq \mathcal{E}^{(n-q,n-p)}(X).$$

Let $\partial^* = -\bar{*}\partial\bar{*}$ and $\bar{\partial}^* = -\bar{*}\bar{\partial}\bar{*}$. These are adjoints of ∂ and $\bar{\partial}$. Then we can define the operators

$$\Delta_\partial = \partial^*\partial + \partial\partial^*,$$

$$\Delta_{\bar{\partial}} = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$$

$$L = \omega \wedge$$

$$\Lambda = - * L *$$

A form is called $\bar{\partial}$-harmonic if it lies in the kernel of $\Delta_{\bar{\partial}}$. For a general hermitean manifold there is no relation between harmonicity and $\bar{\partial}$-harmonicity. However, these notions do coincide for the class the class of Kähler manifolds, which we will define shortly.

A hermitean metric on X is called Kähler if there exist analytic coordinates about any point, for which the metric becomes Euclidean

$$h_{ij} = \delta_{ij} + O(z_1^2, z_1 z_2 \ldots)$$

up to second order. This implies that the Kähler form is closed: $d\omega = 0$, and in fact this is an equivalent condition [GH, p 107]. A Kähler manifold is a complex manifold with a Kähler metric. Standard examples of compact Kähler manifolds are:

Example 9.1.1. *Riemann surfaces with any hermitean metric since $d\omega$ vanishes for trivial reasons.*

Example 9.1.2. *Complex tori with flat metrics.*

Example 9.1.3. *$\mathbb{P}^n$ with the Fubini-Study metric. This is the unique metric with Kähler form which pulls back to*

$$\frac{i}{2\pi} \partial \bar{\partial} \log(|z_0|^2 + \ldots |z_n|^2)$$

under the canonical map $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$.

Example 9.1.4. *Complex submanifolds of Kähler manifolds inherit Kähler metrics where the Kähler class is the restriction Kähler class of the ambient manifold. In particular, any smooth projective variety possess a (nonunique) Kähler metric.*

9.2 The Hodge decomposition

Fix a Kähler manifold X. Theorem 8.3.3 generalizes:

Theorem 9.2.1. $\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}$.

Proof. Since Kähler metrics are Euclidean up to second order, any identity involving geometrically defined first order operators on Euclidean space can be automatically extended to Kähler manifolds. Thus proposition 8.3.4 extends. The rest of the argument is the same as in section 8.3. □

This actually characterizes Kähler manifolds.

Corollary 9.2.2. *If X is compact then $H^q(X, \Omega_X^p)$ is isomorphic the space of harmonic (p,q)-forms.*

Proof. See the proof of proposition 8.3.1. $\square$

We obtain the following special case of Serre duality as a consequence:

Corollary 9.2.3. *When X is compact, $H^p(X, \Omega_X^q) \cong H^{n-p}(X, \Omega_X^{n-q})$*

Proof. $\bar{*}$ induces an antilinear isomorphism between the corresponding spaces of harmonic forms. $\square$

Therefore we obtain the Hodge decomposition:

Theorem 9.2.4. *If X is a compact Kähler manifold then a differential form is harmonic if and only if its (p,q) components are. Consequently we have (for the moment) noncanonical isomorphisms*

$$H^i(X, \mathbb{C}) \cong \bigoplus_{p+q=i} H^q(X, \Omega_X^p).$$

Furthermore, complex conjugation induces $\mathbb{R}$-linear isomorphisms between the space of harmonic (p,q) and (q,p) forms. Therefore

$$H^q(X, \Omega_X^p) \cong H^p(X, \Omega_X^q).$$

Proof. The operator $\Delta_{\bar\partial}$ commutes with the projection $\pi_{(p,q)}$ while Δ commutes with complex conjugation. This implies the first and third statements. Together with corollary 9.2.2, we get the second. $\square$

Corollary 9.2.5. *The Hodge numbers $h^{pq}(X) = dim H^q(X, \Omega_X^p)$ are finite dimensional.*

Corollary 9.2.6. *If i is odd then the ith Betti number b_i of X is even.*

Proof.

$$b_i = 2\sum_{p<q} h^{pq}$$

$\square$

The first corollary is true for compact complex non Kähler manifolds, however the second may fail (the Hopf surface [GH] has $b_1 = 1$). The Hodge numbers gives a set of holomorphic invariants which can be arranged in a diamond:

$$\begin{array}{ccccccc}
 & & & h^{00}=1 & & & \\
 & h^{10} & & & & h^{01} & \\
h^{20} & & & h^{11} & & & h^{02} \\
 & & & \ldots & & & \\
 & h^{n,n-1} & & & & h^{n-1,n} & \\
 & & & h^{nn}=1 & & &
\end{array}$$

The previous results implies that this picture has both vertical and lateral symmetry (e.g $h^{10} = h^{01} = h^{n,n-1} = h^{n-1,n}$).

9.3 Picard groups

Theorem 9.3.1. *Let X be compact Kähler manifold. Then $Pic(X)$ fits into an extension*

$$0 \to Pic^0(X) \to Pic(X) \to H^2(X,\mathbb{Z}) \cap H^{11}(X) \to 0$$

with $Pic^0(X)$ a complex torus of dimension $h^{01}(X)$

Proof. There are two assertions. First is that $Pic^0(X)$ is torus, and second that the image of c_1 is desribed as above. The last assertion is called the Lefschetz $(1,1)$ theorem.

From the exponential sequence, we obtain

$$H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to Pic(X) \to H^2(X,\mathbb{Z})$$

Pic^0 is the cokernel of the first map. This map can be factored through $H^1(X,\mathbb{R}_X)$, and certainly $H^1(X,\mathbb{R})/H^1(X,\mathbb{Z})$ is a torus. Thus for the first part, it suffices to prove that

$$\pi : H^1(X,\mathbb{R}) \to H^1(X,\mathcal{O}_X)$$

is an isomorphism of real vector spaces. We know that that these space have the same real dimension $b_1 = 2h^{10}$, so it is enough to check that π is injective. The Hodge decomposition implies that $\alpha \in H^1(X,\mathbb{C})$ can be represented by the class of a $(1,0)$-form α_1 and a $(0,1)$-form α_2. If $\alpha = \bar{\alpha}$ then $\alpha_1 = \alpha_2$. $\pi(\alpha)$ is just α_1. Therefore $\pi(\alpha) = 0$ implies that $\alpha = 0$.

□

$Pic^0(X)$ is called the Picard torus. When X is a compact Riemann surface, $Pic^0(X)$ is usually called the Jacobian and denoted by $J(X)$

Example 9.3.2. *When $X = V/L$ is a complex torus, $Pic^0(X)$ is a new torus called the dual torus. Using corollary 8.3.2, it can be seen to be isomorphic to $\bar{V}^*/L^*$, where $\bar{V}^*$ is the antilinear dual and*

$$L^* = \{\lambda \in V^* \,|, Im(\lambda)(L) \subseteq \mathbb{Z}\}$$

The dual of the Picard torus is called the Albanese torus $Alb(X)$. Since $H^{01} = \bar{H}^{01}$, $Alb(X)$ is isomorphic to

$$\frac{H^0(X,\Omega^1_X)^*}{H_1(X,\mathbb{Z})/torsion},$$

where the elements γ of the denominator are identified with the integrals $\int_\gamma$.

Exercise 9.3.3.

1. *Let X be a compact Riemann surface. Show that the pairing $< \alpha, \beta >= \int_X \alpha \wedge \beta$ induces an isomorphism $J(X) \cong Alb(X)$; in other words $J(X)$ is self dual.*

2. *Fix a base point $x_0 \in X$. Show that the so called Abel-Jacobi map $X \to Alb(X)$ given by $x \mapsto \int_{x_0}^{x}$ (which is well defined modulo H_1) is holomorphic.*

3. *Show that every holomorphic 1-form on X is the pullback of a holomorphic 1-form from $Alb(X)$. In particular, the Abel-Jacobi map cannot be constant if $h^{01} \neq 0$.*

Chapter 10

Homological methods in Hodge theory

We introduce some homological tools which will allow us to extend and refine the results of the previous chapter.

10.1 Pure Hodge structures

It is useful isolate the purely linear algebraic features of the Hodge decomposition. We define a (pure) Hodge structure of weight m to be finitely generated abelian group H together with a bigrading

$$H_{\mathbb{C}} = H \otimes \mathbb{C} = \bigoplus_{p+q=m} H^{pq}$$

satisfying $\bar{H}^{pq} = H^{qp}$. We generally use the same symbol for Hodge structure and the underlying abelian group. Note that H can have torsion although won't usually consider such examples. One way to avoid torsion issues is to work with rational Hodge structures, where H is replaced by a finite dimensional rational vector space. Given a pure Hodge structure, define the Hodge filtration by

$$F^p H_{\mathbb{C}} = \bigoplus_{p' \geq p} H^{pq}.$$

The following is elementary.

Lemma 10.1.1. *If H is a pure Hodge structure of weight m then*

$$H_{\mathbb{C}} = F^p \bigoplus \bar{F}^{m-p+1}$$

for all p. Conversely if $F^\bullet$ is a descending filtration satisfying $F^a = H_{\mathbb{C}}$ and $F^b = 0$ for some $a, b \in \mathbb{Z}$ and satisfying the above identity, then

$$H^{pq} = F^p \cap \bar{F}^q$$

defines a pure Hodge structure of weight m.

The most natural examples come from the mth cohomology of a compact Kähler manifold, but it is easy to manufacture other examples. For example, we can define Hodge structures of negative weight. In fact, there is a unique rank one Hodge structure $\mathbb{Z}(i)$ of weight $-2i$ for any integer i. Here the underlying group is $\mathbb{Z}$, with $H^{(-i,-i)} = \mathbb{C}$. The collection of Hodge structures of forms a category, where a morphism is homomorphism f of abelian groups such that $f\otimes \mathbb{C}$ preserves the bigradings, or equivalently the Hodge filtrations. In particular, morphisms between Hodge structure with different weights must vanish. This category is abelian with a tensor product and duals. Explicitly, given Hodge structures H and G of weights n and m. their tensor product $H\otimes_{\mathbb{Z}} G$ is equipped with a weight $n+m$ Hodge structure with bigrading

$$(H\otimes G)^{pq} = \bigoplus_{p'+p''=p,q'+q''=q} H^{p'q'} \otimes G^{p''q''}$$

The dual $H^* = Hom(H,\mathbb{Z})$ is equipped with a weight $-n$ Hodge structure with bigrading

$$(H^*)^{pq} = (H^{-p,-q})^*.$$

The operation $H \mapsto H(i)$ is called the Tate twist. It has the effect of leaving H unchanged and shifting the bigrading by $(-i,-i)$.

The obvious isomorphism invariants of a Hodge structure H are its Hodge numbers $\dim H^{pq}$. However, this the doesn't completely characterize them. Consider, the set of $2g$ dimensional Hodge structures H of weight 1 and level 1. This means that the Hodge numbers are as follows: $\dim H^{10} = \dim H^{01} = g$ and the others zero. There are uncountably many isomorphism classes, in fact:

Lemma 10.1.2. *There is a one to one correspondence between isomorphism classes of Hodge structures as above and g dimensional complex tori given by*

$$H \mapsto \frac{H_{\mathbb{C}}}{H_{\mathbb{Z}} + F^1}$$

Exercise 10.1.3.

1. *If H is a weight one (not necessarily) level one Hodge structure, show that the construction of lemma 10.1.2 is a torus.*

2. *Prove lemma 10.1.2.*

10.2 Canonical Hodge Decomposition

The question remains as to what extent the Hodge decomposition can be made independent of the choice of Kähler metric.

Let X be a compact Riemann surface. then we have an exact sequence

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to \Omega^1_X \to 0$$

and we saw in lemma 5.4.6 that the induced map

$$H^0(X, \Omega^1_X) \to H^1(X, \mathbb{C})$$

is injective. If we define

$$F^0H^1(X, \mathbb{C}) = H^1(X, \mathbb{C})$$
$$F^1H^1(X, \mathbb{C}) = im(H^0(X, \Omega^1_X))$$
$$F^2H^1(X, \mathbb{C}) = 0$$

then this together with the isomorphism $H^1(X, \mathbb{C}) = H^1(X, \mathbb{Z}) \otimes \mathbb{C}$ determines a pure Hodge structure of weight 1. To see this choose a metric which is automatically Kähler because $dimX = 1$ (in fact $*$ is a conformal invariant so it's not necessary to choose a metric at all). Then $H^1(X, \mathbb{C})$ is isomorphic to direct sum of the space of harmonic $(1, 0)$-forms which maps to F^1 and the space of harmonic $(0, 1)$ forms which maps to $\bar{F}^1$.

Before proceeding with the higher dimensional version, we need some facts from homological algebra. Let $C^\bullet$ be a bounded below complex of modules over some ring (or even objects in an abelian category). Let

$$C^\bullet \supseteq F^pC^\bullet \supseteq F^{p+1}C^\bullet \ldots$$

be a biregular filtration by subcomplexes, i.e. for each i there exists a and b with $F^aC^i = C^i$ and $F^bC^i = 0$. We get a map on cohomology

$$\phi^p : H^\bullet(F^pC^\bullet) \to H^\bullet(C^\bullet)$$

and we let $F^pH^\bullet(C^\bullet)$ be the image. The filtration is said to be strictly compatible with differentials of $C^\bullet$, or simply just strict if all the ϕ's are injective.

Proposition 10.2.1. *The following are equivalent*

1. *F is strict.*
2. *$F^pC^{i+1} \cap dC^i = dF^pC^i$ for all i and p.*
3. *The connecting maps $H^i(Gr^pC^\bullet) \to H^{i+1}(F^{p+1}C^\bullet)$ associated to*

$$0 \to F^{p+1} \to F^p \to Gr^p \to 0$$

vanish for all i and p.

Proof. We first show (1) $\Rightarrow$ (2). Suppose the differential of $x \in C^i$ lies in F^p, then dx defines an element of $ker\phi^p$. Therefore $dx = dy$ for some $y \in F^p$ if F is strict. The proof that (2) $\Rightarrow$ (1) is similar.

The connecting map in (3) can be described explicitly as follows. Given $\bar{x} \in H^i(Gr^p)$, it can be lifted to an element $x \in F^p$ such that $dx \in F^{p+1}$ and the class of dx is the image of $\bar{x}$ under the connecting map. If (2) holds, then $dx = dy$ for some $y \in F^{p+1}$. Replacing x by $x - y$ shows that the image of $\bar{x}$ is zero.

(3) is equivalent to the injectivity of all the maps $H^i(F^{p+1}) \to H^i(F^p)$, and this implies (1). $\square$

Corollary 10.2.2. *$H^i(Gr^p C^\bullet)$ contains a canonical submodule $I = I^{i,p}$ together with a canonical surjection $I \to Gr^p H^i(C^\bullet)$. Isomorphisms $Gr^p H^i(C^\bullet) \cong I \cong H^i(Gr^p C^\bullet)$ hold, for all i, p, if and only if F is strict.*

Proof. Let $I = image[H^i(F^p) \to H^i(Gr^p C^\bullet)]$ then the surjection $H^i(F^p) \to Gr^p H^i(C^\bullet)$ factors through I. The remaining statement follows from (3) and a diagram chase. □

Corollary 10.2.3. *Suppose that $C^\bullet$ is a complex of vector spaces over a field such that $dim H^i(Gr^p) < \infty$ for all i, p. Then*

$$dim H^i(C^\bullet) \leq \sum_p dim H^i(Gr^p),$$

and equality holds for all i if and if F is strict.

Proof. We have

$$dim H^i(C^\bullet) = \sum_p dim Gr^p H^i(C^\bullet) \leq \sum_p dim I^{ip} \leq \sum_p dim H^i(Gr^p)$$

and equallity is equivalent to strictness of F by the previous corollary. □

These results are usually formulated in terms of spectral sequences which we have chosen to avoid. In this language these corollaries say that F is strict if and only if the associated spectral sequence degenerates at E_1.

Let X be a complex manifold, then the de Rham complex $\mathcal{E}^\bullet(X)$ has a filtration called the Hodge filtration:

$$F^p \mathcal{E}^\bullet(X) = \sum_{p' \geq p} \mathcal{E}^{p'q}(X)$$

The Hodge decomposition and corollary 10.2.3 imply:

Theorem 10.2.4. *If X is compact Kähler, the Hodge filtration is strict. The associated filtration $F^\bullet H^i(X, \mathbb{C})$ gives a canonical Hodge structure*

$$H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{pq}(X)$$

of weight i, where

$$H^{pq}(X) = F^p H^i(X, \mathbb{C}) \cap \bar{F}^q H^i(X, \mathbb{C}) \cong H^q(X, \Omega^p_X).$$

Even though harmonic theory lies is behind this. It should be clear that the final result does not involve the metric. The following is not so much a corollary as an explanation of what the term canonical means:

Corollary 10.2.5. *If $f : X \to Y$ is a holomorphic map of compact Kähler manifolds, then the pullback map $f^* : H^i(Y, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ is compatible with the Hodge structures.*

Corollary 10.2.6. *Global holomorphic differential forms on X are closed.*

Proof. Strictness implies that the maps

$$d : H^0(X, \Omega^p_X) \to H^0(X, \Omega^{p+1}_X)$$

vanish by proposition 10.2.1. □

This corollary, and hence the theorem, can fail for compact complex non Kähler manifolds [GH, p. 444].

Theorem 10.2.7. *If X is a compact Kähler manifold, the cup product*

$$H^i(X, \mathbb{Z}) \otimes H^j(X, \mathbb{Z}) \to H^{i+j}(Z, \mathbb{Z})$$

is a morphism of Hodge structures.

The proof comes down to the observation that

$$F^p\mathcal{E}^\bullet \wedge F^q\mathcal{E}^\bullet \subseteq F^{p+q}\mathcal{E}^\bullet$$

For the corollaries, we work with rational Hodge structures. We have compatability with Poincaré duality:

Corollary 10.2.8. *If $\dim X = n$, then Poincaré duality gives an isomorphism of Hodge structures*

$$H^i(X) \cong [H^{2n-i}(X)^*](-n)$$

We have compatability with the Künneth formula:

Corollary 10.2.9. *If X and Y are compact manifolds, then*

$$\bigoplus_{i+j=k} H^i(X) \otimes H^j(Y) \cong H^k(X \times Y)$$

is an isomorphism of Hodge structures.

We have compatability with Gysin map:

Corollary 10.2.10. *If $f : X \to Y$ is a holomorphic map of compact Kähler manifolds of dimension n and m respectively, the Gysin map is a morphism*

$$H^i(X) \to H^{i+2(m-n)}(Y)(n-m)$$

10.3 The $\partial\bar{\partial}$-lemma

It is possible to generalize theorem 10.2.4 to certain non Kähler manifolds such as nonprojective smooth proper algebraic varieties. The key ingredient is.

Lemma 10.3.1. ($\partial\bar{\partial}$-lemma) *If α is a $\bar{\partial}$-closed ∂-exact form on a compact Kähler manifold, then there exists a form β such that $\alpha = \partial\bar{\partial}\beta$.*

Proof. Let G be Green's operator and H harmonic projection. Then for any form η

$$\partial\eta = \partial H(\eta) + \partial\Delta G(\eta) = \Delta\partial G(\eta)$$

Furthermore $\partial G(\eta)$ is orthogonal to the harmonic forms, therefore it coincides with $G(\partial\eta)$. Similarly G commutes with $\bar{\partial}$. Harmonic forms are orthogonal to ∂-exact forms, thus

$$\alpha = \Delta G(\alpha) = 2\bar{\partial}\bar{\partial}^* G(\alpha) + 2\bar{\partial}^* G(\bar{\partial}\alpha) = 2\bar{\partial}\bar{\partial}^* G(\alpha).$$

Upon substitution of $\bar{\partial}^* = -i[\Lambda, \partial]$ and further simplification, we obtain

$$\alpha = \partial\bar{\partial}(const\Lambda G(\alpha))$$

□

Lemma 10.3.2. *If $f : X \to Y$ is a surjective holomorphic map of compact manifolds with X Kähler, then the $\partial\bar{\partial}$-lemma holds for Y.*

Proof. See [D2, 4.3]. □

A complex manifold X is called Moishezon if its field of meromorphic functions has transcendence degree equal to $\dim X$. These include smooth proper algebraic varieties. Moishezon [Mo] proved that such manifolds need not be Kähler but can always be blown up to a smooth projective variety. Hence:

Corollary 10.3.3. *The $\partial\bar{\partial}$-lemma holds for Moishezon manifolds.*

To see the power of this result, let us prove corollary 10.2.6 directly.

Lemma 10.3.4. *Global holomorphic differential forms are closed on compact manifolds for which the $\partial\bar{\partial}$-lemma holds.*

Proof. Suppose that α is a global holomorphic form. Then $d\alpha = \partial\alpha$ would lie in the image of $\partial\bar{\partial}$. This is impossible unless $d\alpha = 0$. □

A refinement of this yields:

Lemma 10.3.5. *F is strict for compact manifolds for which the $\partial\bar{\partial}$-lemma holds.*

Proof. See [DGMS]. □

Theorem 10.3.6. *If X is Moishezon manifold or more generally a manifold satisfying the hypothesis of lemma 10.3.2, then the conclusion of theorem 10.2.4 holds.*

10.4 Hypercohomology

It is possible to describe the relationship between the De Rham and Dolbeault cohomologies in more direct terms. But first, we need to generalize the constructions given in chapter 3. Recall that a complex of sheaves is a possibly infinite sequence of sheaves

$$\ldots \mathcal{F}^i \xrightarrow{d^i} \mathcal{F}^{i+1} \xrightarrow{d^{i+1}} \ldots$$

satisfying $d^{i+1}d^i = 0$. We say that the complex is bounded if only finitely many of these sheaves are nonzero. Given any sheaf $\mathcal{F}$ and natural number n, we get a complex

$$\mathcal{F}[n] = \ldots 0 \to \mathcal{F} \to 0 \ldots$$

with $\mathcal{F}$ in the nth position. The collection of bounded complexes of sheaves on a space X forms a category $C^b(X)$, where a morphism of complexes $f : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ is defined to be a collection of sheaf maps $\mathcal{E}^i \to \mathcal{F}^i$ which commute with the differentials. This category is abelian. We define additive functors $\mathcal{H}^i : C^b(X) \to Ab(X)$ by

$$\mathcal{H}^i(\mathcal{F}^\bullet) = ker(d^i)/image(d^{i-1})$$

A morphism in $C^b(X)$ is a called a quasi-isomorphism if it induces isomorphisms on all the sheaves $\mathcal{H}^i$.

Theorem 10.4.1. *Let X be a topological space, then there are additive functors $\mathbb{H}^i : C^b(X) \to Ab$, with $i \in \mathbb{N}$, such that*

1. *For any sheaf $\mathcal{F}$, $\mathbb{H}^i(X, \mathcal{F}[n]) = H^{i+n}(X, \mathcal{F})$*

2. *If $\mathcal{F}^\bullet$ is a complex of acyclic sheaves, $\mathbb{H}^i(X, \mathcal{F}^\bullet) = H^i(\Gamma(X, \mathcal{F}^\bullet)))$.*

3. *If $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ is a quasi-isomorphism, then the induced map $\mathbb{H}^i(X, \mathcal{E}^\bullet) \to \mathbb{H}^i(X, \mathcal{F}^\bullet)$ is an isomorphism.*

4. *If $0 \to \mathcal{E}^\bullet \to \mathcal{F}^\bullet \to \mathcal{G}^\bullet \to 0$ is exact, then there is an exact sequence*

$$0 \to \mathbb{H}^0(X, \mathcal{E}^\bullet) \to \mathbb{H}^0(X, \mathcal{F}^\bullet) \to \mathbb{H}^0(X, \mathcal{G}^\bullet) \to \mathbb{H}^1(X, \mathcal{E}^\bullet) \to \ldots$$

We merely give the construction and indicate the proofs of some of these statements. Complete proofs can be found in [GM] or [I]. We start by redoing the construction of cohomology for a single sheaf $\mathcal{F}$. The functor G defined in 3.1, gives a flabby sheaf $G(\mathcal{F})$ with an injective map $\mathcal{F} \to G(\mathcal{F})$. $C^1(\mathcal{F})$ is the cokernel of this map. Applying G again yields a sequence

$$\mathcal{F} \to G(\mathcal{F}) \to G(C^1(\mathcal{F}))$$

By continuing as above, we get a resolution by flabby sheaves

$$\mathcal{F} \to G^0(\mathcal{F}) \to G^1(\mathcal{F}) \to \ldots$$

Theorem 4.1.1 shows that $H^i(X, \mathcal{F})$ is the cohomology of the complex $\Gamma(X, G^\bullet(\mathcal{F}))$, and this gives the clue how to generalize the construction. The complex $G^\bullet$ is functorial. Given a complex

$$\ldots \mathcal{F}^i \xrightarrow{d} \mathcal{F}^{i+1} \ldots$$

we get a commutative diagram

$$\begin{array}{cccc} \ldots & \mathcal{F}^i & \xrightarrow{d} & \mathcal{F}^{i+1} \ldots \\ & \downarrow & & \downarrow \\ & G^0(\mathcal{F}^i) & \xrightarrow{d} & G^0(\mathcal{F}^{i+1}) \\ & \downarrow \partial & & \downarrow \partial \\ & \ldots & & \ldots \end{array}$$

We define, the total complex

$$T^i = \bigoplus_{p+q=i} G^p(\mathcal{F}^q)$$

with a differential $\delta = d + (-1)^q \partial$. We can now define

$$\mathbb{H}^i(X, \mathcal{F}^\bullet) = H^i(\Gamma(X, T^\bullet))$$

When applied to $\mathcal{F}[0]$, this yields $H^i(\Gamma(X, G^\bullet(\mathcal{F})))$ which as we have seen is $H^i(X, \mathcal{F})$, and this proves 1 when $n = 0$.

The precise relationship between the various (hyper)cohomology groups is usually expressed by the spectral sequence

$$E_1^{pq} = H^q(X, \mathcal{E}^p) \Rightarrow \mathbb{H}^{p+q}(\mathcal{E}^\bullet)$$

This has the following consequences that we can prove directly:

Corollary 10.4.2. *Suppose that $\mathcal{E}^\bullet$ is a bounded complex of sheaves of vector spaces then*

$$\dim \mathbb{H}^i(\mathcal{E}^\bullet) \leq \sum_{p+q=i} \dim H^q(X, \mathcal{E}^p)$$

and

$$\sum (-1)^i \dim \mathbb{H}^i(\mathcal{E}^\bullet) = \sum_{p+q=i} (-1)^{p+q} \dim H^q(X, \mathcal{E}^p)$$

and

Proof. Suppose that $\mathcal{E}^N$ is the last nonzero term of $\mathcal{E}^\bullet$. Let $\mathcal{F}^\bullet$ be the complex obtained by replacing $\mathcal{E}^N$ by zero in $\mathcal{E}^\bullet$. There is an exact sequence

$$0 \to \mathcal{E}^N[N] \to \mathcal{E}^\bullet \to \mathcal{F}^\bullet \to 0$$

which induces a long exact sequence of hypercohomology. Then the corollary follows by induction on the length (number of nonzero entries) of $\mathcal{E}^\bullet$. □

Corollary 10.4.3. *Suppose that $\mathcal{E}^\bullet$ is a bounded complex with $H^q(X, \mathcal{E}^p) = 0$ for all $p + q = i$, then $\mathbb{H}^i(\mathcal{E}^\bullet) = 0$.*

In order to facilitate the computation of hypercohomology, we need a criterion for when two complexes are quasi-isomorphic. We will say that a filtration

$$\mathcal{E}^\bullet \supseteq F^p\mathcal{E}^\bullet \supseteq F^{p+1}\mathcal{E}^\bullet \ldots$$

which is finite if $\mathcal{E}^\bullet = F^a\mathcal{E}^\bullet$ and $F^b\mathcal{E}^\bullet = 0$ for some a, b.

Lemma 10.4.4. *Let $f : \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of bounded complexes. Suppose that $F^p\mathcal{E}^\bullet$ and $G^p\mathcal{F}^\bullet$ are finite filtrations by subcomplexes such that $f(F^p\mathcal{E}^\bullet) \subseteq G^p\mathcal{F}^\bullet$. If the induced maps*

$$F^p\mathcal{E}^\bullet/F^{p+1}\mathcal{E}^\bullet \to G^p\mathcal{F}^\bullet/G^{p+1}\mathcal{F}^\bullet$$

are quasi-isomorphisms for all p, then f is a quasi-isomorphism.

10.5 Holomorphic de Rham complex

To illustrate the ideas from the previous section, let us reprove de Rham's theorem. Let X be a C^∞ manifold. We can resolve $\mathbb{C}_X$ by the complex of C^∞ forms $\mathcal{E}^\bullet_X$, which is acyclic. In other words, $\mathbb{C}_X$ and $\mathcal{E}^\bullet_X$ are quasi-isomorphic. It follows that

$$H^i(X, \mathbb{C}_X) = \mathbb{H}^i(X, \mathbb{C}_X[0]) \cong \mathbb{H}^i(X, \mathcal{E}^\bullet_X) \cong H^i(\Gamma(X, \mathcal{E}^\bullet_X))$$

The last group is just de Rham cohomology.

Now suppose that X is a (not necessarily compact) complex manifold. Then we define a subcomplex

$$F^p\mathcal{E}^\bullet_X = \sum_{p' \geq p} \mathcal{E}^{p'q}_X$$

The image of the map

$$\mathbb{H}^i(X, F^p\mathcal{E}^\bullet_X) \to \mathbb{H}^i(X, \mathcal{E}^\bullet_X)$$

is the filtration introduced just before theorem 10.2.4. We want to reinterpret this purely in terms of holomorphic forms. We define the holomorphic de Rham complex by

$$\mathcal{O}_X \to \Omega^1_X \to \Omega^2_X \ldots$$

This has a filtration (sometimes called the "stupid" filtration)

$$\sigma^p\Omega^\bullet_X = \ldots 0 \to \Omega^p_X \to \Omega^{p+1}_X \to \ldots$$

gotten by dropping the first $p - 1$ terms. We have a natural map $\Omega^\bullet_X \to \mathcal{E}^\bullet_X$ which takes σ^p to F^p. Dolbeaut's theorem 8.2.3 implies that F^p/F^{p+1} is quasi-isomorphic to $\sigma^p/\sigma^{p+1} = \Omega^p_X[p]$. Therefore, lemma 10.4.4 implies that $\Omega^\bullet_X \to \mathcal{E}^\bullet_X$, and more generally $\sigma^p\Omega^\bullet_X \to F^p\mathcal{E}^\bullet_X$, are quasi-isomorphisms. Thus:

Lemma 10.5.1. *$H^i(X,\mathbb{C}) \cong \mathbb{H}^i(X,\Omega_X^\bullet)$ and $F^pH^i(X,\mathbb{C})$ is the image of $\mathbb{H}^i(X,\sigma^p\Omega_X^\bullet)$.*

When X is compact Kähler, theorem 10.2.4 implies that the map

$$\mathbb{H}^i(X,\sigma^p\Omega_X^\bullet) \to \mathbb{H}^i(X,\Omega_X^\bullet)$$

is injective.

When X is compact, the Dolbeault groups are always finite dimensional. From corollaries 10.4.2 and 10.4.3, we obtain

Corollary 10.5.2. *If X is compact, the ith Betti number*

$$b_i(X) \leq \sum_{p+q=i} \dim H^q(X,\Omega_X^p)$$

and the Euler characteristic

$$\sum(-1)^i b_i(X) = \sum(-1)^{p+q} \dim H^q(X,\Omega_X^p)$$

Corollary 10.5.3. *If $H^q(X,\Omega_X^p) = 0$ for all $p+q=i$, then $H^i(X,\mathbb{C}) = 0$.*

Corollary 10.5.4. *Let X be a Stein manifold (see section 16.2), then $H^i(X,\mathbb{C}) = 0$ for $i > \dim X$.*

Proof. This follows from theorem 16.2.6. □

Chapter 11

Algebraic Surfaces

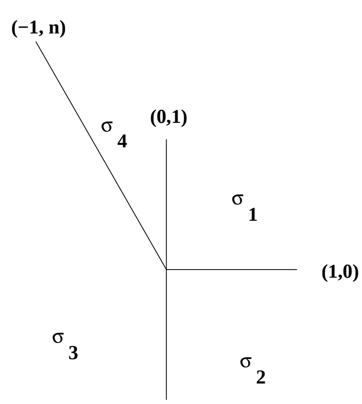
A (nonsingular) complex surface is a two dimensional complex manifold. By an algebraic surface, we will mean a two dimensional smooth projective surface. To be consistent, Riemann surfaces will be referred to as complex curves from now on.

11.1 Examples

Let X be an algebraic surface. We know from the previous chapter that there are three interesting Hodge numbers h^{10}, h^{20}, h^{11}. The first two are traditionally called (and denoted by) the irregularity (q) and geometric genus (p_g).

Example 11.1.1. *If $X = \mathbb{P}^2$, then $q = p_g = 0$ and $h^{11} = 1$. This will be checked in the next chapter.*

Example 11.1.2. *The next example is the rational ruled surface F_n, with n a nonnegative integer. This can be conveniently described as the toric variety (section 2.4) associated to the fan*



These surfaces are not isomorphic for different n. However they have the same invariants $q = p_g = 0$ and $h^{11} = 2$.

Example 11.1.3. *If $X = C_1 \times C_2$ is a product of two nonsingular curves of genus g_1 and g_2. Then by Künneth's formula $q = g_1 + g_2$, $p_g = g_1 g_2$ and $h^{11} = 2g_1g_2 + 4g_1 + 4g_2 + 6$.*

Example 11.1.4. *Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d. Then $q = 0$. We will list the first few values:*

d	p_g	h^{11}
2	0	2
3	0	7
4	1	20
5	4	45
6	10	86

These can be calculated using formulas given later (15.2.3).

A method of generating new examples from old is by blowing up. The basic construction is: Let

$$Bl_0\mathbb{C}^2 = \{(x, \ell) \in \mathbb{C}^2 \times \mathbb{P}^1 \,|\, x \in \ell\}$$

The projection $p_1 : Bl_0\mathbb{C}^2 \to \mathbb{C}^2$ is birational. This is called the blow up of $\mathbb{C}^2$ at 0. This can be generalized to yield the blow up $Bl_pX \to X$ of the surface X at point p. This is an algebraic surface which can be described analytically as follows. Let $B \subset X$ be a coordinate ball centered at p. After identifying B with a ball in $\mathbb{C}^2$ centered at 0, we can form let Bl_0B be the preimage of B in $Bl_0\mathbb{C}^2$. The boundary of Bl_0B can be identified with the boundary of B. Thus we can glue $X - B \cup Bl_0B$ to form Bl_pX.

Using the above description, we can compute $H^*(Y, \mathbb{Z})$ where $Y = Bl_pX$ by comparing Mayer-Vietoris sequences

$$\begin{array}{ccccccccc}
\to & H^i(X) & \to & H^i(X - B') & \oplus & H^i(B) & \to & H^i(X - B' \cap B) \\
 & \downarrow & & || & \oplus & \downarrow & & || \\
\to & H^i(Y) & \to & H^i(Y - Bl_pB') & \oplus & H^i(Bl_pB) & \to & H^i(Y - Bl_pB' \cap Bl_pB)
\end{array}$$

where $B' \subset B$ is a smaller ball. It follows that

Lemma 11.1.5. *$H^1(Y) \cong H^1(X)$ and $H^2(Y) = H^2(X) \oplus \mathbb{Z}$.*

Corollary 11.1.6. *q and p_g are invariant under blowing up. $h^{11}(Y) = h^{11}(X) + 1$.*

Proof. The lemma implies that $b_1 = 2q$ is invariant and $b_2(Y) = b_1(X)+1$. Since $b_2 = 2p_g + h^{11}$, the only possiblities are $h^{11}(Y) = h^{11}(X) + 1$, $p_g(Y) = p_g(X)$, or that $p_g(Y) < p_g(X)$. The last inequallity means that there is a nonzero holomorphic 2-form on X that vanishes on $X - p$, but this is impossible. $\square$

A morphism of varieties $f : V \to W$ is called birational if it is an isomorphism when restricted to nonempty Zariski open sets of X and Y. A birational map $V \dashrightarrow W$ is simply an isomorphism of Zariski open sets. Blow ups and their inverses ("blow downs") are examples of birational morphisms and maps. For the record, we point out

Theorem 11.1.7 (Castelnuovo). *Any birational map between algebraic surfaces is a given by a finite sequence of blow ups and downs.*

Therefore, we get the birational invariance of q and p_g. However, there are easier ways to prove this. Blow ups can also be used to extend meromorphic functions. A meromorphic function $f : X \dashrightarrow \mathbb{P}^1$ is a holomorphic map from an open subset $U \subseteq \mathbb{P}^1$ which can be expressed locally as a ratio of holomorphic functions.

Theorem 11.1.8 (Zariski). *If $f : X \dashrightarrow \mathbb{P}^1$ meromorphic function on an algebraic surface. Then there is a finite sequence of blow ups such that $Y \to X$ such that f extends to a holomorphic map $f' : X \to \mathbb{P}^1$.*

Exercise 11.1.9.

1. *Compare the construction of the blow up given here and in section 2.4 (identify $((x,t)$ with $((x,xt),[t,1])$).*
2. *Finish the proof of lemma 11.1.5.*

11.2 Castenuovo-de Franchis' theorem

When can try to study varieties by mapping them onto lower dimensional varieties. In the case of surfaces, the target should be a curve. A very useful criterion for this is

Theorem 11.2.1 (Castenuovo-de Franchis). *Suppose X is an algebraic surface. A necessary and sufficient condition for X to admit a constant holomorphic map to a smooth curve of genus $g \geq 2$ is that there exists two linear independant forms $\omega_i \in H^0(X, \Omega^1_X)$ such that $\omega_1 \wedge \omega_2 = 0$.*

The necessity is clear. We will sketch part of the argument of the converse, since it gives a nice application of corollary 10.2.6. A complete proof can be found in [BPV, pp123-125]. Choosing local coordinates, we can write

$$\omega_i = f_i(z_1, z_2)dz_1 + g_i(z_1, z_2)dz_2$$

The condtion $\omega_1 \wedge \omega_2 = 0$ is

$$(f_1 g_2 - f_2 g_1)dz_1 \wedge dz_2 = 0$$

which implies that

$$f_2/f_1 = g_2/g_1.$$

Thus $\omega_2 = F(z_1, z_2)\omega_1$. Since ω_i are globally defined, $F = \omega_2/\omega_1$ defines a global meromorphic function $X \dashrightarrow \mathbb{P}^1$. Since ω_i are closed (corollary 10.2.6),

$$dF \wedge \omega_1 = d\omega_2 = 0 \tag{11.1}$$

Choose a "general" point $p \in X$ and let t_1 be a local coordinate centered at $F(p)$. Let us also use t_1 for the pullback of this function to neighbbourhood of p. Then we can choose a function t_2 such that t_1, t_2 give local coordinates at p. (11.1) becomes $dt_1 \wedge \omega_1 = 0$. Consequently $\omega_1 = f(t_1, t_2)dt_1$. The relation $d\omega_1 = 0$, implies f is a function of t_1 alone. Thus ω_1 is locally the pullback of a 1-form on $\mathbb{P}^1$, and likewise for ω_2. This not true globally, since in fact there are no nonzero holomorphic 1-forms on $\mathbb{P}^1$. What we can do is the following. By theorem 11.1.8, there exists a blow up $Y \to X$ such that F extends to a holomorphic function $F' : Y \to \mathbb{P}^1$. The fibers of F' will not be connected. A theorem of Stein shows that the map can be factored as $Y \to C \to \mathbb{P}^1$, where the first map has connected fibers and $C \to \mathbb{P}^1$ is a finite to one map of curves. The argument above can be used to show that ω_i are pullback from holomorphic 1-forms on C. The final step is to prove that, the blow up is unnecessary, i.e. we can take $Y = X$.

An obvious corollary is:

Corollary 11.2.2. *If $q \geq 2$ and $p_g = 0$, then X admits a constant map to a curve as above.*

This can be improved substatially [BPV, IV, 4.2].

11.3 The Neron-Severi group

Let X be an algebraic surface once again. The image of the first Chern class map is the Neron-Severi group $NS(X) = H^2(X, \mathbb{Z}) \cap H^{11}(X)$. The rank of this group is called Picard number $\rho(X)$. We have $\rho \leq h^{11}$ with equality if $p_g = 0$.

A divisor on X is a finite integer linear combination $\sum n_i D_i$ of possibly singular curves D_i. We can define a line bundle $\mathcal{O}_X(D)$ as we did for Riemann surfaces in section 5.5. If f_i are local equations of $D_i \cap U$ in some open set U,

$$\mathcal{O}_X(D)(U) = \mathcal{O}_X(U)\frac{1}{f_1^{n_1} f_2^{n_2} \dots}$$

is a fractional ideal.

Proposition 11.3.1. *If D is a smooth curve, then $c_1(\mathcal{O}_X(D)) = [D]$.*

The cup product pairing

$$H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to H^4(X, \mathbb{Z}) \to \mathbb{Z}$$

restricts to $NS(X)$ denoted by "$\cdot$".

Lemma 11.3.2. *Given a pair of transverse smooth curves D and E,*

$$D \cdot E = \int_X c_1(\mathcal{O}(D)) \cup c_1(\mathcal{O}(E)) = \#(D \cap E)$$

Proof. This follows from the propositions 11.3.1 and 4.4.2. The number $i_p(D, E)$ is seen to be always $+1$ in this case. $\square$

If the intersection of the curves D and E is finite but not transverse, it is still possible to give a geometric meaning to the above product. For each $p \in X$, let $\mathcal{O}_p^{an}$ and $\mathcal{O}_p$ be the rings of germs of holomorphic respectively regular functions at p. Let $f, g \in \mathcal{O}_p \subset \mathcal{O}_p^{an}$ be the local equations of D and E respectively. Define $i_p(D, E) = \dim \mathcal{O}_p/(f, g)$. Then

Lemma 11.3.3.

$$i_p(D, E) = \dim \mathcal{O}_p^{an}/(f, g) = \frac{1}{(2\pi\sqrt{-1})^2} \int_{|f(z)|=|g(z)|=\epsilon} \frac{df \wedge dg}{fg}$$

Proof. [GH, pp. 662-670]. $\square$

Proposition 11.3.4. *If D, E are smooth curves such $D \cap E$ is finite, then*

$$D \cdot E = \sum_{p \in X} i_p(D, E)$$

Recall that $H^2(\mathbb{P}^2, \mathbb{Z}) = H^4(\mathbb{P}^2, \mathbb{Z}) = \mathbb{Z}$, and the generator of $H^2(\mathbb{P}^2, \mathbb{Z})$ is the class of line $[L]$. Given a curve D defined by a polynomial f, we define $\deg D = \deg f$. Then $[D] = \deg D[L]$.

Corollary 11.3.5 (Bezout). *If D, E are smooth curves on $\mathbb{P}^2$ with a finite intersection,*

$$\sum_{p \in X} i_p(D, E) = \deg(D) \deg(E)$$

Proof. We have $[D] = \deg D[L]$, and $[E] = \deg E[L]$, so that $D \cdot E = \deg D \deg E L \cdot L = \deg D \deg E$. $\square$

Exercise 11.3.6.

1. *Let $X = C \times C$ be product of a curve with itself. Consider, the divisors $H = C \times \{p\}$, $V = \{p\} \times C$ and the diagonal Δ. Show that these are independent in $NS(X) \otimes \mathbb{Q}$.*

2. *$E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be an elliptic curve, and let $X = E \times E$. Calculate the Picard number, and show that it could be 3 or 4 depending on τ.*

11.4 The Hodge index theorem

Let X a compact Kähler surface. Then the Kähler for ω is closed real $(1,1)$ form, therefore it defines an element of $H^{11}(X) \cap H^2(X, \mathbb{R})$.

Theorem 11.4.1. *Let X be a compact Kähler surface. Then the restriction of the cup product to $(H^{11}(X) \cap H^2(X, \mathbb{R})) \cap (\mathbb{R}[\omega])^\perp$ is negative definite.*

Proof. Locally, we can find an orthonormal basis $\{\phi_1, \phi_2\}$ of $\mathcal{E}^{(1,0)}$. In this basis,

$$\omega = \frac{\sqrt{-1}}{2}(\phi_1 \wedge \bar{\phi}_1 + \phi_2 \wedge \bar{\phi}_2)$$

and the volume form

$$dvol = \frac{\omega^2}{2} = \left(\frac{2}{\sqrt{-1}}\right)^2 \phi_1 \wedge \bar{\phi}_1 \wedge \phi_2 \wedge \bar{\phi}_2$$

Choose an element of $(H^{11}(X) \cap H^2(X, \mathbb{R})) \cap (\mathbb{R}[\omega])^\perp$ and represent it by a real $(1,1)$ form

$$\alpha = \sum a_{ij} \phi_i \wedge \bar{\phi}_j$$

Since $\bar{\alpha} = \alpha$, $a_{ji} = \bar{a}_{ij}$.

$$\int_X \alpha \wedge \omega = \frac{2}{\sqrt{-1}} \int_X (a_{11} + a_{22}) dvol = 0.$$

which shows $a_{11} + a_{22} = 0$. Therefore

$$\int_X \alpha \wedge \alpha = 2\left(\frac{2}{\sqrt{-1}}\right)^2 \int_X (|a_{11}|^2 + |a_{12}|^2) dvol < 0$$

□

Let X be a smooth projective surface. A divisor H on X is called if called very ample if there is an embedding $X \subset \mathbb{P}^n$ such that $\mathcal{O}_X(H) \cong \mathcal{O}_{\mathbb{P}^n}(1)|_X$. H is ample if some positive multiple is very ample. Since $H^2(\mathbb{P}^n, \mathbb{C})$ is one dimensional, the cohomology class associated to the Kähler class ω of the Fubini-Study metric 9.1.3. would have to be a nonzero multiple of $c_1(\mathcal{O}(1))$. In fact, the constants in ω are *chosen* so that these coincide. Thus $[H] = c_1(\mathcal{O}(H))$ will be the Kähler class for the induced metric, when H is very ample.

Corollary 11.4.2. *If H is an ample divisor on an algebraic surface, the intersection pairing is negative definite on $(NS(X) \otimes \mathbb{R}) \cap (\mathbb{R}H)^\perp$*

If H is ample, then $H^2 > 0$. The converse is false. Therefore when the cup product form is diagonalized, the diagonal consists of one 1 followed by $\rho - 1$ -1's. So we obtain

Corollary 11.4.3. *If H, D are divisors on an algebraic surface such that $H^2 > 0$ and $D \cdot H = 0$, then $D^2 < 0$*

Exercise 11.4.4.

1. *Prove that the restriction of the cup product to* $(H^{20}(X) + H^{02}(X)) \cap H^2(X, \mathbb{R})$ *is positive definite.*

2. *Conclude that (the matrix representing) the cup product pairing has* $2p_g+1$ *positive eigenvalues. Therefore* p_g *is a topological invariant.*

Chapter 12

Topology of families

12.1 Fiber bundles

A C^∞ map $f : X \to Y$ of manifolds is called a *fiber bundle* if it is locally a product of Y with another manifold (called the fiber). It is called trivial if f is a product. Nontrivial bundles over S^1 can be constructed as follows. Let F be a manifold with a diffeomorphism $\phi : F \to F$, then glue $F \times \{0\}$ in $F \times [0,1]$ to $F \times \{1\}$ by identifying $(x, 0)$ to $(\phi(x), 1)$. This includes the familiar example of the Mobius strip where $F = \mathbb{R}$ and ϕ is multiplication by -1. If the induced map $\phi^* : H^*(F) \to H^*(F)$, called monodromy, is nontrivial then the fiber bundle is nontrivial.

A C^∞ map $f : X \to Y$ is called a *submersion* if the map on tangent spaces is surjective. The fibers of such a map are submanifolds. A continuous map of topological spaces is called proper if the preimage of any compact set is compact.

Theorem 12.1.1 (Ehresmann). *Let $f : X \to Y$ be a proper smooth map of C^∞ connected manifolds. Then f is a C^∞ fiber bundle; in particular, the fibers are diffeomorphic.*

Proof. A complete proof can be found in [MK, 4.1]. Here we sketch the proof of the last statement. Since Y is connected, we can join any two points, say 0 and 1 by a path. Thus we replace Y by an interval, and hence by $\mathbb{R}$ which is diffeomorphic to it. Choose a Riemmanian metric on X, then by the inner products allow us to define a vector field (the gradient) ∇f dual to the differential df. By assumption df and therefore ∇f is nowhere zero. The existence and uniqueness theorem of ordinary linear differential equations allows us to define, for each $p \in f^{-1}(0)$, a C^∞ path $\gamma_p : \mathbb{R} \to X$ passing through p at time 0 and with velocity ∇f. Then the gradient flow $p \mapsto \gamma_p(1)$ gives the desired diffeomorphism. $\square$

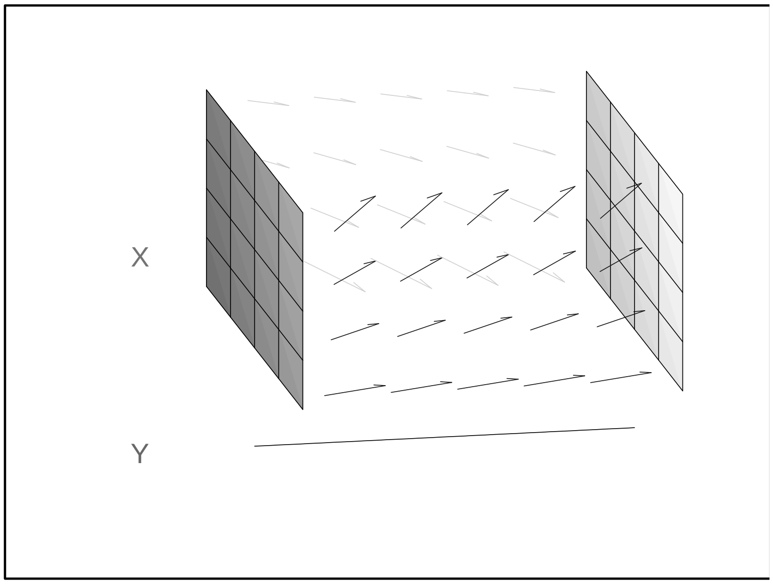


Figure 12.1: Gradient flow

12.2 A family of elliptic curves

Let's take a quick look at a complex analytic example. Legendre's family of projective cubic curves is

$$p_2 : E = \{([x, y, z], t) \in \mathbb{P}^2 \times \mathbb{C} \mid y^2 z - x(x - z)(x - tz) = 0\} \to \mathbb{C}$$

The curves are singular if and only if $t = 0, 1$, in which case, they rational curves with a single node. E restricts to a family of elliptic curves E^o over $\mathbb{C} - \{0, 1\}$. The map $E^o \to \mathbb{C} - \{0, 1\}$ is a submersion, and hence a C^∞ fiber bundle with a torus T as fiber. (Note that $\mathcal{E}^o$ is not locally trivial analytically, since the fibers are not even isomorphic.) Each smooth fiber E_t is a double cover of $\mathbb{P}^1$ branched over $0, t, 1, \infty$. Let $\bar{a}(t)$ and $\bar{b}(t)$ be noncrossing paths in $\mathbb{P}^1$ joining 0 to 1 and 0 to t respectively. Then the preimages of these paths in E_t form closed paths $a(t)$ and $b(t)$. If we orient these so that $a \cdot b = 1$, these form a basis of $H_1(E_t, \mathbb{Z})$. $b(t)$ is called a *vanishing cycle* since it shrinks to the node as $t \to 0$, see figure 12.2

The restriction of E^o to a bundle over the circle $S_\epsilon = \{t \mid |t| = \epsilon\}$ can be described by taking a trivial torus bundle $T \times [0, 1]$ and gluing the ends $T \times \{0\}$ and together $T \times \{1\}$ using a so called Dehn twist about the vanishing cycle $b = b(t)$. This is a diffeomorphism which is the identity outside a neighbourhood U of b and twists "once around" along b (see figures 12.3 and 12.3, U is the shaded region).

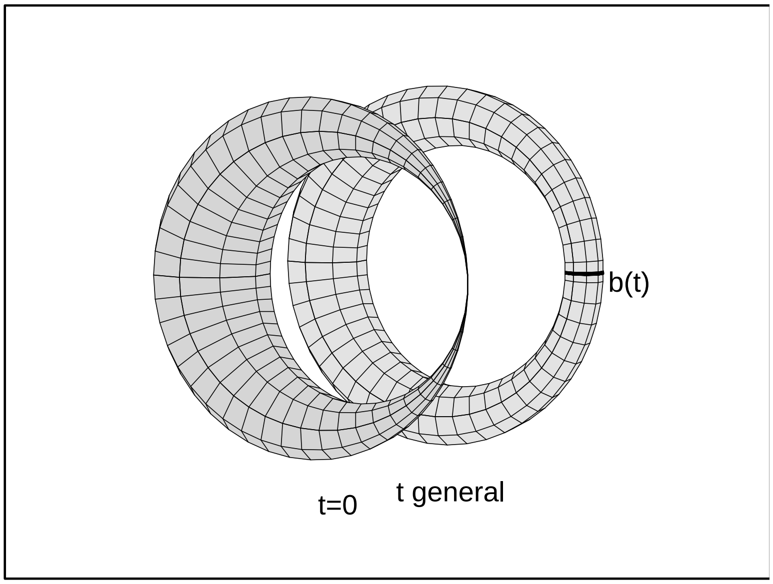


Figure 12.2: Legendre family

Note that the Dehn twist involves choices, however its effect on (co)homology, called the monodromy transformation μ associated to the loop, is independent of them. The matrix of the monodromy transformation is determined by the Picard-Lefschetz formula:

Theorem 12.2.1. *$\mu(a) = a + b$, and $\mu(b) = b$.*

Recall that $H/\Gamma(2) = \mathbb{C} - \{0, 1\}$, and E^0 can also be realized as a quotient of $\mathbb{C} \times H$ by an action of the semidirect product $\mathbb{Z}^2 \rtimes \Gamma(2)$. E is an example of an elliptic modular surface.

12.3 Local systems

In this section, we give a more formal treatment of monodromy.

Let X be a topological space, a path from $x \in X$ to $y \in X$ is a is a continuous map $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Two paths γ, η are homotopic if there is a continuous map $\Gamma : [0, 1] \times [0, 1] \to X$ such that $\gamma(t) = \Gamma(t, 0)$, $\eta(t) = \Gamma(t, 1)$ $\Gamma(0, s) = x$ and $\Gamma(1, s) = y$. We can compose paths: If γ is a path from x to y and η is a path from y to z, then $\gamma \cdot \eta$ is the path given by following one by the other at twice the speed. More formally,

$$\gamma \cdot \eta(t) = \begin{cases} \gamma(2t) & \text{if } t \leq 1/2 \\ \eta(2t - 1) & \text{if } t > 1/2 \end{cases}$$

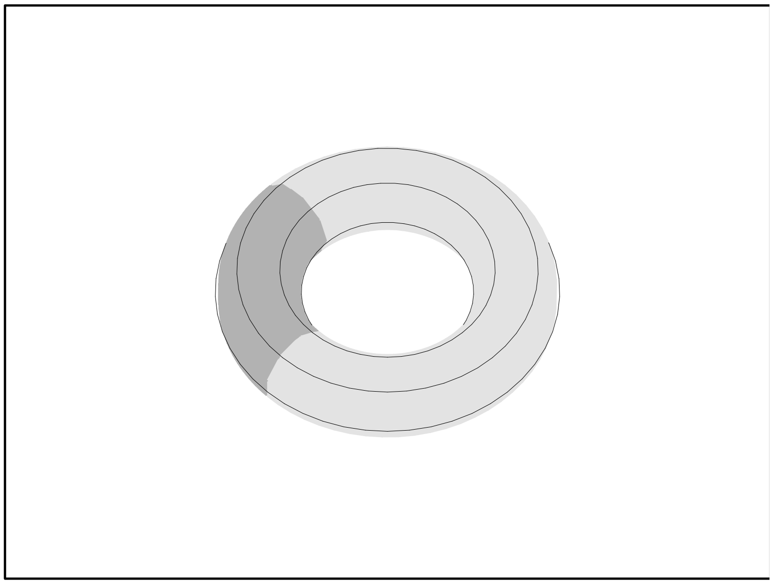


Figure 12.3: T foliated by meridians

This operation is compatible with homotopy in the obvious sense, and the induced operation on homotopy classes is associative. It almost a group law. To make this precise, we define a category $\Pi(X)$ whose objects are points of X and whose morphisms are homotopy classes of paths. This makes $\Pi(X)$ into a *groupiod* which means that every morphism is an isomorphism. In other words every (homotopy class of a) path has a inverse. This is not a group because it is not possible to compose any two paths. To get around this, we can consider loops, i. e. paths which start and end at the same place. Let $\pi_1(X, x)$ be the set of homotopy classes of loops based (starting and ending)at x. This is just $Hom_{\Pi(X)}(x, x)$ and as such it inherits a composition law which makes it a group called the fundamental group of (X, x). We summarize the standard properties that can be found in almost any toplogy textbook, e.g. [Sp]:

1. π_1 is a functor on the category of path connected spaces with base point and base point preserving continous maps.

2. If X is path connected, $\pi_1(X, x) \cong \pi_1(X, y)$ (consequently, we usually suppress the base point).

3. Two homotopy equivalent path connected spaces have isomorphic fundamental groups.

4. Van Kampen's theorem: If X is a path connected simplicial complex which is the union two subcomplexes $X_1 \cup X_2$, then $\pi_1(X)$ is the free product of $\pi_1(X_1)$ and $\pi_1(X_2)$ amalgamated along the image of $\pi_1(X_1 \cup X_2)$.

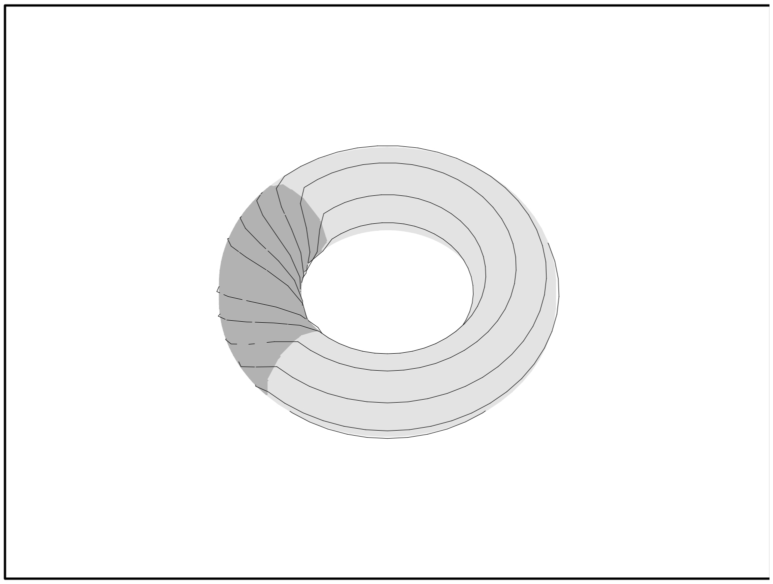


Figure 12.4: T foliated by images of meridians under Dehn twist

5. A X is a connected locally path connected space has a universal cover $\pi : \tilde{X} \to X$, and $\pi_1(X)$ is isomorphic to the group of deck transformations, i.e. selfhomeomorphisms of $\tilde{X}$ which commute with π.

This already suffices to calculate the examples of interest to us. It is easy to see that the fundmental group of the circle $\mathbb{R}/\mathbb{Z}$ is $\mathbb{Z}$. The complement in $\mathbb{C}$ of a set S of k points is homotopic to a wedge of k circles. Therefore $\pi_1(\mathbb{C} - S)$ is a free group on k generators.

Let X be a topological space. A *local system* of abelian groups is a functor $F : \Pi(X) \to Ab$. A local system F gives rise to a $\pi_1(X)$-modules i.e. abelian group $F(x)$ with a $\pi_1(X, x)$ action. We can also define a sheaf $\mathcal{F}$ as follows

$$\mathcal{F}(U) = \{f : U \to \cup F(x) \mid f(x) \in F(x) \text{ and } f(\gamma(1)) = F(\gamma)(f(\gamma(0)))\}$$

This sheaf is *locally constant* which means that every $x \in X$ has an open neighbourhood U such that $\mathcal{F}|_U$ is constant.

Theorem 12.3.1. *Let X be a connected good (i.e. locally path connected semilocally simply connected) topological space. There is an equivalence of categories between*

1. *The category of $\pi_1(X)$-modules.*

2. *The category of local systems and natural transformations.*

3. *The full subcategory of $Sh(X)$ of locally constant sheaves on X.*

In view of this theorem, we will treat local systems and locally constant sheaves as the same.

Let $f : X \to Y$ be a fiber bundle. We are going to construct a local system which takes $y \to H^i(X_y, \mathbb{Z})$. Given a path $\gamma : [0,1] \to Y$ joining y_0 and y_1, the pullback $\gamma^* X = \{(x,t) \mid f(x) = \gamma(t)\}$ will be a trivial bundle over $[0,1]$. Therefore $\gamma^* X$ will be homotopic to both X_{y_0} and X_{y_1}, and so we have isomorphisms

$$H^i(X_{y_0}) \overset{\sim}{\leftarrow} H^i(\gamma^* X) \overset{\sim}{\rightarrow} H^i(X_{y_1})$$

The map $H^i(X_{y_0}) \to H^i(X_{y_1})$ can be seen to depend only on the homotopy class of the path, thus we have a local system which gives rise to a locally constant sheaf which will be constructed directly in the next section.

12.4 Higher direct images

Let us start with a rather general situation. Let $f : X \to Y$ be a continuous map and $\mathcal{F} \in Sh(X)$ a sheaf. We can define the higher direct images by imitating the definition of $H^i(X, \mathcal{F})$ in section 3.2.

$$\begin{aligned} R^0 f_* \mathcal{F} &= f_* \mathcal{F} \\ R^1 f_* \mathcal{F} &= coker[f_* G(\mathcal{F}) \to f_* C^1(\mathcal{F})] \\ R^{n+1} f_* \mathcal{F} &= R^1 f_* C^n(\mathcal{F}) \end{aligned}$$

Then we have an analogue of theorem 3.2.1.

Theorem 12.4.1. *Given an exact sequence of sheaves*

$$0 \to A \to B \to C \to 0,$$

there is a long exact sequence

$$0 \to R^0 f_* A \to R^0 f_* B \to R^0 f_* C \to R^1 f_* A \dots$$

There is an alternative description which is a bit more convenient.

Lemma 12.4.2. *If $f : X \to Y$ is a continuous map, and $\mathcal{F} \in Sh(X)$, then $R^i f_* \mathcal{F}(U)$ is the sheafification of $U \mapsto H^i(U, \mathcal{F})$*

Each element of $H^i(X, \mathcal{F})$ determines an a global section of the presheaf $U \mapsto H^i(U, \mathcal{F})$ and hence of the sheaf $R^i f_* \mathcal{F}$. This map $H^i(X, \mathcal{F}) \to H^0(X, R^i f_* \mathcal{F})$ is often called an edge homomorphism.

Let us now assume that $f : X \to Y$ is a fiber bundle of triangulable spaces. Then choosing a contractible neighbourhood U of y, we see that $H^i(U, \mathbb{Z}) \cong H^i(X_y, \mathbb{Z})$. Since such neigbourhoods are cofinal, it follows that $R^i f_* \mathbb{Z}$ is locally constant. This coincides with the sheaf associated to the local system $H^i(X_y, \mathbb{Z})$ constructed in the previous section. The global sections of $H^0(Y, R^i f_* \mathbb{Z})$ is the space $H^i(X_y, \mathbb{Z})^{\pi_1(Y,y)}$ of invariant cohomology classes on the fiber. We can construct elements of this space using the edge homomorphism. For more general maps, we still have $(R^i f_*\mathbb{Z})_y \cong H^i(X_y, \mathbb{Z})$, but the ranks can jump, so these need not be locally constant.

The importance of the higher direct images is that it allows the cohomology of X to be computed on Y. The precise relationship is given by the Leray spectral sequence

$$E_2^{pq} = H^p(Y, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

which we won't go into. However, we will explain one of its consequences.

Proposition 12.4.3. *If $\mathcal{F}$ is a sheaf of vector spaces over a field, such that $\dim H^p(Y, R^q f_*\mathcal{F}) < \infty$, then*

$$\dim H^i(X, \mathcal{F}) \leq \sum_{p+q=i} H^p(Y, R^q f_*\mathcal{F})$$

It is possible to give a self contained proof of this using corollary 10.4.2.

Exercise 12.4.4.

1. *Prove theorem 12.4.1.*

Chapter 13

The Hard Lefschetz Theorem

13.1 Hard Lefschetz and its consequences

Let X be an n dimensional compact Kähler manifold. Recall that L was defined by wedging with Kähler class ω. Since ω is closed, L takes closed (respectively exact) forms to closed (exact) forms. Therefore this operation is defined on cohomology. The space

$$P^i(X) = ker[L^{n-i+1} : H^i(X, \mathbb{C}) \to H^{2n-i+2}(X, \mathbb{C})]$$

is called the primitive part of cohomology.

Theorem 13.1.1. *For every i,*

$$L^i : H^{n-i}(X, \mathbb{C}) \to H^{n+i}(X, \mathbb{C})$$

is an isomorphism. For every i,

$$H^i(X, \mathbb{C}) = \bigoplus_{j=0}^{[i/2]} L^j P^{i-2j}(X)$$

We indicate the proof in the next section. As a simple corollary we find that the Betti numbers $b_{n-i} = b_{n+i}$. Of course, this is nothing new since this also follows Poincaré duality. However, it is easy to extract some less trivial "numerology".

Corollary 13.1.2. *The Betti numbers satisfy $b_{i-2} \leq b_i$ for $i \leq n/2$.*

Proof. The theorem implies that the map $L : H^{i-2}(X) \to H^i(X)$ is injective. $\square$

Suppose that $X = \mathbb{P}^n$ with the Fubini-Study metric 9.1.3. the Kähler class $\omega = c_1(\mathcal{O}(1))$. The class $c_1(\mathcal{O}(1))^i \neq 0$ is the fundamental class of a codimension i linear space (see sections 6.2 and 6.5), so it is nonzero. Since all the cohomology groups of $\mathbb{P}^n$ are either 0 or 1 dimensional, this implies the Hard Lefschetz theorem for $\mathbb{P}^n$. Things get much more interesting when $X \subset \mathbb{P}^n$ is a nonsingular subvariety with induced metric. By Poinaré duality and the previous remarks, we get a statement closer to what Lefschetz would have stated[1]. Namely, that any element of $H_{n-i}(X, \mathbb{Q})$ is homologous to the intersection of a class in $H_{n+i}(X, \mathbb{Q})$ with a codimension i subspace.

The Hodge index for surfaces admits the following generalization called the Hodge-Riemman relations to an dimensional compact Kähler manifold X. Consider the pairing

$$H^i(X, \mathbb{C}) \times H^i(X, \mathbb{C}) \to \mathbb{C}$$

defined by

$$Q(\alpha, \beta) = (-1)^{i(i-1)/2} \int_X \alpha \wedge \beta \wedge \omega^{n-i}$$

Theorem 13.1.3. *$H^i(X) = \oplus H^{pq}(X)$ is an orthogonal decomposition. If $\alpha \in P^{p+q}(X) \cap H^{pq}(X)$ is nonzero,*

$$\sqrt{-1}^{p-q} Q(\alpha, \bar{\alpha}) > 0.$$

Proof. See [GH, p.123]. ☐

Let us introduce the Weil operator $C : P^i(X) \to P^i(X)$ which acts on $P^{p+q}(X) \cap H^{pq}(X)$ by multiplication by $\sqrt{-1}^{p-q}$.

Corollary 13.1.4. *The form $\tilde{Q}(\alpha, \beta) = Q(\alpha, C\bar{\beta})$ on $P^i(X)$ is positive definite Hermitean.*

13.2 More identities

Let X be as in the previous section. We defined the operators L, Λ acting forms $\mathcal{E}^\bullet(X)$ in section 9.1. We define a new operator H which acts by multiplication by $n - i$ on $\mathcal{E}^i(X)$. Then:

Proposition 13.2.1. *The following hold:*

1. $[\Lambda, L] = H$
2. $[H, L] = -2L$
3. $[H, \Lambda] = 2\Lambda$

[1] Lefschetz stated a version of this theorem for varieties in his book [L], but to my knowledge no one has ever made his proof rigorous. The first correct proof is due to Hodge using harmonic forms. A subsequent arithemetic proof was given by Deligne. However, there is still no geometric proof.

Furthermore these operators commute with Δ.

Proof. See [GH, p 115, 121]. $\square$

This plus the following theorem of linear algebra will prove the Hard Lefschetz theorem.

Theorem 13.2.2. *Let V be a vector space with endomorphisms L, Λ, H satisfying the above idenities. Then*

1. *H is diagonalizable with integer eigenvalues.*
2. *For each $a \in \mathbb{Z}$ let V_a be the space of eigenvectors of H with eigenvalue a. Then L^i induces an isomomorphism between V_i and V_{-i}.*
3. *If $P = ker(\Lambda)$, then*
$$V = P \oplus LP \oplus L^2P \oplus \dots$$
4. *$\alpha \in P \cap V_i$ then $L^{i+1}\alpha = 0$.*

We should say a few words about this mysterious theorem of linear algebra. Consider Lie algebra $sl_2(\mathbb{C})$ of space of traceless 2×2 complex matrices. This is a Lie algebra with a basis is given by

$$\lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\ell = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These matrices satisfy

$$[\lambda, \ell] = h,\ [h, \lambda] = 2\lambda,\ [h, \ell] = -2\ell$$

So the hypothesis of the theorem is simply that V is a representation of $sl_2(\mathbb{C})$. The theorem can then be deduced from the following two facts from representation theory of $sl_2(\mathbb{C})$ (which can be found in almost any book on Lie theory):

1. Every representation of $sl_2(\mathbb{C})$ is a direct sum of irreducible representations.
2. An irreducible representation is classified by an integer $N \geq 0$, and has the following structure

$$0 \xleftarrow[\lambda]{} V_{-N} = \mathbb{C} \underset{\lambda \neq 0}{\overset{\ell \neq 0}{\rightleftarrows}} V_{-N+2} = \mathbb{C} \underset{\lambda \neq 0}{\overset{\ell \neq 0}{\rightleftarrows}} \cdots \underset{\lambda \neq 0}{\overset{\ell \neq 0}{\rightleftarrows}} V_N = \mathbb{C} \xrightarrow{\ell} 0$$

13.3 Lefschetz Pencils

In this section, we are going to reverse history and use the hard Lefschetz theorem to deduce the original statement that Lefschetz used in his attempt to prove it. Choose a smooth n dimensional projective variety $X \subset \mathbb{P}^N$. We can and will assume that X is nondegenerate which means that X does lie on a hyperplane. Let $\check{\mathbb{P}}^N$ be the dual projective space whose points correspond to hyperplanes of $\mathbb{P}^N$. A line $\{H_t\}_{t\in\mathbb{P}^1} \subset \check{\mathbb{P}}^N$ is called a pencil of hyperplanes. A pencil $\{H_t\}$ is called Lefschetz if $H_t \cap X$ has at worst a single node (i.e. singularity $x \in H \cap X$ with $\hat{O}_x \cong \mathbb{C}[[x_1, \dots x_n]]/(x_1^2 + \dots x_n^2)$) for all $t \in \mathbb{P}^1$.

Lemma 13.3.1. *Lefschetz pencils exist.*

Given a pencil, we form an incidence variety $\tilde{X} = \{(x,t) \in X \times \mathbb{P}^1 \mid x \in H_t\}$ The second projection gives a map onto $\mathbb{P}^1$ whose fibers are intersections $H \cap X$. There is a finite set S of $t \in \mathbb{P}^1$ with $p^{-1}t$ singular. Let $U = \mathbb{P}^1 - S$. Choose $t_0 \in U$, set $Y_0 = p^{-1}(t_0)$ and consider the diagram

$$\begin{tikzcd} X & \tilde{X} \arrow[l, "\pi"'] \arrow[d, "p"] \\ Y_0 \arrow[u, "\tau"] \arrow[ur, "\tilde{\tau}"] & \mathbb{P}^1 \end{tikzcd}$$

Choose small disks Δ_i around each $t_i \in S$, and connect these by paths γ_i to the base point t_0 (figure 13.1).

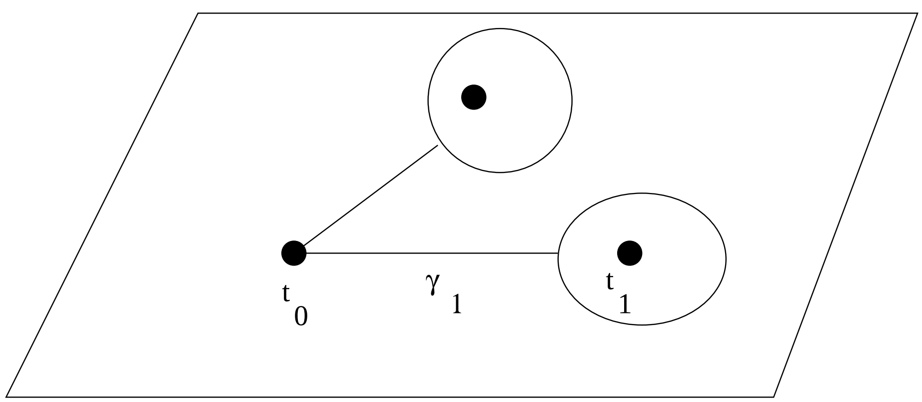


Figure 13.1: Loops

The space $p^{-1}(\gamma_i \cup \Delta_i)$ is homotopic to the singular fiber $Y_i = p^{-1}(t_i)$. The kernel $H^{n-1}(Y_0, \mathbb{Q}) \to H^{n-1}((\gamma_i \cup \Delta_i), \mathbb{Q})$ is generated by a class $\delta_i \in H^{n-1}(Y_0, \mathbb{Q})$ called a vanishing cycle (see figure 12.2). Let μ_i denote the monodromy of going once around Δ_i.

Theorem 13.3.2 (Picard-Lefschetz). *$\mu_i(\alpha) = \alpha \pm \langle \alpha, \delta_i \rangle \delta_i$ where $\langle , \rangle$ denotes the cup product pairing on $H^{n-1}(Y_0, \mathbb{Q})$*

Proof. [La]. $\square$

Let $V \subset H^{n-1}(Y_0, \mathbb{Q})$ be the subspace spanned by the δ_i.

Corollary 13.3.3. *The orthogonal complement $V^\perp$ coincides with the subspace $H^{n-1}(Y_0)^{\pi_1(U)} \subset H^{n-1}(Y_0)$ of classes invariant under $\pi_1(U)$.*

The image of $H^{n-1}(X, \mathbb{Q})$ lies in $H^{n-1}(Y_0)^{\pi_1(U)}$ since it factors through $H^{n-1}(\tilde{X}, \mathbb{Q})$.

Theorem 13.3.4.

(a) V is an irreducible $\pi_1(U)$-module.

(b) $H^{n-1}(Y_0)^{\pi_1(U)} = \tau^* H^{n-1}(X, \mathbb{Q})$.

(c) $\tau^* H^{n-1}(X, \mathbb{Q}) \cap V = 0$.

(d) $H^{n-1}(Y, \mathbb{Q}) = \tau^* H^{n-1}(X, \mathbb{Q}) \oplus V$

Proof. We prove the last two statements. For the previous statements, see [La]. By Poincaré duality $\dim H^{n-1}(Y, \mathbb{Q}) = \dim V + \dim V^\perp$. By part (b) and corollary 13.3.3, $\tau^* H^{n-1}(X) = V^\perp$. The restriction of the cup product pairing to $\tau^* H^{n-1}(X)$ coincides with the pairing

$$(\alpha, \beta) \mapsto \int_Y \tau^*\alpha \cup \tau^*\beta = \pm Q(\alpha, \beta)$$

which is nondegenerate by corollary 13.1.4. This implies that $V \cap V^\perp = 0$ which proves (c) and (d). $\square$

13.4 Barth's theorem

Hartshorne [H1] gave a short elegent proof of a theorem of Barth based on the hard Lefschetz theorem. His reproduce his argument here.

Proposition 13.4.1. *If X is an n dimensional nonsingular complex projective variety, such there positive integers $m < n$ and $N \leq 2m - n$ for which*

$$1 = b_2 = b_4 = \ldots b_{2([N/2]+n-m)}$$

$$b_1 = b_3 = \ldots b_{1+2([(N-1)/2]+n-m)}$$

Then if $Y \subset X$ a nonsingular m dimensional subvariety, the restriction map $H^i(X, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$ is an isomorphism for $i < N$.

Proof. The assumtions imply that $H^{2(n-m)}(X)$ is generated by L^{n-m}. Therefore $[Y] = d[H]^{n-m}$ with $d \neq 0$. Let $\iota : Y \to X$ denote the inclusion. Let L be the Lefschetz operator associated to H and $H|_Y$ (it will be clear from context, which is which). Consider the diagram

$$\begin{tikzcd}
H^i(X) \arrow[r, "L^{n-m}"] \arrow[d, "\iota^*"] & H^{i+2(n-m)}(X) \arrow[d, "\iota^*"] \\
H^i(Y) \arrow[r, "L^{n-m}"] \arrow[ur, "(1/d)\iota_!"] & H^{i+2(n-m)}(X)
\end{tikzcd}$$

The diagram commutes thanks to the identities

$$\iota^*\iota_!\alpha = i^*[Y] \cup \alpha$$

$$\iota_!\iota^*\beta = [Y] \cup \beta$$

$$\iota^*(\beta \cup \gamma) = \iota^*\beta \cup \iota^*\gamma$$

Hard Lefschetz for X implies that $L^{n-m} : H^i(Y) \to H^{i+2(n-m)}(Y)$ is injective and hence an isomorphism by our assumptions. It follows that the restriction $\iota^* : H^i(X) \to H^i(Y)$ is injective. It's enough to prove equality of dimension. Hard Lefschetz for Y implies that $L^{n-m} : H^i(Y) \to H^{i+2(n-m)}(Y)$ is injective. Therefore the same is true of $\iota_!$. Therefore

$$b_i(X) \leq b_i(Y) \leq b_{i+2(n-m)}(X) = b_i(X)$$

$\square$

Corollary 13.4.2 (Barth). *If $Y \subset \mathbb{P}^n$ is a nonsingular, complex projective variety, then $H^i(\mathbb{P}^n, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$ is an isomorphism for $i < \dim Y$.*

When Y has codimension one, this is the Lefschetz hyperplane theorem (or more acurately a special case of it).

13.5 Hodge conjecture

Let X be an n dimensional nonsingular complex projective variety. We define the space of codimension p Hodge cycles on X to be

$$Hodge^p(X) = H^{2p}(X, \mathbb{Z}) \cap H^{pp}(X)$$

Lemma 13.5.1. *Given a nonsingular subvariety $i : Y \to X$ of codimension p, the fundamental class $[Y] \in Hodge^p(X)$*

Proof. $[Y]$ corresponds under Poincaré duality to the functional $\alpha \mapsto \int_Y \alpha$. If α has type (a, b), then $\alpha|_Y = 0$ unless $a = b = p$. Therefore $[Y]$ has type (p, p) thanks to corollary 10.2.8. The class $[Y]$ is also integral, hence the lemma. $\square$

Even if Y has singularities, a fundamental class can be defined with the above properties. Here we give a quick but nonelementary definition. Let us first observe.

Lemma 13.5.2. *$\alpha \in Hodge^p(X)$ if and only if $1 \mapsto \alpha$ defines a morphism of Hodge structures $\mathbb{Z} \to H^{2p}(X, \mathbb{Z})(p)$. Consequently, $Hodge^p(X) \cong Hom(\mathbb{Z}, H^{2p}(X, \mathbb{Z})(p))$.*

By Hironaka's famous theorem [Ha], there exists a smooth projective variety $\tilde{Y}$ with a birational map $p : \tilde{Y} \to Y$. Let $\tilde{i} : \tilde{Y} \to X$ denote the composition of p and the inclusion. By corollary 10.2.10, we have a morphism

$$\tilde{i}_! : H^0(\tilde{Y}) = \mathbb{Z} \to\in H^{2p}(X)(-p)$$

This defines a class $[Y] \in Hodge^p(X)$ which is easily seen to be independent of $\tilde{Y}$.

Let $A^p(X) \subseteq Hodge^p(X) \otimes \mathbb{Q}$ be the subspace spanned by fundamental classes of codimension p subvarieties. The (in)famous Hodge conjecture asserts:

Conjecture 13.5.3 (Hodge). *$A^p(X)$ and $Hodge^p(X) \otimes \mathbb{Q}$ coincide.*

Note that in the original formulation $\mathbb{Z}$ was used in place of $\mathbb{Q}$, but this is known to be false [AH]. For $p = 1$, the conjecture is true by the Lefschetz $(1, 1)$ theorem 9.3.1. We prove it holds for $p = \dim, X - 1$.

Proposition 13.5.4. *If the Hodge conjecture holds for X in degree p with $p < n = \dim X$, i.e. if $A^p(X) = Hodge^p(X) \otimes \mathbb{Q}$, then it holds in degree $n - p$.*

Proof. Let L be the Lefshetz operator corresponding to a projective embedding $X \subset \mathbb{P}^N$. Then for any subvariety Y $L[Y] = [Y \cap H]$ where H is a hyperplane section chosen in general position. It follows thatL^{n-2p} takes $A^p(X)$ to $A^{n-p}(X)$ and the map is injective. Thus

$$\dim Hodge^p(X) \otimes \mathbb{Q} = \dim A^p(X) \leq \dim A^{n-p}(X) \leq \dim Hodge^{n-p}(X) \otimes \mathbb{Q}$$

On the other hand L^{n-2p} induces an isomorphism of Hodge structures $H^{2p}(X, \mathbb{Q})(p-n) \cong H^{2n-2p}(X, \mathbb{Q})$, and therefore an isomorphism $Hodge^p(X) \otimes \mathbb{Q} \cong Hodge^{n-p}(X \otimes \mathbb{Q})$. Thus forces equality of the above dimensions. $\square$

Corollary 13.5.5. *The Hodge conjecture holds in degree $n - 1$.*

Further information can be found in [Lw].

13.6 Degeneration of Leray

We want to mention one last consequence of the Hard Lefschetz theorem due to Deligne [D1]. A projective morphism of smooth complex algebraic varieties is called smooth if the induced maps on (algbraic) tangent spaces are surjective. In particular, such maps are C^∞ fiber bundles.

Theorem 13.6.1. *Let $f : X \to Y$ be smooth projective map of smooth complex algberaic varieties, then the inequalities in proposition 12.4.3 for $\mathcal{F} = \mathbb{Q}$ are equalities, i.e.*

$$\dim H^i(X, \mathbb{Q}) = \sum_{p+q=i} H^p(Y, R^q f_* \mathbb{Q})$$

Note that this generally fails for C^∞ fiber bundles.

Chapter 14

Coherent sheaves on Projective Space

In this chapter, we develop some algebraic tools for studying sheaves on projective spaces. In the last section, we discuss Serre's GAGA theorems which shows that on projective varieties there is an equivalence between the algebraic and analytic theories.

14.1 Cohomology of line bundles on $\mathbb{P}^n$

Let k be a field, and let $(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}})$ denote projective space over k viewed as an algebraic variety 1.3 (even if $k = \mathbb{C}$). Then $\mathbb{P}^n_k$ has a covering by $n+1$ open affine sets $U_i = \{x_i \neq 0\}$, where x_i are the homogenous coordinates. U_i can be identified with affine n-space with coordinates

$$\frac{x_0}{x_i}, \ldots \frac{\widehat{x_i}}{x_i} \ldots \frac{x_n}{x_i}$$

Our goal is to compute cohomology of $\mathcal{O}_{\mathbb{P}^n}(i)$ using the Cech complex with respect to this covering. But we need to show that this a Leray covering. Since the interesction of a finite number of the U_i's is affine, this will follow from:

Theorem 14.1.1 (Serre). *If X is an affine variety, then*

$$H^i(X, \mathcal{O}_X) = 0$$

for all $i > 0$.

This will be proven in a more general form in section 16.2.

Theorem 14.1.2 (Serre). *Let S_i be the space of homogeneous degree i polynomials in the variables $x_0, \ldots x_n$, then*

a) $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(d)) \cong S_d$ (in particular it is 0 if $d < 0$).

b) $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(d)) = 0$ *if* $0 \neq i \neq n$.

c) $H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(d)) \cong S_{-d-n-1}$.

A complete proof can be found in [H, III.5], although from a somewhat more general point of view. We use Čech cohomology with respect to the covering $\{U_i\}$. We use the description of $\mathcal{O}(d)$ given in section 6.3, where we identify a section of $\mathcal{O}(d)(U)$ with collection of functions $f_i \in \mathcal{O}(U \cap U_i)$ satisfying

$$f_i = (x_j/x_i)^d f_j$$

Formally cross multiplying yields

$$x_i^d f_i = x_j^d f_j \tag{14.1}$$

which will be more convenient for us. In order, to interpret this step, note $\mathcal{O}(U_i)$ can be identified with the polynomial ring $k[x_0/x_i, \ldots x_n/x_i]$. Let $S = k[x_0, \ldots x_n]$. The localization

$$S\left[\frac{1}{x_i}\right] = \bigoplus_d S\left[\frac{1}{x_i}\right]_d$$

has a natural $\mathbb{Z}$-grading, where

$$S\left[\frac{1}{x_i}\right]_d = \sum_e \frac{S_{d+e}}{x_i^e}.$$

Similar statements apply to the other localizations $S[1/x_i x_j \ldots]$ along monomials. Note the that degree 0 piece is exactly $\mathcal{O}(U_i)$. The sections $\phi_i = x_i^d f_i$ should be viewed as elements of $S[1/x_i]_d$. At this point, we may well identify $\mathcal{O}(d)(U_i) = S[1/x_i]_d$, equation 14.1 then simply becomes $\phi = \phi_j$.

In general, the Čech complex for $\mathcal{O}(d)$ is the degree d piece of the complex

$$\bigoplus S\left[\frac{1}{x_i}\right] \to \bigoplus S\left[\frac{1}{x_i x_j}\right] \to \ldots$$

where the maps are as defined alternating sums of the natural maps along the lines of section 6.3.

Notice that $H^i(\mathbb{P}^n, \mathcal{O}(d))$ is automatically zero when $i > n$ because the Cech complex has length n, this takes care of part of (b). An element of $H^0(\mathbb{P}^n, \mathcal{O}(d))$ can be represented by a triple $(p_0/x_0^e, p_1/x_1^e, \ldots)$ of rational expressions of degree d, where the p's are polynomials, satisfying

$$\frac{p_0}{x_0^e} = \frac{p_1}{x_1^e} = \frac{p_2}{x_2^e} \cdots$$

This forces the polynomials p_i to be divisible by x_i^e. Thus the $H^0(\mathbb{P}^n, \mathcal{O}(d))$ can be identified with S_d as claimed in (a).

$H^n(\mathbb{P}^n, \mathcal{O}(d))$ is $S[\frac{1}{x_0 \ldots x_n}]$ modulo the the space of coboundaries B, i. e. the image of the previous term. $S[\frac{1}{x_0 \ldots x_n}]$ is spanned by monomials $x_0^{i_0} \ldots x_n^{i_n}$ with arbitrary integer exponents. The image B is the space spanned by those monomials where at least one of the exponents is nonnegative. Therefore the quotient can be identified with the complementary submodule spanned by monomials with negative exponents. In particular

$$H^n(\mathbb{P}^n, \mathcal{O}(d)) \cong \bigoplus_{i_0 + \ldots i_n = d; i_1, \ldots i_n < 0} k x_0^{i_0} \ldots x_n^{i_n}.$$

This is isomorphic to S_{-d-n-1} via

$$x_0^{i_0} \ldots x_n^{i_n} \mapsto x_0^{-i_0-1} \ldots x_n^{-i_n-1}$$

This proves (c).

It remains to finish the proof (b). The reader can look up the argument in [H, pp 227-228]. We will be content to work this out in for $n = 2$. This amount to the assertion that $H^1(\mathbb{P}^2, \mathcal{O}(d))$ vanishes. Our treatment will be elementary but a little messy. A 1-cocycle is given by a triple $\{p_{ij}/(x_i x_j)^k\}$ of rational expressions satisfying

$$\frac{p_{02}}{(x_0 x_2)^k} = \frac{p_{01}}{(x_0 x_1)^k} + \frac{p_{12}}{(x_1 x_2)^k}.$$

We have to find polynomials $\{q_i\}$ satisfying

$$\frac{p_{ij}}{(x_i x_j)^k} = \frac{q_i}{x_i^k} - \frac{q_j}{x_j^k}.$$

The above relation is equivalent to

$$x_1^k p_{02} = x_2^k p_{01} + x_0^k p_{12},$$

which implies $p_{02} \in (x_0^k, x_2^k)$ and similarly $p_{01} \in (x_0^k, x_1^k)$ and $p_{12} \in (x_1^k, x_2^k)$. Therefore, we can find polynomials such that

$$p_{01} = x_1^k q_0 - x_0^k q_1$$

$$p_{12} = x_2^k q_1' - x_1^k q_2'$$

Substituting back into the above relation shows that $q_1 - q_1' = x_1^k s$ for some polynomial s. Now setting $q_2 = q_2' + s$ should work.

14.2 Coherence in general

Let $(X, \mathcal{R})$ be a ringed space. We need to isolate a class of $\mathcal{R}$-modules with good finiteness properties.

Definition 14.2.1. *Given a ringed space $(X, \mathcal{R})$, an $\mathcal{R}$-module $\mathcal{E}$ is coherent if*

1. *$\mathcal{E}$ is locally finitely generated, i. e., each point has a neighbourhood U such that $\mathcal{R}|_U^n$ surjects onto $\mathcal{E}|_U$ for some $n < \infty$.*

2. *If $\mathcal{R}|_U^n \to \mathcal{E}|_U$ is a surjection, the kernel is finitely generated.*

This definition has been given for completeness. We are really only interested in certain standard examples. Standard properties of coherent sheaves on ringed spaces can be found in [EGA] (the definition given in [H] is only valid for noetherian schemes).

Example 14.2.2. *If M is a finitely generated module over a noetherian ring R, then $\tilde{M}$ is coherent on $Spec\, R$.*

Theorem 14.2.3 (Oka). *If $(X, \mathcal{O}_X)$ is a complex manifold, then $\mathcal{O}_X$ is coherent.*

Corollary 14.2.4. *Any locally free $\mathcal{O}_X$-module is coherent.*

In the next section, we will describe all coherent sheaves on projective space.

14.3 Coherent Sheaves on $\mathbb{P}^n$

Let us revert to the notation of section 14.1. The ring $S = k[x_0, \dots x_n] = \oplus S_i$ is graded, where S_i is the space of homogenous polynomials of degree i. Let

$$S_{(j)} = k\left[\frac{x_0}{x_j}, \dots \frac{x_n}{x_j}\right]$$

We can identify $Spec\, S_{(j)}$ with the affine space $U_j = \{x_j \neq 0\}$.

Let $M = \oplus M_i$ be a (finitely generated) graded S-module (in additional to being an S-module it satisfies $S_i M_j \subset M_{i+j}$). Then

$$M_{(j)} = \sum_i M_i[\frac{1}{x_j^i}] \subset M[\frac{1}{x_j^i}]$$

is naturally a (finitely generated) $S_{j)}$-module. We can construct sheaves $\mathcal{O}_{U_j}$-modules $\tilde{M}_{(j)}$ as in section 2.5. These can be glued toegether:

Proposition 14.3.1. *There exist an $\mathcal{O}_{\mathbb{P}^n}$-module $\tilde{M}$ such that $\tilde{M}|_{U_j} \cong \tilde{M}_{(j)}$. The functor $M \to \tilde{M}$ from graded S-modules to $\mathcal{O}_{\mathbb{P}^n}$-modules is exact.*

Proof. See [H, II, 5.11] for the first statement. Since exactness is a local property, the second statement follows from lemma 2.5.2. □

It follows from this that $\tilde{M}$ is coherent for any finitely generated graded S-module M. Conversely:

Proposition 14.3.2. *Any coherent sheaf on $\mathbb{P}^n$ is isomorphic to $\tilde{M}$ for some finitely generated graded S-module M.*

Proof. [H, II, ex. 5.9] $\square$

Example 14.3.3. *Let $S(i)$ denote S with the shifted grading $(S(i))_j = S_{j+i}$. Then $\mathcal{O}(i) = \tilde{S}(i)$. In particular, it is coherent.*

Example 14.3.4. *Let $X \subset \mathbb{P}^n$ be a closed subvariety and let I_X be the homogenous ideal generated by homogenous polynomials vanishing along X. Then $\tilde{I}$ is the ideal sheaf $\mathcal{I}_X$, example 2.2.5.*

As a corollary of proposition 14.3.1, we get

Corollary 14.3.5. *The class of coherent sheaves is closed under kernels, cokernels and extensions.*

It is possible to the impression that $M \mapsto \tilde{M}$ is one to one. But this is wrong as the following example shows.

Example 14.3.6. *Let M be a graded module which is finite dimensional over k, so $M_i =$ for $i >> 0$. Then $\tilde{M} = 0$.*

If one "mods out" the above modules, then one does get an equivalence of categories. However, we won't try to make this precise.

Any vector bundle is coherent, although this is far from clear with our definition. For standard vector bundles, this can be checked directly using the above corollary.

Example 14.3.7. *The tangent sheaf fits into an exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\begin{pmatrix} x_0 \\ x_1 \\ \ldots \end{pmatrix}} \mathcal{O}_{\mathbb{P}^n}(1)^{n+1} \to \mathcal{T}_{\mathbb{P}^n} \to 0$$

If we pull this back via $\pi : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$, we can identitfy the second map with the derivative $\mathcal{T}_{\mathbb{A}^n - \{0\}} \to \pi^*\mathcal{T}_{\mathbb{P}^n}$, and the first map sends 1 to the Euler vector field $\sum x_i \frac{\partial}{\partial x_i}$

The Hilbert syzygy theorem [E, 1.13] says that any finitely generated graded S-module has a finite graded resolution of length at most n. This immediately gives:

Theorem 14.3.8. *Any coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n$ fits into an exact sequence*

$$0 \to \mathcal{E}_r \to \mathcal{E}_{n-1} \to \ldots \mathcal{E}_0 \to \mathcal{E} \to 0$$

where $r \le n$ and each $\mathcal{E}_i$ is a sum of a finite number of line bundles $\mathcal{O}_{\mathbb{P}^n}(j)$.

Theorem 14.3.9. *If $\mathcal{E}$ is a coherent sheaf on $\mathbb{P}^n$, then $H^i(\mathbb{P}^n, \mathcal{E})$ is finite dimensional for each i, and is zero for $i > n$.*

Proof. We prove this by induction on r, where r is the length of the shortest resolution given by theorem 14.3.8. If $r = 0$, we are done by theorem 14.1.2. Suppose, we know the theorem for all $r' < r$. Choose a resolution

$$0 \to \mathcal{E}_r \to \mathcal{E}_{n-1} \to \dots \mathcal{E}_0 \to \mathcal{E} \to 0$$

and let $\mathcal{E}' = ker[\mathcal{E}_0 \to \mathcal{E}]$. We have exact sequences

$$0 \to \mathcal{E}_r \to \mathcal{E}_{n-1} \to \dots \mathcal{E}_1 \to \mathcal{E}' \to 0$$

and

$$0 \to \mathcal{E}' \to \mathcal{E}_0 \to \mathcal{E} \to 0$$

The first sequence implies that $H^*(\mathbb{P}^n, \mathcal{E}')$ is finite dimensional by the induction hypothesis. The second sequences yields a long exact sequence

$$\dots H^i(\mathbb{P}^n, \mathcal{E}_0) \to H^i(\mathbb{P}^n, \mathcal{E}) \to H^{i+1}(\mathbb{P}^n, \mathcal{E}') \dots$$

which finishes the proof. The vanishing for $i > n$ can be proved by induction in similar manner. Details are left as exercise. □

Exercise 14.3.10.

1. *Let f be a homogeneous polynomial of degree d in $S = k[x_0, x_1, x_2]$. Then corresponding to the exact sequence of graded modules*

$$0 \to S(-d) \cong Sf \to S \to S/(f) \to 0$$

there is an exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_X \to 0.$$

Prove that

$$dim H^1(X, \mathcal{O}_X) = \frac{(d-1)(d-2)}{2}$$

2. *Let f be a homogeneous polynomial of degree d in 4 variables. Repeat the above for the first and second cohomology groups.*

3. *Prove the last part of theorem 14.3.9.*

14.4 Hilbert Polynomial

The Euler characteristic of a sheaf $\mathcal{F}$ on a space X is

$$\chi(\mathcal{F}) = \sum (-1)^i \dim H^i(X, \mathcal{F})$$

provided that the sum is finite. The advantage of working with the Euler characteristic is the following:

Lemma 14.4.1. *If*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is an exact sequence,

$$\chi(\mathcal{F}_2) = \chi(\mathcal{F}_1) + \chi(\mathcal{F}_3)$$

Given a coherent sheaf, $\mathcal{E}$ on $\mathbb{P}^n$, let $\mathcal{E}(i) = \mathcal{E} \otimes \mathcal{O}(i)$. This is again coherent. If $\mathcal{E} = \tilde{M}$ for a finitely generated graded S-module M, then $\mathcal{E}(i) = \widetilde{M(i)}$, where $M(i)$ denote M with grading shifted by i. A corollary of theorem 14.3.8 is:

Theorem 14.4.2. *If $\mathcal{E}$ is a coherent sheaf on $\mathbb{P}^n$, then $i \to \chi(\mathcal{E}(i))$ is a polynomial in i. If X is a subvariety of $\mathbb{P}^n_k$ and $\mathcal{E} = \mathcal{O}_X$, then this polynomial has degree* $\dim X$*.*

Proof. From theorem 14.1.2,

$$\chi(\mathcal{O}_{\mathbb{P}^n}(i)) = \begin{cases} \dim S_i & \text{if } i \geq 0 \\ (-1)^n \dim S_{-d-n-1} & \text{otherwise} \end{cases}$$

$$= \binom{n+i}{n},$$

which is a polynomial of degree n. This implies that $\chi(\mathcal{E}(i))$ is degree n polynomial when $\mathcal{E}$ is sum of line bundles $\mathcal{O}_{\mathbb{P}^n}(j)$. In general, theorem 14.3.8 implies that for any coherent $\mathcal{E}$

$$\chi(\mathcal{E}(i)) = \sum_{j=0}^{r} (-1)^j \chi(\mathcal{E}_j(i))$$

where $\mathcal{E}_j$ are sums of line bundles. This shows $\chi(\mathcal{E}(i))$ is degree n polynomial in general.

For the second, we use induction on $\dim X$ and the relation

$$\chi(\mathcal{O}_{H\cap X}(i)) = \chi(\mathcal{O}_X(i)) - \chi(\mathcal{O}_X(i-1))$$

which follows from the sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_{H\cap X} \to 0,$$

where H is a general hyperplane. $\square$

$\chi(\mathcal{E}(i))$ is called the Hilbert polynomial of $\mathcal{E}$. It has a elementary algebraic interpretation

Corollary 14.4.3 (Hilbert). *Let M be a finitely generated graded S-module, then for $i >> 0$, $\dim M_i$ is given by a polynomial, and this coincides with the Hilbert polynomial $\chi(\tilde{M}(i))$.*

14.5 GAGA

Let $k = \mathbb{C}$. Let $(\mathbb{P}^n, \mathcal{O}_\mathbb{P})$ denote complex projective space as an algebraic variety, and $(\mathbb{P}^n_{an}, \mathcal{O}_{\mathbb{P}_{an}})$ projective space as a complex manifold. In other words, $\mathbb{P}^n$ (resp. $\mathbb{P}^n_{an}$) is endowed with the Zariski (strong) topology and $\mathcal{O}_{\mathbb{P}^n}$ (resp. $\mathcal{O}_{\mathbb{P}^n_{an}}$) is the sheaf of algebraic (resp. holomorphic) functions. We have a morphism of ringed spaces

$$\iota : (\mathbb{P}^n, \mathcal{O}_\mathbb{P}) \to (\mathbb{P}^n_{an}, \mathcal{O}_{\mathbb{P}_{an}})$$

Given an $\mathcal{O}_\mathbb{P}$-module $\mathcal{E}$, define the $\mathcal{O}_{\mathbb{P}_{an}}$-module $\mathcal{E}^{an} = \iota_*\mathcal{E}$. $\mathcal{E}^{an}$ is coherent if $\mathcal{E}$ is.

Theorem 14.5.1 (Serre). *If $\mathcal{E}$ is a coherent $\mathcal{O}_{\mathbb{P}_{an}}$-module, then there exists a unique coherent $\mathcal{O}_\mathbb{P}$-module $\mathcal{F}$ such that $\mathcal{E} \cong \mathcal{F}^{an}$. There is an isomorphism*

$$H^i(\mathbb{P}^n, \mathcal{F}) \cong H^i(\mathbb{P}^n_{an}, \mathcal{E})$$

The GAGA theorem can fail for nonprojective varieties. For example, $H^0(\mathcal{O}_{\mathbb{C}^n_{an}})$ is the space of holomorphic functions on $\mathbb{C}^n$ which is much bigger than the space $H^0(\mathcal{O}_{\mathbb{C}^n})$ of polynomials.

Let $X \subseteq \mathbb{P}^n$ be a nonsingular subvariety. Then the sheaf of algebraic p-forms Ω^p_X can be viewed as a coherent $\mathcal{O}_\mathbb{P}$-module. Then $(\Omega^p_X)^{an}$ is the sheaf of holomorphic p-forms. In particular, $\mathcal{O}^{an}_X$ is the sheaf of holomorphic functions of the associated complex manifold X^{an}. Thus:

Corollary 14.5.2 (Chow). *Every complex submanifold of $\mathbb{P}^n$ is a projective algebraic subvariety.*

Corollary 14.5.3. *$dim H^q(X, \Omega^p_X)$ coincides with the Hodge number h^{pq} of the manifold X^{an}.*

Chapter 15

Computation of some Hodge numbers

The GAGA theorem 14.5.1 allows us compute Hodge numbers by working in the algebraic setting. We look a few examples.

15.1 Hodge numbers of $\mathbb{P}^n$

Let $S = k[x_0, \dots x_n]$ be a polynomial ring over a field and $\mathbb{P} = \mathbb{P}^n_k$ for some field k. The basic result (which is dual to 14.3.7) is:

Proposition 15.1.1. *There is an exact sequence*

$$0 \to \Omega^1_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}} \to 0$$

Proof. Let $F = S^{n+1}$ be the free module of rank $n+1$ with basis vectors e_i. We define a map of graded modules $F(-1) \to S$ by sending e_i to x_i. Let K be the kernel. We have an exact sequence of graded modules

$$0 \to K \to F(-1) \to m \to 0$$

where $m = (x_0, \dots x_n)$....

□

Proposition 15.1.2. *Given an exact sequence of locally free sheaves*

$$0 \to A \to B \to C \to 0$$

If A has rank one, then

$$0 \to A \otimes \wedge^{p-1} C \to \wedge^p B \to \wedge^p C \to 0$$

is exact for any p. If C has rank one, then

$$0 \to \wedge^p C \to \wedge^p B \to \wedge^{p-1} A \otimes C \to 0$$

is exact for any p.

Proof. See [H, II, 5.16]. $\square$

Corollary 15.1.3. *There is an exact sequence*

$$0 \to \Omega^p_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(-p)^{\binom{n+1}{p}} \to \Omega^{p-1}_{\mathbb{P}} \to 0$$

Proof. This follows from the above proposition and proposition 15.1.1, together with the isomorphism

$$\wedge^p[\mathcal{O}_{\mathbb{P}}(-1)^{n+1}] \cong \mathcal{O}_{\mathbb{P}}(-p)^{\binom{n+1}{p}}$$

$\square$

This corollary can be understood from another point of view. Using the notation introduced in the proof of proposition 15.1.1, we can extend the map $F(-1) \to m$ to an exact sequence

$$0 \to [\wedge^{n+1}F](-n-1) \xrightarrow{\delta} \dots [\wedge^2 F](-2) \xrightarrow{\delta} F(-1) \xrightarrow{\delta} m \to 0 \tag{15.1}$$

where

$$\delta(e_{i_1} \wedge \dots e_{i_p}) = \sum (-1)^p x_{i_j} e_{i_1} \wedge \dots \hat{e}_{i_j} \wedge \dots e_{i_p}$$

This is the so called Koszul resolution [E, chap 17] which is one of the basic workhorses of homological algebra. The associated sequence of sheaves is

$$0 \to [\wedge^{n+1}\mathcal{O}^{n+1}_{\mathbb{P}}](-n-1) \to \dots [\wedge^2\mathcal{O}^{n+1}_{\mathbb{P}}](-2) \to [\mathcal{O}^{n+1}_{\mathbb{P}}](-1) \to \mathcal{O}_{\mathbb{P}} \to 0$$

If we break this up into short exact sequences, then we obtain exactly the sequences in corollary 15.1.3.

Proposition 15.1.4.

$$H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}) = \begin{cases} k & \text{if } p = q \le n \\ 0 & \text{otherwise} \end{cases}$$

When $k = \mathbb{C}$, this gives a new proof of the formula for Betti numbers of $\mathbb{P}^n$ given in section 6.2.

15.2 Hodge numbers of a hypersurface

We now let $X \subset \mathbb{P}^n$ be a nonsingular hypersurface defined a degree d polynomial.

Proposition 15.2.1. *The restriction map*

$$H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}}) \to H^q(X^n, \Omega^p_X)$$

is an isomorphism when $p + q < n - 1$.

When $k = \mathbb{C}$ (which we assume for the remainder of this section), this proposition can be deduced from corollary 13.4.2, the canonical Hodge decomposition (10.2.4) and GAGA. As corollary, we can calculate many of the Hodge numbers of X.

Corollary 15.2.2. *The Hodge numbers $h^{pq}(X) = \delta_{pq}$, where δ_{pq} is the Kronecker symbol, when $n-1 \neq p+q < 2n-2$.*

Proof. For $p+q < n-1$, this follows from the above proposition and proposition 15.1.4. For $p+q > n-1$, this follows from GAGA and 9.2.3. $\square$

This leaves the middle Hodge numbers. For $h^{0,n-1}(X)$, this is equivalent to computing the Euler characteristic $\chi(\mathcal{O}_X)$. From the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

we obtain

$$\chi(\mathcal{O}_X) = \binom{i+n}{n} - \binom{i+n-d}{n}.$$

In general, the formulas can be given in terms of a generating function. Let $h^{pq}(d)$ denote the pqth Hodge number of smooth hypersurface of degree d in $\mathbb{P}^{p+q+1}$. Define the formal power series

$$H(d) = \sum_{pq} (h^{pq}(d) - \delta_{pq}) x^p y^q$$

in x and y. Then:

Theorem 15.2.3 (Hirzebruch).

$$H(d) = \frac{1}{(1+x)(1+y)} \left[\frac{(1+x)^d - (1+y)^d}{(1+y)^d x - (1+x)^d y} - 1 \right]$$

Hirzebruch [Hh] deduces a slightly different identity from his general Riemman-Roch theorem. The above form of the identity appears in [SGA7, exp X1]. Similar formulas are available for complete intersections.

Exercise 15.2.4.

1. *Show that the generating for $h^{0,q}$ obtained by setting $x = 0$ in $H(d)$ is correct.*

15.3 Machine computations

The formulas of the previous section are easily implemented on computer and are rather efficient to use when X is a hypersurface (or a complete intersection). There is, in principle, a method for computing Hodge numbers of a general $X \subset \mathbb{P}^n$ given explicit equations for it. In rough outline, one can proceed as follows:

- View the sheaves Ω_X^p as coherent sheaves on $\mathbb{P}^n$. These can be given an explicit presentation. For example by combining

$$\Omega_X^1 = \Omega_{\mathbb{P}^n}^1/(I\Omega_{\mathbb{P}^n}^1 + dI),$$

with the formulas from section 15.1, where I is the ideal of X.

- Resolve this as in theorem 14.3.8.

- Calculate cohomology using the resolution.

Nowdays, it is possible to do these calculations on a machine using packages such as Macaulay2 [M2]. Below is part of a Macaulay 2 session for computing H^{12} and H^{22} of the Fermat quartic threefold $x^4 + y^4 + z^4 + u^4 + v^4 = 0$ over $\mathbb{Q}$. The commands should be more or less self explanitory. The answer $H^{12} = \mathbb{Q}^{30}$ and $H^{22} = \mathbb{Q}$ can be checked against the formulas from the previous section.

```
i1 : S = QQ[x,y,z,u,v];

i2 : I = ideal (x^4+y^4+z^4+u^4+v^4);

i3 : X = Proj(S/I);

i4 : Om1 = cotangentSheaf(1,X);

i5 : HH^2 Om1

       30
o5 = QQ

i6 : Om2 = cotangentSheaf(2,X);

i7 : HH^2 Om2

       1
o7 = QQ
```

Exercise 15.3.1.

1. *Do more examples in Macaulay 2. (Look up the commands* HH, cotangentSheaf... *first.)*

Chapter 16

Deformation invariance of Hodge numbers

In this chapter, the various strands (algebra, analysis, topology) will converge. Our goal is to prove that Hodge numbers of a family of smooth complex projective varieties stays constant. While this can be proved purely analytically, we develop much of background using Grothendieck's language of schemes which gives a particularly elegant approach to families.

16.1 Families of varieties via schemes

We gave a definition of schemes 2.3 earlier without giving much geometric motivation. Varieties often occur in families as we have already seen. As a simple example, let $f(x,y,t) = y^2 - xt$ be a polynomial over a field k. We can view this as defining a family of parabolas in the xy-plane $\mathbb{A}^2$ parameterized by $t \in \mathbb{A}^1$. When $t = 0$, we get a degenerate parabola $y^2 = 0$ which is the "doubled" x-axis. It is impossible to capture this fully within the category of varieties, but it makes perfect sense with schemes. Here, we have a map of rings

$$k[t] \to k[x,y,t]/(f(x,y,t))$$

which induces a morphism of schemes

$$\pi : Spec\, k[x,y,t]/(f) \to Spec\, k[t] = \mathbb{A}^1$$

We can view this as the family of schemes

$$\{Spec\, k[x,y,t]/f(x,y,a) \mid \ a \in k\}$$

given by the fibers of π.

More generally, let $R = \mathcal{O}(Y)$ be the coordinate ring be an affine variety Y over an algebraically closed field k. Affine space over R is

$$\mathbb{A}^n_R = Spec\, R[x_1, \dots x_n] \cong \mathbb{A}^n_k \times Y.$$

Let $I \subset R[x_1, \dots x_n]$ be an ideal. It is generated by polynomials

$$f_j(x,y) = \sum f_{j,i_1,\dots i_n}(y) x_1^{i_1} \dots x_n^{i_n}.$$

We get a morphism

$$Spec\, R[x_1, \dots x_n]/I \to Spec\, R = Y$$

We can view this as a family of affine subschemes of $\mathbb{A}_k^n$

$$Spec\, R[x_1, \dots x_n]/I \otimes R/m_a = Spec\, k[x_1, \dots x_n]/(f_j(x,a))$$

parameterized by points $a \in Y$. If $(f_j(x,a))$ is a radical ideal in $k[x_1, \dots x_n]$, this would be a subvariety of $\mathbb{A}_k^n$, but in general it be a closed subscheme.

Similarly, a (not necessarily radical) homogenous ideal $I \subset k[x_1, \dots x_n]$ gives rise to a closed subscheme called $Proj\, k[x_1, \dots x_n]/I$ of $\mathbb{P}_k^n$. More generally if $I \subset R[x_1, \dots x_n] = S$ is an ideal with homogenous generators $f_j(x,y)$. We can loosely define $Proj\, S/I$ as the family of subschemes of $\mathbb{P}_k^n$ defined by the ideals $(f_j(x,a))$ for points $a \in Y$. A bit more formally, the projective space over R $\mathbb{P}_R^n = \mathbb{P}_k^n \times Y$ can be constructed by gluing $n+1$ copies of affine space

$$U_{i,R} = Spec\, R[x_0/x_i, \dots \widehat{x_i/x_i} \dots x_n/x_i]$$

together. Given a homogenous polynomial in $f(x,y) \in S = R[x_1, \dots x_n]$, we get a polynomial in $f(x_1/x_i, \dots x_n/x_i, y)$ in the above ring. Thus a homogenous ideal I defines ideals in $I_i \subset R[x_1/x_i, \dots]$ by applying this substitution on the generators. The scheme $ProjS/I$ is defined by gluing the affine schemes defined I_j together. See [H] for a more precise account of $Proj$.

16.2 Cohomology of Affine Schemes

Let R be a commutative ring.

Definition 16.2.1. *An R-module I is injective iff for any for injective map $N \to M$ of R-modules, the induced map $Hom_R(M,I) \to Hom_R(N,I)$ is surjective.*

Example 16.2.2. *If $R = \mathbb{Z}$ then I is injective provided that it is divisible i.e. if $a = nb$ has a solution for every $a \in I$ and $n \in \mathbb{Z} - 0$.*

A standard result of algebra is the following. A proof can be found in several places, for example [E, p. 627].

Theorem 16.2.3 (Baer). *Every module is (isomorphic to) a submodule of an injective module.*

Recall construction of the sheaf $\tilde{M}$ of section 2.5.

Proposition 16.2.4. *If I is injective and R is noetherian, then $\tilde{I}$ is flabby.*

Proof. Suppose that $U = D(f)$ is a basic open set. Given an element of $\tilde{I}(U)$, it can be written as a fraction x/f^n. Consider the map $R \to I$ which sends $1 \mapsto x$. Since I is injective, this extends to a map $R[1/f^n] \to I$. In other words, x/f^n lies in I.

For a general open set U, express $U = \cup D(f_i)$. We have a diagram

$$\begin{tikzcd} \tilde{I}(U) \arrow[r, "i"] & \prod \tilde{I}((D(f_i)) \\ \tilde{I}(X) \arrow[u, "r"] \arrow[ur, "s"] & \end{tikzcd}$$

where i is injective. s is surjective by the previous paragraph, therefore r is surjective. □

Theorem 16.2.5 (Grothendieck). *Let $X = \operatorname{spec} R$, then for any R-module,*

$$H^i(X, \tilde{M}) = 0$$

Proof. Let M be an R-module, then it embeds into an injective module I. Let $N = I/M$. By lemma 2.5.2, $M \mapsto \tilde{M}$ is exact, therefore

$$0 \to \tilde{M} \to \tilde{I} \to \tilde{N} \to 0$$

is exact. As $H^i(X, \tilde{I}) = 0$ for $i > 0$, we obtain

$$I \to N \to H^1(X, \tilde{M}) \to 0,$$

and

$$H^{i+1}(X, \tilde{M}) \cong H^i(X, \tilde{N}).$$

$I \to N$ is surjective, therefore $H^1(X, \tilde{M}) = 0$. Note that this argument can be applied to any module, in particular to N. This implies that

$$H^2(X, \tilde{M}) \cong H^1(X, \tilde{N}) = 0.$$

We can kill the all higher cohomology groups in the same fashion. □

There is an analogous theorem in the analytic category which predates this. A complex manifold is called Stein[1] if it can be embedded into some $\mathbb{C}^N$. For example, any nonsingular complex affine variety is Stein.

Theorem 16.2.6 (Cartan). *Let $\mathcal{E}$ be a coherent sheaf on the Stein manifold then $H^i(X, \mathcal{E}) = 0$ for all $i > 0$.*

Proof. [Hr, 7.4.3] □

[1] This definition is nonstandard but equivalent to the usual ones, see [Hr, 5.1.5, 5.3.9].

16.3 Semicontinuity of coherent cohomology

Let $R = \mathcal{O}(Y)$ be the coordinate ring of an affine variety, and let $S = R[x_0, \ldots x_n] = \oplus S_i$ with the usual grading. Given a finited generated graded R-module M. We can construct a coherent sheaf $\tilde{M}$ on $\mathbb{P}^n_R = \mathbb{P}^n_k \times Y$ as in section 14.3. Proposition 14.3.2 generalizes to this setting [H, II, ex. 5.9]. For each point $a \in Y$, we get a a graded $k[x_0, \ldots x_n]$-module $M_a = M \otimes R/m_a$. Thus we get a family of coherent sheaves $\tilde{M}_a$ parameterized by Y. This can be constructed sheaf theoretically. Let $i_a : \mathbb{P}^n_k \to \mathbb{P}^n_k \times Y$ be the inclusion $x \mapsto (x, a)$, then $\tilde{M}_a = i_a^* \tilde{M}$.

Example 16.3.1. *Let M be a finitely generated graded $k[x_0, \ldots x_n]$-module, then $R \otimes M$ is a finitely generated graded S-module. $\widetilde{R \otimes M}$ is a "constant" family sheaves; we have $\widetilde{R \otimes M}_a = \tilde{M}$. This can be constructed geometrically. Let $\pi : \mathbb{P}^n_k \times Y \to Y$ be the projection. Then $\widetilde{R \otimes M} = \pi^*(\tilde{M})$. A special case is $\mathcal{O}_{\mathbb{P}^n_k \times Y}(i)$ which is the constant family $\pi^* \mathcal{O}_{\mathbb{P}^n_k}$.*

Example 16.3.2. *Let $I \subset S$ be a homogeneous ideal, this gives rise to a family of projective schemes of $\mathbb{P}^n_k$ with ideal sheaves $\tilde{I}_a$.*

We now ask the basic question, suppose that M as above, how do the dimensions of the cohomology groups

$$h^i(\tilde{M}_a) = \dim H^i(\tilde{M}_a)$$

vary with a? We look at some examples.

Example 16.3.3. *If M is constant family, then clearly $a \mapsto h^i(\tilde{M}_a)$ is a constant function of a.*

In general, this is not constant. Here is a more typical example:

Example 16.3.4. *Set $R = k[s, t]$, and choose three points*

$$p_1 = [0, 0, 1],\ p_2 = [s, 0, 1],\ p_3 = [0, t, 1]$$

in $\mathbb{P}^2_k$ with s, t variable. Let $\mathcal{I}$ be the ideal sheaf of the union of these points in $\mathbb{P}^2_k \times \mathbb{A}^2_k$. This can also be described as $\tilde{I}$, where I is product

$$(x, y, z - 1)(x - s, y, z - 1)(x, y - t, z - 1) \subset R[x, y, z]$$

Consider $\mathcal{I}(1) = \tilde{I}(1)$. The global sections of this sheaf correspond to the space of linear forms vanishing at p_1, p_2, p_3. Such a form can exist only when the points are colinear, and it is unique upto scalars unless the points coincide. Thus

$$h^0(\mathcal{I}(1)_{(s,t)}) = \begin{cases} 2 & \textit{if } s = t = 0 \\ 1 & \textit{if } s = 0 \textit{ or } t = 0 \textit{ but not both} \\ 0 & \textit{if } s \neq 0 \textit{ and } t \neq 0 \end{cases}$$

In the above example, the sets where cohomology jumps are Zariski closed. This can be reformulated as saying that the function $h^0(\mathcal{I}(1)_{(s,t)})$ is upper semicontinuous for the Zariski topology. Let's try to show this in general for the top cohomology (which is technically easier). Suppose that M is a finitely graded module $S = R[x_0, \dots x_n]$-module, with $R = O(Y)$. Choose a graded presentation

$$\bigoplus_j S(i_j) \to \bigoplus_m S(\ell_m) \to M \to 0$$

The first map is given by a matrix of polynomials. In order for this presentation to be useful to us, we want

$$\bigoplus_j S(i_j)_y \to \bigoplus_m S(\ell_m)_y \to M_y \to 0$$

to stay exact for any $y \in Y$. This to leads to concept of flatness.

Definition 16.3.5. *An R-module T is flat if the functor $N \mapsto T \otimes_R N$ is exact (note that it always right exact).*

Lemma 16.3.6. *If T is flat, and*

$$0 \to A \to B \to T \to 0$$

is exact, then

$$0 \to A \otimes C \to B \otimes C \to T \otimes C \to 0$$

is exact for any module C.

Proof. This follows immediately from properties of the Tor functor. See for example [E, pp162-172]. □

Now suppose that M is a flat R-module. Break the above presentation into exact sequences

$$0 \to K \to \bigoplus_m S(\ell_m) \to M \to 0$$

$$\bigoplus_j S(i_j) \to K \to 0$$

The above lemma, shows that

$$0 \to K_y \to \bigoplus_m S(\ell_m)_y \to M_y \to 0$$

is exact, and we also have a surjection

$$\bigoplus_j S(i_j)_y \to K_y \to 0$$

Applying H^n to the corresponding sequences of sheaves, and splicing the resulting sequences yields

$$\bigoplus_j H^n(\mathbb{P}^n_k, \mathcal{O}(i_j)) \xrightarrow{A(y)} \bigoplus_m H^n(\mathbb{P}^n_k, \mathcal{O}(\ell_m)) \to H^n(\mathbb{P}^n, \tilde{M}_y) \to 0$$

Note that the first two spaces are constant. Thus we can view the map between them, $A(y)$, as matrix depending algebraically on y (i.e. it comes from the evaluation of a matrix over R). The $(r+1)\times(r+1)$ minors of $A(y)$ define a closed subset $Y_r \subset Y$. Set $N = \sum h^n(\mathcal{O}\ell_m)$. Then we obtain

$$h^n(\tilde{M}_y) = N - rank(A(y)) \geq N - r$$

if and only if $y \in Y_r$. Thus we have shown that $h^n(\tilde{M}_y)$ is upper semicontinuous. In general,

Theorem 16.3.7 (Grothendieck). *If M is a flat finitely generated graded R-module, then*

$$y \mapsto h^i(\tilde{M}_y)$$

is upper semicontinuous, and the function

$$y \mapsto \chi(M_y) = \sum(-1)^i h^i(\tilde{M}_y)$$

is locally constant.

Proof. See [H, 12.8] and [EGA, III]. □

Grauert has established analogous results in the analytic setting, see [GPR]. We say that the subscheme $Z \subset \mathbb{P}^n_k \times Y$ is flat over Y, if $\mathcal{O}_Z$ is given by a flat R-module The geometric meaning of flatness is a little elusive. However the above theorem implies

Corollary 16.3.8. *If Y is flat over Z, then the Hilbert polynomial $\chi(\mathcal{O}_Y(d))$ is locally constant.*

It follows from this that the fibers of $Y \to Z$ have the same dimension. Thus, for example, the blow up of point on surface is not flat. If fact, constancy of the Hilbert polynomial characterizes flatness [H][9.9].

16.4 Smooth families

We introduced flatness as techinical assumption without really giving any examples. We now introduce a stronger notion, where the geometric meaning is clear. To simply matters, we restrict out attention to nonsingular varieties.

Definition 16.4.1. *A morphism $f : X \to Y$ of nonsingular varieties is smooth if for every point x, the induced map $T_x \to T_y$ is surjective.*

Example 16.4.2. *Let $f : \mathbb{A}^m_k \to \mathbb{A}^n_k$ be given by $f(x) = (f_1(x), \ldots f_n(x))$. where f_i are polynomials. Then f is smooth if and only if the Jacobian $(\partial f_i/\partial x_j)$ has rank n. When this is satisfied, it is clear that the fibers $f^{-1}(y)$ are nonsingular varieties of dimension $m - n.n$*

Theorem 16.4.3. *If $f : X \to Y$ is a smooth morphism of nonsingular varieties, then f is flat and the fibers $f^{-1}(y)$ are all nonsingular varieties.*

Thus we can view a smooth morphism as a family of nonsingular varieties over a nonsingular parameter space. We can define an equivalence relation on nonsingular projective varieties weaker than isomorphism. We say that two nonsingular projective varieties X_1, X_2 are *deformations* of each other if there is a smooth map of nonsingular varieties $f : Z \to Y$ such that the fibers are all projective varieties and $X_i = f^{-1}(y_i)$ for points $y_i \in Y$. This generates an equivalence relation that we will call deformation equivalence. For example, any two nonsingular hypersurfaces $X_i \subset \mathbb{P}^n$ of degree d are deformation equivalent, since they members of the the family constructed below. Let V_d be the space of homogeneous polynomials of degree d in $n+1$ variables. $\Delta \subset \mathbb{P}(V_d)$ the closed set of singular hypersurfaces. Then

$$\mathcal{U} = \{(x, [f]) \in \mathbb{P}^n \times (\mathbb{P}(V_d) - \Delta) \mid f(x) = 0\}$$

Then $\mathcal{U} \to \mathbb{P}(V_d) - \Delta$ is a smooth map containing all nonsingular hypersurfaces as fibers.

Since any elliptic curve can be realized as a smooth cubic in $\mathbb{P}^2$, it follows that any two elliptic curves are deformation equivalent. Considerably deeper is the fact that any two smooth projective curves of the same genus $g \geq 2$ are deformation equivalent. This can be proved with the help of the Hilbert scheme. A weak formulation of its defining property is as follows:

Theorem 16.4.4 (Grothendieck). *Fix a polynomial $p \in \mathbb{Q}[t]$. There is a projective scheme $H = Hilb^p_{\mathbb{P}^n}$ with a flat family $\mathcal{U} \to H$ of closed subschemes of $\mathbb{P}^n$ with Hilbert polynomial p, such that every closed subscheme with Hilbert polynomial p occurs as a fiber of $\mathcal{U}$ exactly once.*

For example, the space $\mathbb{P}(V_d)$ above is the Hilbert scheme $Hilb^p_{\mathbb{P}^n}$ with

$$p(i) = \chi(\mathcal{O}_{\mathbb{P}^n}(i)) - \chi(\mathcal{O}_{\mathbb{P}^n}(i-d)) = \binom{n+i}{n} - \binom{n+i-d}{n}.$$

Choose $N \geq 3$, then for a smooth projective curve X of genus g, $\omega_X^{\otimes N}$ is very ample. The set of such curves in $\mathbb{P}^{(g-1)(2N-1)}$ is parameterized by an open subset of the Hilbert scheme with $p(t) = (2g-2)Nt + (1-g)$. This set can be shown to be irreducible. See [HM] for further details.

16.5 Deformation invariance of Hodge numbers

In this section, we revert to working over $\mathbb{C}$.

Theorem 16.5.1 (Kodaira-Spencer). *If two complex nonsingular projective varieties are deformation equivalent, then their Hodge numbers are the same.*

Proof. Let $f : X \to Y$ be a smooth projective morphism of nonsingular varieties. Then by theorem 16.3.7, there there are constants g^{pq} such that

$$h^{pq}(X_t) \geq g^{qp} \tag{16.1}$$

for all $t \in X$ with equality on an a nonempty open set U. Choose a points $t \in U$ and $s \notin U$. Then since X_t and X_s are diffeomorphic 12.1.1, they have the same Betti numbers. Therefore by the Hodge decomposition (theorem 9.2.4),

$$\sum_{pq} h^{pq}(X_s) = \sum_{pq} g^{pq}$$

and this implies that (16.1) is an equality. $\square$

This kind of result is not true of all the invariants considered so far. Any elliptic curve can be embedded as a cubic in $\mathbb{P}^2$, thus any two elliptic curves are deformation equivalent. Likewise for the products of elliptic curves with themselves. But we saw in section 11.3 that the Picard number of $E \times E$ was not constant. Therefore it not a deformation invariant. Other examples of this phenomenon are provided by:

Theorem 16.5.2 (Noether-Lefschetz). *Let $d \geq 4$ then there exists a surface $X \subset \mathbb{P}^3$ of degree d with Picard number 1.*

Remark 16.5.3. *This is true "for almost all" surfaces.*

Proof. We sketch the proof. Since $H^1(X, \mathcal{O}_X) = 0$ for any surface in $\mathbb{P}^3$, $c_1 : Pic(X) \to H^2(X, \mathbb{Z})$ is injective. On the other hand, $H^2(X, \mathcal{O}_X) \neq 0$ if X is has degree $d \geq 4$, therefore c_1 is not onto. Choose a Lefschetz pencil $\{X_t\}_{t \in \mathbb{P}^1}$ of surfaces, let $\tilde{X} \to \mathbb{P}^1$ be the incidence variety and let $U \subset \mathbb{P}^1$ parameterize X_t smooth 13.3. The set of all curves lying on some X_t is a parameterized by a countable union of Hilbert schemes. Each irreducible component will either parameterize curves lying on a fixed X_t or curves varying over all X_t. Therefore for all but countably many t, any curve lying on a an X_t will propogate to all the members of the pencil. Choose such a nonexceptional $t_0 \in U$. The group $c_1(Pic(X_{t_0}))$ is the group of curves on X_{t_0}, and this is stable under the action of $\pi_1(U, t_0)$ since any such curve can be moved along a loop avoiding the exceptional t's. By theorem 13.3.4, $H^2(X_{t_0}, \mathbb{Q}) = image(H^2(\mathbb{P}^3)) \oplus V$, where V is the generated by vanishing cycles, and $\pi_1(U)$ acts irreducibly on this. Since $C = c_1(Pic(X_{t_0})) \otimes \mathbb{Q}$ contains $image(H^2(\mathbb{P}^3))$, it follows that either $c_1(Pic(X_{t_0}))$ equals $image(H^2(\mathbb{P}^3))$ or it equals $H^2(X_{t_0})$. The last case is impossible, so $c_1(Pic(X_{t_0})) = image(H^2(\mathbb{P}^3)) = \mathbb{Q}$. $\square$

Exercise 16.5.4.

1. *Construct a smooth quartic* $X \in \mathbb{P}^3$ *containing a line* L*. It can be shown that* $L^2 = -2$*, assume this and prove that the Picard number of* X *is at least* 2*.*

Chapter 17

Mixed Hodge Numbers

Deligne has extended Hodge theory to algebraic varieties which may be be singular or noncompact. Surprisingly, some of these results were motivated by certain analogies with varieties over finite fields. Here we give a brief introduction to these ideas by concentrating on the purely numerical aspects.

17.1 Mixed Hodge numbers

We defined compactly supported cohomology of manifolds with real coefficients using differential forms in section 4.3. We can define this with $\mathbb{Z}$ coefficients for any (locally compact Hausdorff) topological space X by

$$H^i_c(X,\mathbb{Z}) = H^i(\bar{X}/\{*\},\mathbb{Z})$$

where $\bar{X} = X \cup \{*\}$ is the one point compactification. Note that $\bar{X}$ can be replaced by any (reasonable) compactification, say $\tilde{X}$. Then from (6.2), we get a long exact sequence

$$\dots H^i_c(X,\mathbb{Z}) \to H^i(\tilde{X},\mathbb{Z}) \to H^i(\tilde{X}-X,\mathbb{Z}) \to H^{i+1}_c(X,\mathbb{Z}) \to \dots \tag{17.1}$$

In De Rham cohomology, the first first map is given by extending a compactly supported form by 0.

The following is really an amalgam of various theorems in [D2].

Theorem 17.1.1 (Deligne). *To every complex algebraic variety X, there is a canonical bigrading*

$$H^i_c(X,\mathbb{C}) = \bigoplus_{p,q\in T_i} H^i(X)^{(p,q)}$$

where $T = \{(p,q) \mid 0 \le p,q \le i,\ p+q \le i\}$ such that

1. *if X is smooth and projective $H^{p+q}(X)^{(p,q)} \cong H^q(X,\Omega^p_X)$*
2. *$dim H^i(X)^{(p,q)} = \dim H^i(X)^{(q,p)}$*

3. *If $U \subset X$ is a Zariski open subset of a projective variety, with $Z = X - U$, then the long exact sequence is compatible with the bigrading:*

$$\dots H^i_c(U)^{(p,q)} \to H^i(X)^{(p,q)} \to H^i(Z)^{(p,q)} \to H^{i+1}_c(U)^{(p,q)} \dots$$

We introduce the mixed Hodge and Betti numbers

$$h^{i;(p,q)}(X) = \dim H^i(X)^{(p,q)}$$

by

$$b_i^{(m)} = \sum_{p+q=m} h^{i;(p,q)}(X)$$

To get some feeling for this, let us calculate the dimension of these invariants for smooth nonprojective curve U. We can find a smooth compactification X of genus g. Let $Z = X - U$, this is a finite set of say s points. The map $H^2_c(U) \to H^2(X)$ is an isomorphisms and $H^0_c(U) = 0$, thus we get

$$0 \to H^0(X)^{(0,0)} \to H^0(Z)^{(p,q)} \to H^1_c(U)^{(p,q)} \to H^1(X)^{(p,q)} \to 0$$

Which gives

$$h^{1;(0,0)}(U) = s - 1,\ h^{1;(1,0)}(U) = h^{1;(0,1)}(U) = g$$

In order to further facilitate calculations, let us introduce the Euler characteristics

$$\chi^{(p,q)}(X) = \sum_i (-1)^i h^{i;(p,q)}(X)$$

$$\chi^{(m)}(X) = \sum_i b_i^{(m)}(X)$$

following [DK]. The theorem yields.

Corollary 17.1.2. *In the above notation, $\chi^{(p,q)}(X) = \chi^{(p,q)}(U) + \chi^{(p,q)}(Z)$ and $\chi^{(m)}(X) = \chi^{(m)}(U) + \chi^{(m)}(Z)$ If X is smooth and projective $\chi^{(p,q)}(X) = (-1)^{p+q}h^{pq}(X)$, and $\chi^{(m)}(X) = (-1)^m b_i(X)$, where $h^{pq} = \dim H^q(X\Omega^p_X)$ and b_i are the usual Hodge and Betti numbers.*

This leads to a practical tool for computing Hodge and Betti numbers for projective varieties that can be decomposed into simpler pieces. As an example of this, let X be smooth projective variety of dimension n. Choose $x \in X$. We can define the blow up Bl_xX by generalizing the construction in 11.1. This is a smooth projective variety with a morphism $\pi : Bl_xX \to X$ which is an isomorphism over $X - \{x\}$ and such that $\pi^{-1}(x) \cong \mathbb{P}^{n-1}$. Then the Hodge numbers of Bl_xX are determined by

$$\begin{aligned}
\chi^{(p,q)}(Bl_xX) &= \chi^{(p,q)}(\mathbb{P}^{n-1}) + \chi^{(p,q)}(X - \{x\}) \\
&= \chi^{(p,q)}(\mathbb{P}^{n-1}) + \chi^{(p,q)}(X) - \chi^{(p,q)}(\{x\}) \\
&= \begin{cases} \chi^{(p,q)}(X) + 1 & \text{if } p = q > 0 \\ \chi^{(p,q)}(X) & \text{otherwise.} \end{cases}
\end{aligned}$$

The previous corollary has a rather surprising consquence:

Corollary 17.1.3 (Durfee). *If $X = \cup X_i$ and $Y = \cup Y_i$ are smooth projective varieties expressable as disjoint unions of locally closed subvarieties such that $X_i \cong Y_i$, then X and Y have the same Hodge and Betti numbers.*

Proof. These decompositions are necessarily finite since the spaces are Noetherian. Therefore repeated applications of corollary 17.1.2 leads to

$$\chi^{(p,q)}(X) = \sum_i \chi^{(p,q)}(X_i) = \sum_i \chi^{(p,q)}(Y_i) = \chi^{(p,q)}(Y).$$

□

Each of the ruled surfaces F_n, described in section 11.1, can be decomposed as a union of $\mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{A}^1$. Thus the Hodge numbers are the same as for $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and this is easy to compute.

17.2 Cohomology of the complement of a smooth hypersurface

While we won't even pretend to prove theorem 17.1.1. It is instructive to see where some of this structure comes from in a special case. Let us consider an n dimensional smooth projective variety X with a nonsingular connected hypersurface $D \subset X$. Let $U = X - D$, and let $i : D \to X$ and $j : U \to X$ denote the inclusions. Let $\pi : T \to D$ be a tubular neigbourhood of D 4.4.1. Any differential form with compact support on U can be extended by 0 to X. Thus the sheaf of compactly supported forms $\mathcal{E}^\bullet_{cU}$ can be regarded as subsheaf of $\mathcal{E}^\bullet_X$. This lies in the kernel $\mathcal{K}^\bullet$ of the restriction map $\mathcal{E}^\bullet_X \to \mathcal{E}^\bullet_D$.

Lemma 17.2.1. *$\mathcal{E}^\bullet_{cU}$ is quasi-isomorphic to $\mathcal{K}^\bullet$.*

Thus the long exact associated to

$$0 \to \mathcal{K}^\bullet \to \mathcal{E}^\bullet_X \to \mathcal{E}^\bullet_D \to 0$$

is just (17.1).

We want to replace these with holomorphic objects. We define $\Omega^p_X(*D) \subset j_*\mathcal{E}^p_U$ to be the sheaf of meromorphic p-forms which are holomorphic on U. $\Omega^p_X(\log D) \subset \Omega^p_X(*D)$ is the subsheaf of meromorphic forms α such that both α and $d\alpha$ have simple poles along D. If we choose local coordinates $z_1, \ldots z_n$ so that D is defined by $z_1 = 0$. Then the sections of $\Omega^p_X(\log D)$, are locally spanned as an $\mathcal{O}_X$ module by

$$\{dz_{i_1} \wedge \ldots dz_{i_p} \mid i_j > 1\} \cup \{\frac{dz_1 \wedge dz_{i_2} \wedge \ldots dz_{i_p}}{z_1}\}$$

$\Omega^p_X(\log D)(-D)$ is the product of the ideal sheaf of D with the previous sheaf. Locally this is spanned by

$$\{z_1 dz_{i_1} \wedge \ldots dz_{i_p} \mid i_j > 1\} \cup \{dz_1 \wedge dz_{i_2} \wedge \ldots dz_{i_p}\}$$

These are exactly the forms vanishing along D. Thus

Lemma 17.2.2. $\Omega^p_X(\log D)(-D) = ker[\Omega^p_X \to \Omega^p_D]$

Clearly $\Omega^\bullet_X(\log D)(-D)$ is a subcomplex of $\Omega^\bullet_X$. It is not difficult to see, using 10.5, that in the diagram

$$\begin{array}{ccccccccc} 0 & \to & \Omega^\bullet_X(\log D)(-D) & \to & \Omega^\bullet_X & \to & \Omega^\bullet_D & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{K}^\bullet & \to & \mathcal{E}^\bullet_X & \to & \mathcal{E}^\bullet_D & \to & \end{array}$$

the vertical maps are quasi-isomorphisms. Thus

$$\ldots \to \mathbb{H}^i(\Omega^\bullet_X(\log D)(-D)) \to \mathbb{H}^i(\Omega^\bullet_X) \to \mathbb{H}^i(\Omega^\bullet_D) \to \ldots$$

coincides with (17.1). We also have sequences

$$\ldots \to H^q(\Omega^p_X(\log D)(-D)) \to H^q(\Omega^p_X) \to H^q(\Omega^p_D) \to \ldots$$

We can now express the mixed Hodge numbers as

$$h^{pq;i}(U) = \begin{cases} \dim ker[H^q(\Omega^p_X) \to H^q(\Omega^p_D)] & \text{if } p+q=i \\ \dim im[H^q(\Omega^p_X) \to H^q(\Omega^p_D)] & \text{if } p+q=i-1 \\ 0 & \text{otherwise} \end{cases}$$

Notice that

$$\dim H^q(X, \Omega^p_X(\log D)(-D)) = h^{pq;p+q}(U) + h^{pq;p+q-1}(U)$$

We have thus proved

Proposition 17.2.3 (Deligne). *There is a noncanonical decomposition*

$$H^i_c(U, \mathbb{C}) \cong \bigoplus_{p+q=i} H^q(X, \Omega^p_X(\log D)(-D))$$

Corollary 17.2.4. $H^i(X, \mathcal{O}_X(-D)) = 0$ *if the* $2n-i$ *the Betti number of* U *vanishes.*

Proof. This follows from Poincaré duality: $H^i_c(U) \cong H^{2n-i}(U)^*$. ☐

This yields a special case of the Kodaira vanishing theorem. The method used is closely related to various "topological" proofs found by Esnault, Viehweg, Kollár and others (see [EV]). We say that

Corollary 17.2.5 (Kodaira). *If* D *is very ample (the intersection of* X *with a hyperplane under a projective embedding), then* $H^i(X, \mathcal{O}_X(-D)) = 0$ *for* $i > 0$.

Proof. Since $X - D$ is affine and hence Stein, this follows corollary 10.5.4 ☐

17.3 Counting points over finite fields

Suppose that X is a complex quasiprojective algebraic variety with a fixed embedding into $\mathbb{P}^N_{\mathbb{C}}$. By adjoining the coefficients of the defining equations (and inequalities) to $\mathbb{Z}$, we obtain a finitely generated algebra $A \subset \mathbb{C}$. Choose a maximal ideal $Q \subset A$, then A/Q will be a finite field. Recall that, up to isomorphism, there is exactly one such field $\mathbb{F}_q$ with q elements, for each prime power q. Fix an isomorphism $A/Q \cong \mathbb{F}_q$. Now we can reduce the equations modulo Q to get a quasiprojective "variety" X_Q defined over $\mathbb{F}_q$. We can define $X_Q(\mathbb{F}_{q^n})$ to be the set of points of $\mathbb{P}^N_{\mathbb{F}_{q^n}}$ satisfying the equations (and inequalities) defining X. Since these sets are finite, we can count their elements. Let $N_n(X_Q) = N_n$ be the cardinality of this set. In more abstract terms, we have a scheme $\mathcal{X} \to Spec\, A$ called a model of X. The original variety X is the fiber product $\mathcal{X} \times_{Spec\, A} Spec\, \mathbb{C}$, and and X_Q is the scheme theoretic fiber over Q. Note that even if X is smooth, X_Q need not be; we say that X has "good reduction" at Q if this is the case. We have $X_Q(\mathbb{F}_{q^n}) = Hom_{schemes}(Spec\, \mathbb{F}_{q^n}, X_Q)$.

Let's consider a few simple examples.

Example 17.3.1. *Let $X = \mathbb{A}^k_{\mathbb{C}}$, then $\mathbb{A}^k_Z$ is the obvious model. $N_n(\mathbb{A}^k_{\mathbb{F}_q}) = q^{nk}$.*

Example 17.3.2. *Let $X = \mathbb{P}^k_{\mathbb{C}}$, then $\mathbb{P}^k_Z$ is a model. Writing $\mathbb{P}^k = \mathbb{A}^k \cup \mathbb{A}^{k-1} \cup \ldots$ gives $N_n(\mathbb{P}^k_{\mathbb{F}_q}) = q^{nk} + q^{n(k-1)} + \ldots q^n$.*

Example 17.3.3. *Consider the elliptic curve E defined by $zy^2 = x^3 - z^3$. This equation gives a model over the integers. Then $N_1(E_p) = p + 1$ if $p \equiv 2 \mod 3$ is an odd prime, but not in general as the following table shows:*

p	N_1	p	N_1	p	N_1	p	N_1
7	4	67	52	127	148	193	192
13	12	73	84	139	124	199	172
19	28	79	76	151	148	211	196
31	28	97	84	157	144	223	196
37	48	103	124	163	172	229	252
43	52	109	108	181	156	241	228

As the last example indicates, it is not usually possible to write down exact formulas. So we should seek qualitative information. A quick inspection of the table suggests $N_1(E_p)$ stays fairly close to $1 + p$. In fact, we always have following estimate:

Theorem 17.3.4 (Weil). *If X is a smooth projective curve of genus g, and suppose that X has good reduction at Q. Then*

$$|N_n(X_Q) - (1 + q^n)| \leq 2g\sqrt{q}$$

This is very remarkable formula which says that topological and arithmetic properties of curves are related. Weil conjectured, and Deligne [D4] subsequently proved, that this phenomenon holds much more generally. To formulate it, let us say that an algebraic number $\lambda \in \bar{\mathbb{Q}}$ has uniform absolute value $x \in \mathbb{R}$ if $|\iota(\lambda)| = x$ for all embeddings $\iota : \bar{\mathbb{Q}} \to \mathbb{C}$.

Theorem 17.3.5 (Deligne). *Let X be a smooth projective d dimensional variety and suppose that X has good reduction at Q. Then*

$$N_n(X_Q) = \sum_{i=0}^{2d} (-1)^i \sum_{j=1}^{b_i} \lambda_{ji}^n,$$

where λ_{ji} are algebraic integers with uniform absolute values $q^{i/2}$ and b_i coincides with the ith Betti number of X.

See [H, appendix C] and especially [K] for a more involved discussion of this and the other Weil conjectures. It is worth noting that the classical method of Lefschetz pencils play an important role in the proof. Deligne [D3], [D5] found a subsequent refinement which gives meaning to the mixed Betti numbers.

Theorem 17.3.6 (Deligne). *Let X be a d dimensional variety Then*

$$N_n(X_q) = \sum_{i=0}^{2d} (-1)^i \sum_{j=1}^{b_i} \lambda_{ji}^n$$

where λ_{ji} are algebraic integers with uniform absolute values in $\{0, q^{1/2}, q, \ldots q^{i/2}\}$. $b_i = \dim H_c^i(X, \mathbb{C})$ and

$$b_i^{(m)}(X) = \#\{j \mid |\lambda_{ij}| = q^{m/2}\}$$

17.4 A transcendental analogue of Weil's conjecture

The discussion in the previous section may have a seemed a bit like black magic. It may be worthwhile to explain a little more about the philosophy behind it. Let X be a smooth projective variety defined over $\mathbb{F}_q$, and $\bar{X}$ be the extension of X to the algebraic closure $\bar{\mathbb{F}}_q$. Fix a prime ℓ different from the characteristic of $\mathbb{F}_q$. If we choose an embedding $X \subset \mathbb{P}^N$, we have $F : \bar{X} \to \bar{X}$ be the Frobenius morphism which acts by raising the coordinates of the qth power (see [H] for a more precise description). Then $N_n(X)$ is just the number of fixed points of F^n. Weil suggested that that these numbers could be computed by an appropriate generalization of Lefschetz's trace formula. This was realized by Grothendieck's ℓ-adic cohomology theory [Mi], which assigns to X a collection of finite dimensional $\mathbb{Q}_\ell$-vector spaces $H^i_{et}(\bar{X}, \mathbb{Q}_l)$ which behaves very much singular cohomology of a complex variety. In particular, if X is obtained as in the previous section by reducing modulo Q, $\dim H^i_{et}(\bar{X}, \mathbb{Q}_l)$ coincides with the Betti number of the original complex variety. The Grothendieck-Lefschetz trace formula shows that

$$N_n(X) = \sum_i (-1)^i trace[F^{n*} : H^i_{et}(\bar{X}, \mathbb{Q}_l) \to H^i_{et}(\bar{X}, \mathbb{Q}_l)]$$

The λ_{ji} of theorem 17.3.5 are precisely the eigenvalues of F acting on these cohomology groups. Thus the real content of this theorem is the estimate on these eigenvalues.

Around 1960, Serre [S3] had proved an analogue of theorem 17.3.5 for complex varieties. This is something that we can prove. To set up the analogy let us replace $\bar{X}$ above by a smooth complex projective variety Y, and F by and endomorphism $f : Y \to Y$. As for q, if we consider, the effect of the Frobenius on $\mathbb{P}^N_{\mathbb{F}_q}$, the pullback of $\mathcal{O}(1)$ under this map is $\mathcal{O}(q)$. To complete the analogy, we require the existence of a very ample line bundle $\mathcal{O}_Y(1)$ on Y, so that $f^*\mathcal{O}_Y(1) \cong \mathcal{O}_Y(1)^{\otimes q}$. We can take $c_1(\mathcal{O}_Y(1))$ to be the Kähler class ω Then we have $f^*\omega = q\omega$.

Theorem 17.4.1 (Serre). *If $f : Y \to Y$ is holomomorphic endomorphism of a compact Kähler manifold with Kähler class ω, such that $f^*\omega = q\omega$ for some $q \in \mathbb{R}$. Then the eigenvalues λ of $f^* : H^i(Y,\mathbb{Z}) \to H^i(Y,\mathbb{Z})$ are algebraic integers with uniform absolute value $q^{i/2}$*

Proof. The theorem holds for $H^{2n}(Y)$ since ω^n generates it. By hypothesis, f^* preserves the Lefschetz decomposition (theorem 13.1.1), thus we can replace $H^i(Y)$ by primitive cohomology $P^i(Y)$. Recall from corollary 13.1.4,

$$\tilde{Q}(\alpha,\beta) = Q(\alpha, C\bar{\beta})$$

is positive definite Hermitean form $P^i(Y)$, where

$$Q(\alpha,\beta) = (-1)^{i(i-1)/2}\int \alpha\wedge\beta\wedge\omega^{n-i}.$$

Consider the endomorphism $F = q^{-i/2}f^*$ of $P^i(Y)$. We have

$$Q(F(\alpha),F(\beta)) = (-1)^{i(i-1)/2}q^{-n}\int f^*(\alpha\wedge\beta\wedge\omega^{n-i}) = Q(\alpha,\beta)$$

Moreover, since f^* is a morphism of Hodge structures, it preserves the Weil operator C. Therefore F is unitary with respect to $\tilde{Q}$, so its eigenvalues have norm 1. This gives the desired estimate on absolute values of the eigenvalues of f^*. Since f^* can be represented by an integer matrix, the set of it eigenvalues is a Galois invariant set of algebraic integers, so these have uniform absolute value $q^{i/2}$. ∎

Grothendieck suggested that one should be able to carry out a similar proof for the Weil conjectures. However making this work would require the full strength of his *standard conjectures* [GSt, Kl] which are, at present, very much open.

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