ARAKELOV INEQUALITIES FOR FAMILIES OF ABELIAN VARIETIES AFTER DELIGNE, VIEHWEG, ZUO

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Talk at IAS, Nov 25, 2014.

I want to report on parts of 2 papers

(1) Deligne, Un théorème de finitude pour la monodromie
(2) Viehweg, Zuo, A characterization of certain Shimura curves...

Here is the basic set up throughout. Let $f : X \to C$ be projective semistable map to a genus q curve which corresponds to a g dimensional abelian scheme over $C - S$, where the discriminant $S \neq \emptyset$. Let $\Delta = f^{-1}S$. Let $E^{10} = f_*\Omega^1_{X/C}(\log \Delta/S)$ and $E^{01} = R^1 f_*\mathcal{O}_X$ denote the Hodge bundles, and $\theta : E^{10} \to E^{01} \otimes \Omega^1_C(\log S)$ the Kodaira-Spencer map. I also want to assume that $\mathcal{H} = R^1 f_*\mathbb{Z}|_{C-S}$ has no nonzero sections over $C-S$. I refer to this last condition as *nondegeneracy*. It is not essential, but it simplifies things.

Theorem 0.1 (Deligne)**.** *The "Arakelov inequalities"*

$$\deg E^{10} \leq \frac{1}{2} g \deg \Omega^1_C(S) = \frac{1}{2} g(2q - 2 + \#S) \tag{0.1}$$

hold.

Proof. Let $d = \deg E^{10}$. Nondegeneracy implies

$$h^1(C - S, \mathcal{H}) = -\chi(C - S, \mathcal{H}) = -2g\chi(C - S) = 2g \deg \Omega^1_C(S) \tag{0.2}$$

By Zucker[1], $H = H^1(C - S, \mathcal{H})$ carries a mixed Hodge structure with graded pieces

$$Gr^F H = \begin{cases} H^1(E^{01} \xrightarrow{0} 0) \\ H^1(E^{10} \xrightarrow{\theta} E^{01} \otimes \Omega^1(S)) \\ H^1(0 \xrightarrow{0} E^{10} \otimes \Omega^1(S)) \end{cases}$$

Discarding the second term yields an inequality

$$\begin{aligned} h^1(C - S, \mathcal{H}) &\geq h^1(E^{01}) + h^0(E^{10} \otimes \Omega^1(S)) \\ &= h^0(E^{10} \otimes \Omega^1) + h^0(E^{10} \otimes \Omega^1(S)) \quad \text{(Serre duality)} \\ &\geq [d + g(q - 1)] + [d + g(q - 1 + \#S)] \quad \text{(Riem. Roch)} \end{aligned}$$

[1]Zucker, Hodge theory with degenerating coefficients

$$= -g\chi(C-S) + 2d$$

The theorem follows by combining with (0.2).

□

I will refer to (AE) as the condition that the equality holds in (0.1).

Theorem 0.2 (Viehweg-Zuo)**.** *If $f : X \to C$ satisfies the previous assumptions and (AE) holds, then there exists an étale cover $C' \to C$ such that $X \times_C C'$ is isogenous to a g-folds fibered product of an elliptic modular surface with itself. (An elliptic modular surface is the universal elliptic curve over a modular curve.)*

Conversely, a fibred product of elliptic modular surfaces satisfies (AE). I will outline the proof of this theorem. It is clear from an examination of the proof of Deligne's theorem that (AE) implies that the vanishing of $H^1(E^{10} \stackrel{\theta}{\to} E^{01} \otimes \Omega^1(S))$. On route to proving the above, Viehweg and Zuo show more:

Proposition 0.3. *(AE) holds if and only if $E^{10} \stackrel{\theta}{\to} E^{01} \otimes \Omega^1(S)$ is an isomorphism.*

Proof. If $E^{01} \cong E^{01} \otimes \Omega^1(S)$, then

$$\deg E^{10} = -\deg E^{10} + g\chi(C-S)$$

which implies (AE).

Conversely, let $E = E^{10} \oplus E^{01}$. This carries a map $E \to E \otimes \Omega^1(S)$ induced by θ that we refer to by the same symbol. We call the pair (E, θ) a Higgs bundle. The key result needed, due to Simpson[2], says *if $F \subset E$ is a sub Higgs bundle, that is a sub bundle stable under θ, then the slope $\deg F \leq 0$ and if equality holds then F is a direct summand.* Let $I \otimes \Omega^1(S)$ denote the image of θ. Since $(E^{10} \oplus I)$ is a sub Higgs bundle, we get

$$\deg E^{10} + \deg I \leq 0 \tag{0.3}$$

When this is applied to $K = \ker\theta$, this implies $\deg K \leq 0$.. Then

$$\deg E^{10} = \deg K + \deg(I \otimes \Omega^1(S)) \leq \deg(I \otimes \Omega^1(S))$$

$$= \deg I + rank(I) \deg \Omega^1(S)$$

$$\leq \deg I + rank(E^{10}) \deg \Omega^1(S)$$

$$\leq -\deg E^{10} + rank(E^{10}) \deg \Omega^1(S) \quad \text{by (0.3)}$$

$$= \deg E^{10} \quad \text{by (AE)}$$

$$\Rightarrow \deg I = -\deg E^{10} = \deg E^{01} \;\&\; \deg K = 0$$

Therefore θ is an isomorphism.

□

[2]Simpson, Harmonic bundles on noncompact curves

They give a direct proof when $g = 1$, which I sketch. In fact, however, an equivalent statement was known before [Mangala Nori, On elliptic surfaces with maximal Picard number].

Proposition 0.4. *The theorem holds when $g = 1$.*

Proof. Let $X \to C$ be a nondegenerate elliptic surface satisfying (AE). Then $L = E^{10}$ is an ample line bundle. The last proposition shows that

$$\theta : L \cong L^{-1} \otimes \Omega^1(S) \tag{0.4}$$

Therefore $\Omega^1(S)$ is ample, and so the universal cover $\tilde{U}$ of $U = C - S$ is abstractly isomorphic to the upper half plane $\mathbb{H}$. In fact, an explicit isomorphism is given by the period map $\phi : \tilde{U} \to \mathbb{H}$. To see this, note that the Kodaira-Spencer map for the universal marked elliptic curve $\tilde{f}$ induces an isomorphism $T_{\mathbb{H}} \cong (R\tilde{f}_*\mathcal{O})^2$. So (0.4) can be interpreted as the condition for ϕ to be everywhere unramified. We have a commutative diagram

$$\begin{tikzcd} \tilde{U} \arrow[r, "\phi", "\cong"'] \arrow[d, "\pi"] & \mathbb{H} \arrow[d, "\psi"] \\ U \arrow[r, "j"] & \mathbb{C} \end{tikzcd}$$

where π is the universal cover, $\psi : \mathbb{H} \to \mathbb{H}/SL_2(\mathbb{Z}) \cong \mathbb{C}$ is the quotient, and j is the j-invariant. From the diagram, $U \cong \mathbb{H}/\Gamma$, where Γ is the monodromy group of $\mathcal{H}$. Hence we may deduce that $\Gamma \subset SL_2(\mathbb{Z})$ has finite index.

$\square$

Higgs bundles form a rigid tensor category, so $F = End(E)$ carries an induced Higgs bundle structure. Writing

$$F^{1,-1} = E^{10} \otimes E^{01*}$$

$$F^{0,0} = E^{10} \otimes E^{10*} \oplus E^{01} \otimes E^{01*}$$

$$F^{-1,0} = E^{01} \otimes E^{10*}$$

the Higgs field for F decomposes as a sum of

$$\phi_{1,-1} : F^{1,-1} \to F^{00} \otimes \Omega^1(S)$$

$$\phi_{0,0} : F^{0,0} \to F^{-1,1} \otimes \Omega^1(S)$$

By direct calculation plus Simpson,

Proposition 0.5. *(F, ϕ) is a direct sum of $(\ker \phi_{00}, 0)$ and another Higgs bundle (F', ϕ'). This corresponds to a sum of local systems $End(\mathcal{H} \otimes \mathbb{C}) = U \oplus W$, where U is a unitary local system on C of rank g^2.*

Remark 0.6. (Belated answer to János) Using prop 0.3, we can see that ϕ_{00} is surjective. So the kernel has rank $2g^2 - g^2$ as asserted above.

One can see more or less immediately that the above decomposition is as $\mathbb{C}$-variations of Hodge structure. A more refined analysis is needed to show that this is defined over $\mathbb{Q}$.

Proposition 0.7. *The above local systems U, W come from $\mathbb{Q}$-variations of Hodge structure. The monodromy group of U is finite.*

We recall that there are moduli spaces, called (PEL) Shimura varieties, parametrizing abelian varieties of a given dimension with specified polarization and endomorphism structure. The final result needed, which is only stated implicitly by the authors, is

Proposition 0.8. *If A is a complex g dimensional abelian variety such that* $\dim End(A)\otimes \mathbb{Q} \supseteq \mathbb{Q}^{g^2}$, *then A is isogenous to either*

(a) *a self product of an elliptic curve, or*
(b) *a self product of an abelian surface of a certain type for which the corresponding Shimura variety is a projective curve.*

Proof of theorem 2. Use proposition 0.7, then after pulling back to an étale cover we can assume that $U = \mathbb{Q}^{g^2}$. Now apply proposition 0.8 to the geometric generic fibre. Case (b) would lead to contradiction since we assume that the discriminant $S \neq \emptyset$. Therefore we are in case (a). It follows easily that $X \to C$ is isogenous to a self fibre product of an elliptic surface satisfying (AE). So we conclude by proposition 0.4. $\square$

If we relax the condition that $S \neq \emptyset$, then there are more examples of (AE) type. For instance, there are complete Shimura curves parametrizing "false elliptic curves" i.e. two dimension abelian varieties with endomorphism algebra equal to an indefinite quaternion algebra. With additional hypotheses, Viehweg and Zuo classify all (AE) families with $S = \emptyset$ as well. I refer to the paper for the precise statement.