Solutions to Midterm for MA514 in Fall09

Solution to Problem 1:

Iterative function $\varphi(x) = x - \lambda f(x)$

$\Rightarrow \varphi'(x) = x - \lambda f'(x)$

$\Rightarrow -1 = 1 - \frac{2}{M} \cdot M < 1 - \lambda M \le \varphi'(x) \le 1 - \lambda m < 1$

Let $\mu = \max(|1 - \lambda M|, |1 - \lambda m|)$ then we have $|\varphi'(x)| \le \mu < 1$ for $\forall x$

So φ is contractive.

$\Rightarrow \varphi$ has a unique fixed point α : $\varphi(\alpha) = \alpha \Rightarrow \alpha = \alpha - \lambda f(\alpha) \Rightarrow f(\alpha) = 0$.

By Contractive Mapping Theorem: $\varphi(x)$ has a unique fixed point $\alpha \in R$ which is the root of $f(x) = 0$, and α is the limit of sequence $\{x_n\}$ generated by $x_{n+1} = x_n - \lambda f(x_n) = \varphi(x_n)$ with any initial $x_0 \in R$.

Solution to Problem 2:

Denote $\varphi(x) = px + qax^{-2} + ra^2 x^{-5}$

Let $\varphi(\sqrt[3]{a}) = \sqrt[3]{a}$ $\Rightarrow$ $a^{1/3} = pa^{1/3} + qa \cdot a^{-2/3} + ra^2 \cdot a^{-5/3} = (p+q+r)a^{1/3}$ $\Rightarrow$

$p + q + r = 1$ (1)

$\varphi'(x) = p + qa \cdot (-2)x^{-3} + ra^2 \cdot (-5)x^{-6} = p - 2qax^{-3} - 5ra^2 x^{-6}$, let $\varphi'(\sqrt[3]{a}) = 0$ $\Rightarrow$

$p - 2q - 5r = 0$ (2)

$\varphi''(x) = -2qa \cdot (-3)x^{-4} - 5ra^2 \cdot (-6)x^{-7} = 6qax^{-4} + 30ra^2 x^{-7}$, let $\varphi''(\sqrt[3]{a}) = 0$ $\Rightarrow$

$6qa^{-1/3} + 30ra^{-1/3} = 0 \Rightarrow$ $q + 5r = 0$ (3)

(1) (2) (3) $\Rightarrow$ $p = q = \frac{5}{9}$, $r = -\frac{1}{9}$.

Solution to Problem 3:

Newton Form:

x_0					
0	6				
1	4	-2			
3	18	7	3		
5	16	-1	-2	-1	

From above we see $p(x) = 6 - 2x + 3x(x-1) - x(x-1)(x-3)$

Lagrange Form:

$$p(x) = \sum_{j=0}^{3} L_{3,j}(x) f(x_j)$$

where $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 5$,

$f(x_0) = 6$, $f(x_1) = 4$, $f(x_2) = 18$, $f(x_3) = 16$

$$L_{3,0}(x) = \frac{(x-1)(x-3)(x-5)}{(-1)(-3)(-5)} = -\frac{(x-1)(x-3)(x-5)}{15}$$

$$L_{3,1}(x) = \frac{x(x-3)(x-5)}{(1)(-2)(-4)} = \frac{x(x-3)(x-5)}{8}$$

$$L_{3,2}(x) = \frac{x(x-1)(x-5)}{(3)(2)(-2)} = -\frac{x(x-1)(x-5)}{12}$$

$$L_{3,3}(x) = \frac{x(x-1)(x-3)}{(5)(4)(2)} = \frac{x(x-1)(x-3)}{40}$$

$$p_3(x) = -\frac{2}{5}(x-1)(x-3)(x-5) + \frac{1}{2}x(x-3)(x-5) - \frac{3}{2}x(x-1)(x-5) + \frac{2}{5}x(x-1)(x-3).$$

Solution to Problem 4:

x_0							
0	-8						
1	2	10					
1	2	3	-7				
2	6	4	1	4			
2	6	7	3	2	-1		
2	6	7	8/2=4	1	-1	0	
2	6	7	8/2=4	6/6=1	0	1	0.5

Hence, the polynomial is:

$$p(x) = -8 + 10x - 7x(x-1) + 4x(x-1)^2 - x(x-1)^2(x-2) + 0.5x(x-1)^2(x-2)^3$$

Solution to Problem 5:

The interpolation conditions yield:

$$0 = S(0) = a_0$$

$$1 = S(2) = a_0 + 2b_0 + 4c_0$$

$$1 = S(2) = a_1$$

$$2 = S(3) = a_1 + b_1 + c_1$$

The continuity of the derivative condition yields:

$$b_1 = S'(2) = b_0 + 4c_0$$

$$2 = S'(0) = b_0$$

From the five equations above, we can get:

$$a_0 = 0$$

$$a_1 = 1$$

$$b_0 = 2$$

$$b_1 = -1$$

$$c_0 = -3/4$$

$$c_1 = 2$$

$$\Rightarrow \quad S(x) = 2x - \frac{3}{4}x^2 \text{ if } 0 \le x < 2,$$

$$S(x) = 1 - (x-2) + 2(x-2)^2 \text{ if } 2 \le x \le 3.$$