Section 1.3

10. The null space of E^r consists of sequences $x = [x_1, x_2, \dots]$ in which $x_i = 0$ for $i > r$. The null space has dimension r.

11. (a) Characteristic equation: $x^3 - 3x^2 + 4 = 0$. Roots: $-1, 2$ (double). Basis: $[-1, 1, -1, 1, \dots, (-1)^n, \dots]$, $[2, 4, 8, 16, \dots, 2^n, \dots, \dots]$, $[1, 4, 12, 32, \dots, n2^{n-1}, \dots]$.

    (c) Characteristic equation: $2x^6 - 9x^5 + 12x^4 - 4x^3 = 0$. Roots: 0 (triple), 1/2 (simple), 2 (double). Basis: $x^{(1)} = [1, 0, 0, \dots]$, $x^{(2)} = [0, 1, 0, \dots]$, $x^{(3)} = [0, 0, 1, \dots]$, $x^{(4)} = [1/2, 1/4, 1/8, \dots]$, $x^{(5)} = [2, 4, 8, \dots]$, $x^{(6)} = [1, 4, 12, \dots]$. Here the general term for $x^{(6)}$ is $x_n^{(6)} = n2^{n-1}$.

13. Notice these are not polynomial difference operators. Solve by inspection for first few terms.

    (a) $x_{n+1} = n!x_1$ (b) $x_{n+1} = (1/2)n(n+1) + x_1$

    (c) $x_{n+1} = 2n + x_1$

14. It is obvious that $\Delta = E - I$. If p is a polynomial of degree n, then by Taylor's Theorem $p(x) = \sum_{j=0}^{n} [p^{(j)}(a)/j!](x-a)^j$. Put $x = E$ and $a = I$ to get $p(E) = \sum_{j=0}^{n} (1/j!)p^{(j)}(I)\Delta^j$.

27. Characteristic equation: $\lambda^2 - 2\lambda - 2 = 0$. Roots: $1 \pm \sqrt{3}$. General solution: $z_n = \alpha(1+\sqrt{3})^n + \beta(1-\sqrt{3})^n$. Initial values give $1 = x_1 = \alpha(1+\sqrt{3}) + \beta(1-\sqrt{3})$ and $1 - \sqrt{3} = x_2 = \alpha(1+\sqrt{3})^2 + \beta(1-\sqrt{3})^2$. So solution is $\alpha = 0$ and $\beta = 1/(1-\sqrt{3})$.

Section 2.1

35. Typical case: $\text{fl}(x^4) = \text{fl}(x \cdot x \cdot x \cdot x) = [x \cdot \text{fl}(x \cdot x \cdot x)](1+\delta_1) =$ $x \cdot x \cdot \text{fl}(x \cdot x)(1+\delta_2)(1+\delta_1) = x \cdot x \cdot x \cdot x(1+\delta_3)(1+\delta_2)(1+\delta_1)$
    General case: $\text{fl}(x^k) = x^k(1+\delta_1)(1+\delta_2)\cdots(1+\delta_{k-1})$ where each δ_i satisfies $|\delta_i| \le 2^{-24}$. The term $(1+\delta_1)\cdots(1+\delta_{k-1})$ lies between $(1-2^{-24})^{k-1}$ and $(1+2^{-24})^{k-1}$. (To see this take all δ_i at their smallest value, -2^{-24}, and then at their largest value, $+2^{-24}$). Find a single δ so that $(1+\delta_1)\cdots(1+\delta_{k-1}) = (1+\delta)^{k-1}$. This is possible because as δ varies from -2^{-24} to $+2^{-24}$, $(1+\delta)^{k-1}$ covers the interval from $(1-2^{-24})^{k-1}$ to $(1+2^{24})^{k-1}$, and this interval contains the term $(1+\delta_1)\cdots(1+\delta_{k-1})$. So $\text{fl}(x^k) = x^k(1+\delta)^{k-1}$ with $|\delta| \le 2^{-24}$.

36. Assume a machine with 5-decimals in its floating point numbers and that it stores numbers in rounded form.

    (a) $x = .63180 \times 10^{-3}$ $y = -.14782 \times 10^5$ $z = .71110 \times 10^{-1}$
    $\text{fl}(xy) = -.93393 \times 10^1$, $\text{fl}(yz) = -.10511 \times 10^4$, $\text{fl}[\text{fl}(xy) \cdot z] = -.66412$, $\text{fl}[x \cdot \text{fl}(yz)] = -.66408$, and $-.66412 \neq .66408$.

    (b) $x = .62209 \times 10^{-3}$ $y = -.20971 \times 10^7$ $z = .53100 \times 10^{-1}$
    $\text{fl}(xy) = -.13046 \times 10^4$, $\text{fl}(yz) = -.11136 \times 10^6$, $\text{fl}[\text{fl}(xy) \cdot z] = -.69274 \times 10^2$, $\text{fl}[x \cdot \text{fl}(yz)] = -.69276 \times 10^2$, and $-.69274 \times 10^2 \neq -.69276 \times 10^2$.

Section 2.2

8. Write $x = [(-b+\sqrt{b^2-4ac})/(2a)][(b+\sqrt{b^2-4ac})/(b+\sqrt{b^2-4ac})]$
$= (-2c)/(b+\sqrt{b^2-4ac})$. This works if $b > 0$. If $b < 0$ the troublesome root is
$x = [(-b-\sqrt{b^2-4ac})/(2a)][(b-\sqrt{b^2-4ac})/(b-\sqrt{b^2-4ac})] = (-2c)/(b-\sqrt{b^2-4ac})$.
Summary: If $b > 0$ then $x_1 = (-b-\sqrt{b^2-4ac})/(2a)$ and
$x_2 = (-2c)/(b+\sqrt{b^2-4ac}) = c/(ax_1)$. If $b < 0$ then $x_1 = (-b+\sqrt{b^2-4ac})/(2a)$ and
$x_2 = (-2c)/(b-\sqrt{b^2-4ac}) = c/(ax_1)$. Here $x_1x_2 = c/a$. This is the standard approach.
In addition, if there is loss of precision in the calculation of $\sqrt{b^2-4ac}$ it could be done in double precision to reduce loss of precision in this subtraction. An accurate and reliable procedure for solving the quadratic equation is discussed at length in Young and Gregory [1972, pp.77–83].

9. (a) $\sqrt{x^2+1}-x = (\sqrt{x^2+1}-x)[(\sqrt{x^2+1}+x)/(\sqrt{x^2+1}+x)] = 1/(\sqrt{x^2+1}+x)$.
   (b) $\log x - \log y = \log(x/y)$.
   (e) $e^x - e = (1-e) + x + x^2/2 + x^3/3! + x^4/4! + \cdots$.
   (f) $\log x - 1 = \log(x/10)$.
   (h) $\sin x - \tan x = \sin x - (\sin x/\cos x)$
   $= \sin x(\cos x - 1)/(\cos x) = \tan x(\cos x - 1)(\cos x + 1)/(\cos x + 1)$
   $= \tan x(\cos^2 x - 1)/(\cos x + 1) = (-\tan x \sin^2 x)/(\cos x + 1)$.

15. (a) $f(0) = 0$ using L'Hospital's rule.
   (b) Near $x = 2\pi n$, loss of significance since $\cos x \approx 1$.
   (c) Use $f(x) = (\sin^2 x)/[x(1+\cos x)]$
   (d) From identity $1 - \cos x = 2\sin^2(x/2)$, use $f(x) = (2/x)\sin^2(x/2)$ or original formula for values when $\cos x \approx -1$.

17. By the Theorem on Loss of Precision, we restrict x so that $1/4 \le 1 - 1/(\sqrt{2}+1)$. Hence, $1/(\sqrt{x^2+1}) \le 3/4 \Rightarrow |x| \ge \sqrt{7}/3 \equiv 0.8819$.

Section 2.3

7. Characteristic equation $\lambda^2 - 2\lambda - 1 = 0$ has roots $1 \pm \sqrt{2}$ so general solution of the form $x_n = A(1+\sqrt{2})^n + B(1-\sqrt{2})^n$. No, unstable since $(1+\sqrt{2}) > 1$.

8. Assume $r_n = \lambda^n$ gives characteristic equation $\lambda^2 - \lambda - 1 = 0$ with roots
$\lambda_1 = [(1+\sqrt{5})/2]$ and $\lambda_2 = [(1-\sqrt{5})/2]$. General solution $r_n = A\lambda_1^n + B\lambda_2^n$ where
$A = [(1+\sqrt{5})/2]/\sqrt{5}$ and $B = [(1-\sqrt{5})/2]/\sqrt{5}$. Now
$r_n/r_{n-1} = (A\lambda_1^n + B\lambda_2^n)/(A\lambda_1^{n-1} + B\lambda_2^{n-1}) = \lambda_1[A + B\theta^n]/[A + B\theta^{n+1}] \to \lambda_1$ since
$\theta = \lambda_2/\lambda_1 <$. The convergence has linear behavior.

9. As above, $r_n = A\lambda_1^n + B\lambda_2^n = A[(1+\sqrt{5})/2]^n + B[(1-\sqrt{5})/2]^n$. Since the root $|\lambda_1| > 1$, the recurrence relation does not provide a stable means for computing r_n. In this case, $A = 0$ and $B = 1$ so $r_n = [(1-\sqrt{5})/2]^n \to 0$ as $n \to \infty$.

Section 3.1

4. By theorem, $|r - c_n| \le 2^{-(n+1)}(b_0 - a_0) \le \varepsilon$. So,
$-(n+1)\log 2 + \log(b_0 - a_0) \le \log\varepsilon \Rightarrow -(n+1) \le [\log\varepsilon - \log(b_0 - a_0)]/\log 2$.
Hence, $n > [\log(b_0 - a_0) - \log\varepsilon]/(\log 2) - 1$.

5. Relative precision $= |r - c_n|/|r| \le \varepsilon$. Since $r \ge a_0 > 0$, we require $|r - c_n|/a_0 \le \varepsilon$,
$2^{-(n+1)}(b_0 - a_0)/a_0 \le \varepsilon$, $-(n+1) \le \log[(\varepsilon a_0)/(b_0 - a_0)]/\log 2$,
$n \ge \log[(b_0 - a_0)/(\varepsilon a_0)]/(\log 2) - 1 = [\log(b_0 - a_0) - \log\varepsilon - \log a_0]/(\log 2) - 1$.

7. By Problem 3.1.4, $|r - c_n| \le \varepsilon$ after $n \ge [\log(b_0 - a_0) - \log\varepsilon]/(\log 2) - 1$. Here $n \ge 6/(\log 2) - 1 = 18.93$. So in 19 steps, we obtain 10^{-6} absolute accuracy. On `MARC-32`, machine precision 2^{-24} is obtained in $n > (24\log 2)/(\log 2) - 1 = 23$ steps for absolute accuracy to full precision.

By Problem 3.1.5, relative precision $|r - c_n|/|r| \le \varepsilon$ requires
$n \ge [\log(b_0 - a_0) - \log\varepsilon - \log a_0)]/(\log 2) - 1$. Here
$n \ge [6 - \log 2]/(\log 2) - 1 = [6/\log 2] - 2 = 17.93$. So in 18 steps, we obtain 10^{-6} relative accuracy. On the `MARC-32`, we have $n \ge [24\log 2 - \log 2]/(\log 2) - 1 = 24 - 2 = 22$ steps for relative accuracy to full precision.

Section 3.2

5. Let $\lim_{n\to\infty} x_n = r$, then $r = 2r - r^2 y \Rightarrow r = 1/y$. So the purpose of the formula is to compute $1/y$. Now
$x_{n+1} = 2x_n - x_n^2 y = x_n + (x_n - x_n^2 y) = x_n + x_n^2(1/x_n - y) = x_n + (x_n^{-1} - y)/x_n^{-2}$. So
$x_{n+1} = x_n - (x_n^{-1} - y)/(-x_n^{-2})$. Thus it is Newton's iteration for $f(x) = x^{-1} - y$.
Alternative Solution: $x_{n+1} = 2x_n - x_n^2 y = x_n - f(x_n)/f'(x_n)$. Change y to a and $f(x)$ to y. Solve $x - ax^2 = -y/y'$. Now $(\log y)' = y'/y = 1/[x(ax-1)] = A/x + B/(ax-1)$ implies $A = -1$ and $B = a$. So $(\log y)' = -1/x + a/(ax-1)$ or
$\log y = -\log x + \log(ax-1) = \log[(ax-1)/x]$ implies $y = a - 1/x$. Hence, original problem to solve $f(x) = y - 1/x$.

10. $f(x) = x^3 - R$; $f'(x) = 3x^2$. Thus the Newton's iteration formula for computing $\sqrt[3]{R}$ is: $x_{n+1} = x_n - (x_n^3 - R)/3x_n^2 = (2x_n + R/x_n^2)/3$. For $x > 0$, $f'(x) > 0$ and $f''(x) > 0$. Then by Theorem 2 the Newton iteration will converge from any point > 0. For $x < 0$, Newton method will not always converge since one of the iterates could nit the orgin. A quick calculation shows that if we start with the point $x = \sqrt[3]{-R/2}$, then the first iterate will be the point $x_1 = 0$ where $f'(x_1) = 0$ and $x_2 = \infty$. So the method fails in this case. Hence, it converges for all $x > 0$.

15. $e_{n+1} = e_n - f(x_n)/f'(x_0)$. Also $0 = f(r) = f(x_n - e_n) = f(x_n) - e_n f'(\xi_n)$
$\Rightarrow f(x_n) = e_n f'(\xi_n)$. Hence $e_{n+1} = e_n(1 - f'(\xi_n)/f'(x_0)) \Rightarrow s = 1$,
$C = 1 - f'(\xi_n)/f'(x_0)$.

19. $f(r) = f'(r) = \cdots = f^{(k-1)}(r) = 0 \neq f^{(k)}(r)$.
$f(x_n) = f(r+e_n) = f(r) + e_n f'(r) + \cdots + e_n^{(k-1)} f^{(k-1)}(r)/(k-1)! + e_n^k f^{(k)}(r)/k!$
$+ e_n^{k+1} f^{(k+1)}(\xi_n)/(k+1)! \Rightarrow f(x_n) = e_n^k f^{(k)}(r)/k! + e_n^{k+1} f^{(k+1)}(\xi_n)/(k+1)!$. Similarly,
$f'(x_n) = e_n^{k-1} f^k(r)/(k-1)! + e_n^k f^{(k+1)}(\eta_n)/k!$. Then
$e_{n+1} = x_{n+1} - r = x_n - r - kf(x_n)/f'(x_n) = e_n - [e_n^k f^{(k)}(r)/(k-1)!$
$+ e_n^{k+1} f^{(k+1)}(\xi_n)/(k+1)(k-1)!]/[e_n^{k-1} f^{(k)}(r)/(k-1)! + e_n^k f^{(k+1)}(\eta_n)/k!]$
$= e_n^2 [f^{(k+1)}(\eta_n)/k! - f^{(k+1)}(\xi_n)/(k+1)(k-1)!]/[f^{(k)}(r)/(k-1)! + e_n f^{(k+1)}(\eta_n)/k!]$,
implying quadratic convergence.

Alternative Solution: $f^{(j)}(r) = 0$ for $0 \leq j \leq m-1$ and $f^{(m)}(r) \neq 0$. So the Taylor formula gives
$f(r+h) = f(r) + hf'(r) + \cdots + [h^{m-1}/(m-1)!] f^{(m-1)}(r) + [h^m/m!] f^{(m)}(r) + \cdots$. Then
$f(x_n) = f(r+e_n) = [e_n^m/m!] f^{(m)}(r) + e_n^{m+1} A$ where $A \equiv f^{(m+1)}(\xi_n)/(m+1)!$. Similarly,
$f'(x_n) = [e_n^{m-1}/(m-1)!] f^{(m)}(r) + e_n^m B$ where $B \equiv f^{(m+1)}(\eta_n)/m!$. Now
$e_{n+1} = x_{n+1} - r = x_n - r - mf(x_n)/f'(x_n)$
$= e_n - mf(x_n)/f'(x_n) = [e_n f'(x_n) - mf(x_n)]/f'(x_n)$
$= \{e_n[e_n^{m-1} f^{(m)}(r)/(m-1)! + e_n^m B] - m[e_n^m f^{(m)}(r)/m! + e^{m+1)} A]\}/$
$\{e_n^{m-1} f^{(m)}(r)/(m-1)! + e_n^m B\} = [e_n^{m+1} B - m e_n^{m+1} A]/[e^{m-1} f^{(m)}(r)/(m-1)! + e_n^m B]$
$= e_n^2 \{(B - mA)/[f^{(m)}(r)/(m-1)! + e_n B]$. We need to assume $f, f', \ldots, f^{(m+1)}$ are continuous and that $f^{(m+1)}(r) \neq 0$.

23. (a) $J = \begin{bmatrix} 8x_1 & -2x_2 \\ 4x_2^2 - 1 & 8x_1 x_1 \end{bmatrix}$. So $J(0,1) = \begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix}$ and $J^{-1}(0,1) = (1/6) \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$.

Thus, $\begin{bmatrix} h_1^{(1)} \\ h_2^{(1)} \end{bmatrix} = -(1/6) \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/2 \end{bmatrix}$. So $\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix}$.

Next $\begin{bmatrix} h_1^{(2)} \\ h_2^{(2)} \end{bmatrix} = -(1/6) \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/6 \end{bmatrix}$. Thus

$\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/2 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix}$.

Section 3.3

7. $x_{n+1} = x_n - f(x_n)[(x_n - x_{n-1})/(f(x_n) - f(x_{n-1}))]$
$= [x_n f(x_n) - x_n f(x_{n-1}) - x_n f(x_n) + x_{n-1} f(x_n)]/[f(x_n) - f(x_{n-1})]$
$= [f(x_n) x_{n-1} - x_n f(x_{n-1})]/[f(x_n) - f(x_{n-1})]$. This is inferior to Eqn. (3) because as $x_n \to x_{n+1} \approx r$, $f(x_n) \to f(x_{n+1}) \approx f(r)$ resulting in $\approx [f(r)r - rf(r)]/[f(r) - f(r)]$ "catastrophic cancellation" with a loss of precision while Eqn. (3) produces
$\approx r - f(r)[(r-r)/(f(r) - f(r))] \approx r$.

Section 3.4

6. $F(x) = x + f(x)g(x)$, $f(\xi) = 0$, $f'(\xi) \neq 0$. The theory developed in this section shows that the sequence defined by $x_{n+1} = F(x_n)$ will converge *cubically* to ξ if $F'(\xi) = 0$, $F''(\xi) = 0$, $F'''(\xi) \neq 0$. (We want *at least* cubically convergent sequence, so we do not insist on $F'''(\xi) \neq 0$.) $F'(x) = 1 + f'(x)g(x) + f(x)g'(x)$
$\Rightarrow F'(\xi) = 1 + f'(\xi)g(\xi) + f(\xi)g'(\xi) = 1 + f'(\xi)g(\xi)$. We want this to be zero. So $g(\xi) = -1/f'(\xi)$. (Condition 1) $F''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$
$\Rightarrow F''(\xi) = f''(\xi)g(\xi) + 2f'(\xi)g'(\xi) + f(\xi)g''(\xi) = f''(\xi)[-1/f'(\xi)] + 2f'(\xi)g'(\xi)$. For this to be zero we require $g'(\xi) = f''(\xi)/\{2[f'(\xi)]^2\}$. (Condition 2) *Note*: If $g(x) = -[f'(x)]^{-1}$ then $g'(x) = [f'(x)]^{-2}[f''(x)]$, so $g'(\xi)$ is off by a factor of 2.

7. Eventually 0.9998477 appears. Let $f(x) = \cos x$. Then $|\cos x - \cos y| = |\sin \xi|\ |x - y|$ for $x < \xi < y$ since $|\sin \xi| < 1$, $F(x)$ is a contraction and thus has a fixed point.

12. $x = \sqrt{p + \sqrt{p + \sqrt{p + \cdots}}}$ Let $x_1 = \sqrt{p}$, $x_2 = \sqrt{p + \sqrt{p}}$, $x_3 = \sqrt{p + \sqrt{p + \sqrt{p}}}$, and so on. Observe that $x_2 = \sqrt{p + x_1}$, $x_3 = \sqrt{p + x_2}$, and so on. In general $x_{n+1} = \sqrt{p + x_n}$ (I). Let $f(x) = \sqrt{p + x}$. Equation (I) is the result of using functional iteration on f. If $\lim x_n$ exists, denote it by x. Take limits in Equation (I) to get $x = \sqrt{p + x}$. Hence $x^2 = p + x$, $x^2 - x - p = 0$, $x = (1 + \sqrt{1 + 4p})/2$. This is the limit of the sequence. For example if $p = 2$, $x = 2$. Try it on your pocket calculator.

13. Use the ideas of Problem 3.4.12. Let $x_1 = 1/p$, $x_2 = 1/(p + (1/p))$, $x_3 = 1/(p + (1/p + (1/p)))$ etc. So $x_2 = 1/(p + x_1)$, $x_3 = 1/(p + x_2)$, and so on. Hence $x_{n+1} = 1/(p + x_n)$. If $\lim_{n\to\infty} x_n = x$ then $x = 1/(p + x)$. Hence $x(p + x) = 1$, $x^2 + px - 1 = 0$, $x = (-p + \sqrt{p^2 + 4})/2$. This illustrates functional iteration with $f(x) = 1/(p + x)$. If $p > 1$, f is a contraction. Use Mean Value Theorem: $|f(x) - f(y)| = |f'(\xi)|\,|x - y| = |-1/(p + \xi)^2|\,|x - y|$. Since $p > 1$, all x_n's will be ≥ 0, and $1/(p + x)^2 \leq 1/p^2 < 1$. So f is a contraction on $[0, \infty]$. f actually maps $[0, 1]$ into $[0, 1]$, so has a fixed point in $[0, 1]$.

40. $F(x) = x(x^2 + 3R)/(3x^2 + r)$, $F(\sqrt{R}) = \sqrt{R}$, $F'(\sqrt{R}) = 0$, $F''(\sqrt{R}) = 0$, $F'''(\sqrt{R}) \neq 0$.

Section 3.5

1.

	3	−7	−5	1	−8	2
4		12	20	60	244	944
	3	5	15	61	236	946

So $p(4) = 946$.

3. $p(4) = 946$, $p'(4) = 1808$ so $z_1 = z_0 - p(z_0)/p'(z_0) = 4 - 946/1808 = 3.47677$

4.

$a_k:$	3	−7	−5	1	−8	2
$u = 3$		9	6	12	45	123
$v = 1$			3	2	4	15
$b_k:$	3	2	4	15	$41 = b_1$	$140 = b_0$
$u = 3$		9	33	120	438	
$v = 1$			3	11	40	
$c_k:$	3	11	$40 = c_2$	$146 = c_1$	$519 = c_0$	

So $J = c_0c_2 - c_1^2 = -556$, $\delta u = (c_1b_1 - c_2b_2)/J = -0.69425$ and $\delta v = (c_1b_0 - c_0b_1)/J = 1.50899$. Hence, $u = u + \delta u = 2.30576$ and $v = v + \delta v = 2.50899$.

10.

	9	−7	1	−2	5
6		54	282	1698	10176
	9	47	283	1696	10181
6		54	606	5334	
	9	101	889	7030	

So $p(6) = 10181$ and $p'(6) - 7030$. Using $z_0 = 6$, we have $z_1 = z_0 - p(z_0)/p'(z_0) = 6 - 10181/7030 = 4.55178$.

16. $\alpha_0 = a_0 + \alpha_1 x$, $\alpha_1 = a_1 + x\alpha_2$, $\alpha_2 = a_2 + x\alpha_3, \ldots$. Thus $\alpha_0 = a_0 + x(a_1 + x(a_2 + x(a_3 + \ldots))) = p(x)$, $\beta_0 = \alpha_1 + x\beta_1$, $\beta_1 = \alpha_2 + x\beta_2$, $\beta_2 = \alpha_3 + x\beta_3, \ldots$. Thus
$\beta_0 = a_1 + x(a_2 + x(a_3 + \ldots)) + x(\alpha_2 + x(\alpha_3 + \ldots)) = a_1 + 2a_2x + 3a_3x^2 + \ldots = p'(x)$,
$\gamma_0 = \beta_1 + x\gamma_1$, $\gamma_1 = \beta_2 + x\gamma_2$, $\gamma_2 = \beta_3 + x\gamma_3 \ldots$. Thus
$\gamma_0 = a_2 + x(a_3 + x(a_4 + \ldots)) + x(\alpha_3 + x(\alpha_4 + \ldots)) + x(\beta_2 + x(\beta_3 + \ldots))$
$= a_2 + 3a_3x + 6a_4x^2 + \ldots$. Therefore, $2\gamma_0 = p''(x)$.

Section 3.6

1. Let $f(x) = \begin{bmatrix} \xi_1 - 2\xi_2 + \xi_2^2 + \xi_2^3 - 4 \\ -\xi_1 - \xi_2 + 2\xi_2^2 - 1 \end{bmatrix}$ and $f(0) = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$. By Equation (4),
$h(t,x) = f(x) + (t-1)f(0) = \begin{bmatrix} \xi_1 - 2\xi_2 + \xi_2^2 + \xi_2^3 - 4t \\ -\xi_1 - \xi_2 + 2\xi_2^2 - t \end{bmatrix}$. By Equation (9),
$\begin{bmatrix} -4 & 1 & -2 + 2\xi_2 + 3\xi_2^2 \\ -1 & -1 & -1 + 4\xi_2 \end{bmatrix} \begin{bmatrix} t' \\ \xi_1' \\ \xi_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. By Equation (10), $t' = -3 + 6\xi_2 + 3\xi_2^2$,
$\xi_1' = -2 + 14\xi_2 - 3\xi_2^2$, $\xi_2' = 5$ with $t(0) = \xi_1(0) = \xi_2(0) = 0$. Clearly, $\xi_2 = 5s$. Thus, $t' = -3 + 30s + 75s^2$ and $\xi_1' = -2 + 70s - 75s^2$. So $t = -3s + 15s^2 + 25s^3$ and $\xi_1 = -2s + 35s^2 - 25s^3$. We want to find values of s so that $25s^3 + 15s^2 - 3s - 1 = 0$. Using *Mathematica*, we find the roots of this cubic are $s = -1/5, -(1 \pm \sqrt{6})/5$. So the solutions of the original system are $(2, -1)$ and $(14 \pm 5\sqrt{6}, -(1 \pm \sqrt{6}))$.

Section 6.1

1.

(c) Move leftmost table entry $(3, 10)$ to rightmost table postion. From part (b), $p(x) = 146 + 24(x-7) + 5(x-7)(x-1) + c_3(x-7)(x-1)(x-2)$. Evaluating at $x = 3$, we find $10 = 10 - 8c_3$ and $c_3 = 0$. Answer the same as in part (b), namely, $p(x) = 146 + 24(x-7) + 5(x-7)(x-1)$.

Alternative Solution:

$p(x) = c_0 + c_1(x-3) + c_2(x-3)(x-7) + c_3(x-3)(x-7)(x-1)$. Solve $c_0 = 10$, $c_0 + 4c_1 = 146$, $c_0 - 2c_1 + 12c_2 = 2$, $c_0 - c_1 + 5c_2 + 5c_3 = 1$ to get $p(x) = 10 + 34(x-3) + 5(x-3)(x-7)$.

8. If $p(x) = a + bx + cx^2$ then we have $p(0) = a$, $p(1) = a + b + c$, $p'(\xi) = b + 2c\xi$. The determinant of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2\xi \end{bmatrix}$ should not be zero, i.e., $\xi \neq \frac{1}{2}$.

9. Let $g(x_i) = f(x_i)$ for $0 \le i \le n-1$ and $h(x_i) = f(x_i)$ for $1 \le i \le n$. Set $k(x) = g(x) + [(x_0 - x)/(x_n - x_0)][g(x) - h(x)]$. Then $k(x_0) = g(x_0) = f(x_0)$ and for $1 \le i \le n-1$ we have $k(x_i) = g(x_i) + [(x_0 - x_i)/(x_n - x_0)][g(x_i) - h(x_i)] = g(x_i) = f(x_i)$ and $k(x_n) = h(x_n) = f(x_n)$.

14. Say $x_j = 0$. $|p(x) - f(x)| = \left|(1/n!) f^{(n)}(\xi_x) \prod_{i=0}^{n-1}(x - x_i)\right| \le (1.5431/n!)|x| \prod_{\substack{i \ne j \\ i=0}}^{n-1}(x - x_i)$ $\le (1.5431/n!)|x| 2^{n-1}$ since node x_j is 0. Note as in Problem 6.1.13, $f^{(n)}(\sinh x) = \begin{cases} \sinh x & n \text{ even} \\ \cosh x & n \text{ odd} \end{cases}$ and $|f^{(n)}(\sinh x)| \le \max\{\sinh 1, \cosh 1\}$, on $[-1, 1]$. So $|p(x) - f(x)|/|f(x)| \le (1.5431/n!)|x/\sinh x| 2^{n-1} \le (1.5431/n!) 2^{n-1} \le (2^n/n!)$ since $x/\sinh x \le 1$.

22. Lagrange form: $p(x) = -(1/2)(x+2)(x-1) - (1/3)x(x+2) = -(1/6)(5x^2 + 7x - 6)$.

Newton form:

x	$f(x)$		
-2	0	$1/2$	$-5/6$
0	1	-2	
1	-1		

$p(x) = (1/2)(x+2) - (5/6)(x+2)x = -(1/6)(5x^2 + 7x - 6)$.

27. $|f(x) - p(x)| \le \left|f^{(13)}(\xi_x)\right| / (13!) \prod_{i=0}^{12} |x - x_i|$. Now $f(x) = f^{(13)}(x) = e^{x-1}$ and $|f(\xi)| \le f(1) = e^0 = 1$ for $\xi \in [-1, 1]$. Also, $\prod_{i=0}^{12} |x - x_i| \le 2^{13}$. Therefore, $|f(x) - p(x)| \le 2^{13}/(13!) = 1.315 \times 10^{-6}$.

Section 6.2

8. LHS is the Lagrange interpolating polynomial of degree $\le n$ for f at nodes $x_0, x_1, \ldots, x_n$. RHS is the Newton interpolating polynomial of degree $\le n$ for f at $x_0, x_1, \ldots, x_n$. Hence, LHS equals RHS by uniqueness.

9. By Problem 6.2.8, the two polynomials are equal and hence the coefficients of x^n in each are equal.

19. At x_0, we have $[(x_n - x_0)u(x_0) + (x_0 - x_0)v(x_0)]/(x_n - x_0) = u(x_0) = f(x_0)$. For $1 \le i \le n-1$, we have $[(x_n - x_i)u(x_i) + (x_i - x_0)v(x_i)]/(x_n - x_0)$ $= [(x_n - x_i)f(x_i) + (x_i - x_0)f(x_i)]/(x_n - x_0) = (x_n - x_0)f(x_i)/(x_n - x_0) = f(x_i)$. At x_n, we have $[(x_n - x_n)u(x_n) + (x_n - x_0)v(x_n)]/(x_n - x_0) = [(x_n - x_0)/(x_n - x_0)]v(x_n)$ $= v(x_n) = f(x_n)$.

24.

x	$f(x)$			
4	63	26	6	1
2	11	2	5	
0	7	7		
3	28			

Thus, $p(x) = 63 + 26(x-4) + 6(x-4)(x-2) + x(x-4)(x-2)$.

Section 6.3

1.

x	$p(x)$				
0	2	−9	3	7	5
0	2	−6	10	17	
1	−4	4	44		
1	−4	48			
2	44				

So $p(x) = 2 - 9x + 3x^2 + 7x^2(x-1) + 5x^2(x-1)^2$.

3. By Theorem 1, there exists a unique polynomial p of degree $\le m$ $(m = 2n+1)$ such that $p(x_i) = y_i$ and $p'(x_i) = 0$ for $0 \le i \le n$. By Equation (9) $p(x) = \sum_{i=0}^{n} y_i[1 - 2(x - x_i)\ell_i'(x_i)]\ell_i^2(x)$ where $\ell_i(x) = \prod_{j=0,\, j\ne i}^{n}(x - x_j)/(x_i - x_j)$ for $0 \le i \le n$.

4. Let us write $p(x) = a + b(x-x_0) + c(x-x_0)^2 + d(x-x_0)^3$. Then $p''(x) = 2c + 6d(x - x_0)$. The four conditions can be written as: $c_{00} = p(x_0) = a$, $c_{02} = p''(x_0) = 2c$, $c_{10} = p(x_1) = a + bh + ch^2 + dh^3$, and $c_{12} = p''(x_1) = 2c + 6dh$ when $h = x_1 - x_0$. So a and c are obtained without restrictions: $a = c_{00}$, $c = c_{02}/2$. d and b can be obtained from last two equations: $\begin{bmatrix} h & h^3 \\ 0 & 6h \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} =$ known vector. $\det \begin{bmatrix} h & h^3 \\ 0 & 6h \end{bmatrix} = 6h^2 \ne 0$ iff $h \ne 0$ $\Rightarrow$ condition: $x_0 \ne x_1$.

12. By definition of the partial derivative, we have,

$$\frac{\partial}{\partial x_i} f[x_0, x_1, \ldots, x_n] = \lim_{h\to 0} \frac{f[x_0, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n] - f[x_0, \ldots, x_n]}{h}$$

since divided differences are symmetric in the inputs, we have,

$$= \lim_{h\to 0} \frac{f[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_i + h] - f[x_i, x_0, \ldots, x_{i-1}, x_{i+1}, x_n]}{(x_i + h) - (x_i)}$$

which is by definition the following divided difference,

$$= \lim_{h\to 0} f[x_i, x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_i + h]$$

and by symmetry, we have,

$$= \lim_{h\to 0} f[x_0, \ldots, x_{i-1}, x_i, x_i + h, x_{i+1}, \ldots, x_n]$$
$$= f[x_0, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_n]$$

16. $p'(t) = -(b-a)[-6(b-t)/(b-a)^2 + 6(b-t)^2/(b-a)^3] = -6(b-t)(a-t)/(b-a)^2$, so $p'((a+b)/2) = -6[(b-a)/2][(a-b)/2]/(b-a)^2 = 3/2$. Therefore $|p'(t)| \leq p'((a+b)/2) = 3/2$. And $p(a) = b-(b-a)[3-2] = a$; $p(b) = b-(b-a)\cdot 0 = b$; $p'(a) = 0$ and $p'(b) = 0$.

Section 6.4

5. $f(1^-) = 1 = f(1^+)$, so f is continuous at $x = 1$. Also $f(2^-) = 3/2 = f(2^+)$, so f is continuous at $x = 2$. $f'(x) = \begin{cases} 1 & x \in (-\infty, 1] \\ 2 - x & x \in [1, 2] \\ 0 & x \in [2, \infty) \end{cases}$ Thus, $f'(1^-) = 1 = f'(1^+)$ and $f'(2^-) = 0 = f'(2^+)$. Therefore, $f'(x)$ is continuous at $x = 1$, $x = 2$. Hence, f is a quadratic spline function.

7. Enforce the continuity of f at knots: 1,3. At $x = 1$, $a(-1)^2 + 0 = c(-1)^2 \Rightarrow a = c$. At $x = 3$, $c(1)^2 = d(1)^2 + 0 \Rightarrow c = d$. Continuity of f' at knots: At $x = 1$,

$2a(-1) + 0 = 2c(-1) \Rightarrow a = c$. At $x = 3$, $2c = 2d + 0 \Rightarrow c = d$. Continuity of f'' at knots: At $x = 1$, $2a + 0 = 2c \Rightarrow a = c$. At $x = 3$, $2c = 2d + 0 \Rightarrow c = d$. Thus, in order that f be a cubic spline: $a = c = d$ and b, e any arbitrary values. Next, determine a, b, c, d, e so that f interpolates the table. At $x = 0$, $a(-2)^2 + b(-1)^3 = 26 \Rightarrow 4a - b = 26$. At $x = 1$, $a(-1)^2 + b \cdot 0 = 7 \Rightarrow a = 7$. So $b = 2$ and $c = d = 7$. At $x = 4$, $d(2)^2 + e(1)^3 = 25 \Rightarrow 28 + e = 25 \Rightarrow e = -3$. Then: $a = c = d = 7$, $b = 2$, $e = -3$.

9. Put $q_i(x) = \frac{1}{2}(z_{i+1}/h_i)(x - t_i)^2 - \frac{1}{2}(z_i/h_i)(t_{i+1} - x)^2 + y_i + \frac{1}{2}z_i h_i$. Then
$q_i(t_i) = -\frac{1}{2}(z_i/h_i)(t_{i+1} - t_i)^2 + y_i + \frac{1}{2}(z_i h_i) = \frac{1}{2}[-(z_i h_i^2)/h_i] + \frac{1}{2}(z_i h_i) + y_i = y_i$ where $h_i = t_{i+1} - t_i$. $q_i'(x) = (z_{i+1}/h_i)(x - t_i) + (z_i/h_i)(t_{i+1} - x)$,
$q_i'(t_i) = (z_i/h_i)(t_{i+1} - t_i) = z_i$, $q_i'(t_{i+1}) = (z_{i+1}/h_i)(t_{i+1} - t_i) = z_{i+1}$.
$q_{i-1}(x) = \frac{1}{2}(z_i/h_{i-1})(x - t_{i-1})^2 - \frac{1}{2}(z_{i-1}/h_{i-1})(t_i - x)^2 + y_{i-1} + \frac{1}{2}(z_{i-1}h_{i-1})$
$q_{i-1}(t_i) = \frac{1}{2}(z_i/h_{i-1})(t_i - t_{i-1})^2 + y_{i-1} + \frac{1}{2}(z_{i-1}h_{i-1}) = \frac{1}{2}(z_i + z_{i-1})h_{i-1} + y_{i-1}$
Continuity Condition $\frac{1}{2}(z_i + z_{i-1})h_{i-1} + y_{i-1} = y_i \Rightarrow z_i + z_{i-1} = (2/h_{i-1})(y_i - y_{i-1})$.
$q_i(x) = \frac{1}{2}(z_{i+1}/h_i)(x - t_i)^2 - \frac{1}{2}(z_i/h_i)(t_{i+1} - x)^2 + y_i + \frac{1}{2}(z_i h_i)$

$= \frac{1}{2}(z_{i+1}/h_i)(x - t_i)^2 - \frac{1}{2}(z_i/h_i)(x - t_i - h_i)^2 + y_i + \frac{1}{2}(z_i h_i)$
$= \frac{1}{2}[(z_{i+1} - z_i)/h_i](x - t_i)^2 + z_i(x - t_i) - \frac{1}{2}(z_i h_i) + y_i + \frac{1}{2}(z_i h_i)$
$= \frac{1}{2}[(z_{i+1} - z_i)/h_i](x - t_i)^2 + z_i(x - t_i) + y_i$. Here $i = 1, 2, \ldots, n - 1$. Q: piecewise quadratic Q, Q' continuous $Q'(t_i) = z_i$ well-defined $q_1(t_2) = q_2(t_2)$ etc.
$q_{n-2}(t_{n-1}) = q_{n-1}(t_{n-1})$, i.e., $q_{i-1}(t_i) = q_i(t_i)$ for $i = 2 \ldots n - 1$
$z_{i-1} + z_i = (2/h_{i-1})(y_i - y_{i-1})$ $(2 \le i \le n - 1)$. Let $z_1 = 0$ and define inductively
$z_i = (2/h_{i-1})(y_i - y_{i-1}) - z_{i-1}$, $i = 2, 3, \ldots, n$. z_i is arbitrary,
$z_i = (2/h_{i-1})(y_i - y_{i-1}) - z_{i-1}$ $i = 2, \ldots, n$. So $z_i = \alpha_i - z_{i-1}$. $z_2 = \alpha_2 - z_1$,
$z_3 = \alpha_3 - z_2 = \alpha_3 - (\alpha_2 - z_1) = \alpha_3 - \alpha_2 + z_1$, $z_4 = \alpha_4 - z_3 = \alpha_4 - \alpha_3 + \alpha_2 - z_1$, Etc..
$z_1 = \alpha_1 - \alpha_{i-1} + \alpha_{i-2} \cdots + (-1)^i(\alpha_2 - z_1)$. So $z_i = \gamma_i - (-1)^i z_1$, $\gamma_2 = \alpha_2$, $\gamma_3 = \alpha_3 - \gamma_2$,
$\gamma_4 = \alpha_3 - \gamma_3$, etc. $\Phi = \sum_{i=2}^{n} z_i 62 = z_2^2 + z_3^2 + z_4^2 + \cdots + z_n^2$
$= (\gamma_2 - z_i)^2 + (\gamma_3 + z_1)^2 + (\gamma_4 - z_1)^2 + \cdots + (\gamma_n - (-1)^n z_1)^2$
$d\Phi/dz_1 = -2(\gamma_2 - z_1) + 2(\gamma_3 + z_1) - 2(\gamma_4 - z_1) - \cdots - 2(-1)^n(\gamma_n - (-1)^n z_1) = 0$
$-\underbrace{\gamma_2 + z_1}_{1} + \underbrace{\gamma_3 + z_1}_{2} - \underbrace{\gamma_4 + z_1}_{3} \cdots - (-1)^n \underbrace{\gamma_n + z_1}_{n-1} = 0$
$(n - 1)z_1 - (\gamma_2 - \gamma_3 + \gamma_4 - \gamma_5 + \cdots + (-1)^n\gamma_n) = 0$

$z_1 = (\gamma_2 - \gamma_3 + \gamma_4 - \gamma_4 + \cdots + (-1)^n \gamma_n)/(n-1)$. Now $\gamma_2 - \gamma_3 = \alpha_2 - (\alpha_3 - \alpha_2) = 2\alpha_2 - \alpha_3$ and $\gamma_4 - \gamma_5 = \gamma_4 - (\alpha_4 - \gamma_4) = 2\gamma_4 - \alpha_4$. $\gamma_2 = \alpha_2$, $\gamma_3 = \alpha_3 - \alpha_2$, $\gamma_4 = \alpha_4 - \alpha_3 + \alpha_2$, $\gamma_5 = \alpha_5 - \alpha_4 + \alpha_3 - \alpha_2$, etc. $\gamma_2 = \alpha_2$, $\gamma_3 = \alpha_2 - \alpha_3$, $\gamma_4 = \alpha_2 - \alpha_3 + \alpha_4$, $-\gamma_5 = \alpha_2 - \alpha_3 + \alpha_4 - \alpha_5$, etc. So $[\gamma_2 - \gamma_3 + \gamma_4 - \gamma_5 + \cdots + (-1)^n \gamma_n]/(n-1)$ $= [(n-1)\alpha_2 - (n-2)\alpha_3 + (n-3)\alpha_4 - \cdots]/(n-1)$. Algorithm: For $i = 2 \cdots n$ define $\alpha_i = (2/h_{i-1})(y_i - y_{i-1})$. For $i = 3 \cdots n$ do $\alpha_i - \alpha_{i-1} \to \alpha_i$. For $i = 2 \cdots n$ do $\alpha_2 + (-1)^i \alpha_{i+1} \to \alpha_2$. $z_1 = \alpha_2/(n-1)$.

13. Let $f_1(x) = 1 + x - x^3$, $x \in [0,1]$; $f_2(x) = 1 - 2(x-1) - 3(x-1)^2 + 4(x-1)^3$, $x \in [1,2]$; $f_3(x) = 4(x-2) + 9(x-2)^2 - 3(x-2)^3$, $x \in [2,3]$. Since $f_1(0) = 1$, $f_1(1) = 1 = f_2(1)$, $f_2(2) = 0 = f_3(2)$, and $f_3(3) = 10$, $f(x)$ interpolates the table and $f(x)$ is continuous at $x = 1$ and $x = 2$. Also $f_1'(1) = -2 = f_2'(1)$, $f_2'(2) = 4 = f_3'(2)$, $f_1''(1) = -6 = f_2''(1)$, $f_2''(2) = 18 = f_3''(2)$, $f_1''(0) = 0$, and $f_3''(3) = 0$. Hence, $f(x)$ is a natural cubic spline which interpolates the table values.

24. The integral $\int_0^1 S(x)\,dx$ becomes an approximation of the area under the curve $f(t)$ on the interval $[0,1]$ where $R_i = \frac{1}{2}(t_i - t_{i-1})[f(t_i) + f(t_{i-1})]$. This is the composite trapezoid rule with non-uniform segments. Thus, $\int_0^1 S(x)\,dx = \sum_{i=1}^n R_i = \frac{1}{2}\sum_{i=1}^n (t_i - t_{i-1})[f(t_i) - f(t_{i-1})]$.

30. In the matrix system for the z_i's in the text, the first equation is replaced by $h_0 z_0 + u_1 z_1 + h_1 z_2 = v_1$ and the last equation is replaced by $h_{n-2} z_{n-2} + u_{n-1} z_{n-1} + h_{n-1} z_n = v_{n-1}$. Using Eq. (8) with $i = 0$, we have $-2h_0 z_0 - h_0 z_1 = 6S'(t_0) - b_0$. Using Eq. (9) with $i = n$, we have $h_{n-1} z_{n-1} + 2h_{n-1} z_n = 6S'(t_n) - b_{n-1}$. Use these latter two equations to expand the linear system and solve for $[z_0, z_1, z_2, \ldots, z_{n-2}, z_{n-1}, z_n]^T$.

Section 6.5

1. By Lemma 1, $B_i^2(x) = 0$ on the complement of (t_i, t_{i+3}). Use Equation (1) with $k = 2$ to compute $B_i^2(t_{i+1}) = (t_{i+1} - t_i)/(t_{i+2} - t_i)$ and $B_i^2(t_{i+2}) = (t_{i+3} - t_{i+2})/(t_{i+3} - t_{i+1})$. All other $B_i^2(t_j)$ are zero.

3. $S(t_m) = \sum_i c_i B_i^2(t_m) = c_{m-2} B_{m-2}^2(t_m) + c_{m-1} B_{m-1}^2(t_m)$
$= c_{m-2} h_m/(h_m + h_{m-1}) + c_{m-1} h_{m-1}/(h_m + h_{m-1})$
$= (c_{m-2} h_m + c_{m-1} h_{m-1})/(h_m + h_{m-1}) = y_m$.

7. Just let x tend to ∞ in Lemma 7, and use Lemma 4.

Section 6.6

8. $S = \sum_{j=-\infty}^{\infty} c_j B_j^3$. Theorem: If $f(x) = \sum_{i=-\infty}^{\infty} c_i B_i^k(x)$ then $f'(x) = \sum_{i=-\infty}^{\infty} d_i B_i^{k-1}(x)$ where $d_i = k(c_i - c_{i-1})/(t_{i+k} - t_i)$. Now, $S' = \sum_{j=-\infty}^{\infty} d_j B_j^2$ where $d_j = 3(c_j - c_{j-1})/(t_{j+3} - t_j)$ and $S'' = \sum_{j=-\infty}^{\infty} e_j B_j^1$ where $e_j = 2(d_j - d_{j-1})/(t_{j+2} - t_j)$,
$e_j = [2/(t_{j+2} - t_j)][d_j - d_{j-1}]$
$= [6/(t_{j+2} - t_j)][(c_j - c_{j-1})/(t_{j+3} - t_j) - (c_j - c_{j-2})/(t_{j+2} - t_{j-1})]$.

9. $S''(t_i) = \sum_{j=-\infty}^{\infty} e_j B_j^1(t_i) = e_{i-1}B_{i-1}^1(t_i) = e_{i-1}$ (using Problem 6.6.8 and definition of B_{i-1}^1).

Section 6.8

6. One Gram matrix has elements $a_{ij} = \int_0^1 x^i x^j \, dx = \int_0^1 x^{i+j} \, dx = (1+i+j)^{-1}$.

7. If we start with the basis $\{v_1, \dots, v_n\}$ and apply the Gram-Schmidt process, each new vector u_j is a linear combination of $v_1, \dots, v_j$. Hence $u_j = \sum_{i=1}^{j} a_{ij} v_i$. The coefficients a_{ij} are zero for $i > j$. Hence the matrix is upper triangular.

21. $p_0 = 1$, $p_1 = x - 1/2$, $p_2 = x^2 - x + 1/6$, $p_3 = x^3 - (23/2)x^2 + (318/30)x - (103/60)$.

22. $p_0 = 1$, $p_1 = x - a_1$, $p_2 = x^2 - (a_1 + a_2)x + (a_1 a_2 - b_2)$,
$p_3 = x^3 - (a_1 + a_2 + a_3)x^2 + (a_1a_2 + a_1a_3 + a_2a_3 - b_2 - b_3)x - (a_1b_3 + a_3b_2 - a_1a_2a_3)$.

Section 6.12

5. Notice that $\frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi ijk/n} = \frac{1}{n}\sum_{j=0}^{n-1} \left(e^{2\pi ik/n}\right)^j$. This geometric series has sum 1 if n divides k, for in that case each term is 1. Otherwise, the standard formula $(r^n - 1)/(r-1)$ for the sum yields 0 since $\left(e^{2\pi ik/n}\right)^n = 1$. The real part on the left is $\sum \frac{1}{n}\cos(2\pi ik/n)$. Since the sum is always real its imaginary part is 0.

Section 7.1

7. (a) By Taylor series, $f(x+h) = f + hf' + (h^2/2)f'' + (h^3/6)f''' + (h^4/24)f^{iv} + \cdots$,
$f(x+2h) = f + 2hf' + (4h^2/2)f'' + (8h^3/6)f''' + (16h^4/24)f^{iv} + \cdots$,
$f(x+3h) = f + 3hf' + (9h^2/2)f'' + (27h^3/6)f''' + (81h^4/24)f^{iv} + \cdots$. Now
$f(x+h) - f(x+2h) = -hf' - (3h^2/2)f'' - (7h^3/6)f''' - (15h^4/24)f^{iv} - \cdots$. So
$f(x+3h) + 3[f(x+h) - f(x+2h)] = f + h^3 f''' + (36h^4/24)f^{iv} + \cdots$ and error term
$-(3h/2)f^{iv}(\xi) = O(h)$.

   (b) Now $f(x+2h) - f(x-2h) = 4hf' + (16h^3/6)f''' + (64h^5/120)f^v + \cdots$ and
$f(x+h) - f(x-h) = 2hf' + (2h^3/6)f''' + (2h^5/120)f^v + \cdots$. So
$[f(x+2h) - f(x-2h)] - 2[f(x+h) - f(x-h)] = 2h^3 f''' + (h^5/2)f^v + \cdots$ and error term $= -(h^2/4)f^v(\xi) = O(h^2)$. Second approximation more accurate.

10. Given $L = x_n + a_1 n^{-1} + a_2 n^{-2} + \cdots$ and replacing n by n^2 we have
$L = x_{n^2} + a_1 n^{-2} + a_2 n^{-4} + \cdots$. Taking n times the latter equation from the former one, we have $(n-1)L = n x_{n^2} - x_n + (n^{-3} - n^{-2})a_2 + \cdots$. So
$L = (n x_{n^2} - x_n)/(n-1) + [n^{-3}(1-n)/(n-1)]a_2 + \cdots$ and
$L = [n/(n-1)]x_{n^2} - [1/(n-1)]x_n + O(n^{-3})$.

12. Given $L = \phi(h) + a_1 h + a_3 h^3 + \cdots$. Replacing h by $h/2$, we have
$L = \phi(h/2) + a_1(h/2) + a_3(h/2)^3 + \cdots$. Multiplying the latter equation by 2 and subtracting the former equation, we obtain $L = 2\phi(h/2) - \phi(h) - (3/4)a_3 h^3 + \cdots$.

13. Given $L - f(h) = c_6h^6 + c_9h^9 + \cdots$. Replacing h by $h/2$, we have $L - f(h/2) = c_6(h/2)^6 + c_9(h/2)^9 + \cdots$. Multiplying the latter by 2^6 and subtracting the former, we obtain $63L - 64f(h/2) + f(h) = (2^{-3} - 1)c_9h^9 + \cdots$. Thus, $L = [64f(h/2) - f(h)]/63 - (1/48)c_9h^9 + \cdots$.

Section 7.2

4. $f(x) = 1 :$ LHS $= \int_0^1 dx = 1$, RHS $= (1/90)[7 + 32 + 12 + 32 + 7] = 1$.
$f(x) = x :$ LHS $= \int_0^1 x dx = 1/2$, RHS $= (1/90)[32(1/4) + 12(1/2) + 32(3/4) + 7] = 1/2$.
$f(x) = x^2 :$ LHS $= \int_0^1 x^2 dx = 1/3$,
RHS $= (1/90)[32(1/4)^2 + 12(1/2)^2 + 32(3/4)^2 + 7] = 1/3$.
$f(x) = x^3 :$ LHS $= \int_0^1 x^3 dx = 1/4$,
RHS $= (1/90)[32(1/4)^3 + 12(1/2)^3 + 32(3/4)^3 + 7] = 1/4$.
$f(x) = x^4 :$ LHS $= \int_0^1 x^4 dx = 1/5$,
RHS $= (1/90)[32(1/4)^4 + 12(1/2)^4 + 32(3/4)^3 + 7] = 1/5$. Since it is exact for $1, x, x^2, x^3$, and x^4, it follows that it is exact for any linear combination of them, namely all polynomials of degree ≤ 4.

8. $f(x) = e^x :$ LHS $= \int_0^1 e^x dx = e - 1$, RHS $= A_0 + A_1 e$.
$f(x) = \cos(x\pi/2) :$ LHS $= \int_0^1 \cos(\pi x/2)dx = \sin(x\pi/2)/(\pi/2)|_0^1 = 2/\pi$, RHS $= A_0$.
Solving $A_0 = 2/\pi$, $A_1 = e^{-1}(e - 1 - A_0) = 1 - 1/e - 2/(\pi e)$.

10. (a) $\ell_0(x) = -3(x - 2/3) :$ LHS $= 1/2$, RHS $= A$.
$\ell_1(x) = 3(x - 1/3) :$ LHS $= 1/2$, RHS $= B$. So $A = B = 1/2$.

(b) Let $x = \lambda(t) = (b - a)t + a$ so $dx = (b - a)dt$. So
$\int_a^b f(x)dx = (b - a)\int_{\lambda(0)}^{\lambda(1)} f(\lambda(t))dt = (1/2)(b - a)[f(\lambda(1/3)) + f(\lambda(2/3))]$
$= [(b - a)/2][f((2a + b)/3) + f((a + 2b)/3)]$.

19. $\int_a^b f(x)dx = \sum_{i=1,\text{ odd}}^{n-1} \int_{x_{i-1}}^{x_{i+1}} f(x)dx \approx \sum_{i=1,\text{ odd}}^{n-1}(x_{i+1} - x_{i-1})f(x_i)$
$= (2h)\sum_{k=1}^{n/2} f(x_{2k-1})$ since n even.

Section 7.3

7. (a) By Theorem 5, Section 6.8, we have $q_0 = 1, q_1 = x - a_1, q_n = (x - a_n)q_{n-1} - b_nq_{n-2}$ where $a_n = \lambda xq_{n-1}, q_{n-1}\rangle/\lambda q_{n-1}, q_{n-1}\rangle$ and $b_n = \lambda xq_{n-1}, q_{n-2}\rangle/\lambda q_{n-2}, q_{n-2}\rangle$. Here $\lambda f, g\rangle = \int_0^1 xfgdx$. For this problem, $a_1 = 2/3, a_2 = 8/15$ and $b_2 = 1/18$ so that $q_1 = x - 2/3$ and $q_2 = x^2 - (6/5)x + (3/10)$. The roots of q_2 are $(6 \pm \sqrt{6})/10$. Using method of undetermined coefficients, we obtain $A_0 + A_1 = 1/2$ and $[(6 - \sqrt{6})/10]A_0 + [(6 + \sqrt{6})/10]A_1 = 1/3$. Solving, we obtain $A_0 = (3 - 2/\sqrt{6})/12$ and $A_1 = (3 + 2/\sqrt{6})/12$. Hence,

$\int_0^1 xf(x)dx \approx [(3-2/\sqrt{6})/12]f((6-\sqrt{6})/10) + [(3+2/\sqrt{6})/12]f((6+\sqrt{6})/10)$.
Note: $x_0 = (6-\sqrt{6})/10 \approx 0.35505\,10259, x_1 = (6+\sqrt{6})/10 \approx 0.84494\,89743, A_0 = (3-2/\sqrt{6})/12 \approx 0.18195\,86183, A_1 = (3+2/\sqrt{6})/12 \approx 0.31804\,13817$. These results agree with those in Table 25.8 in Abramowitz and Stegun [1965, page 921].

(b) Continuing, $a_3 = 18/35$ and $b_3 = 3/50$ so that $q_3 = x^3 - (12/7)x^2 + (6/7)x - (4/35)$. Using a symbolic manipulation package to solve this cubic, we find $x_1 = (4/7) + a$, $x_2 = [(4/7) - (a/2)] + i[b\sqrt{3}/2)$, $x_3 = [(4/7) - (a/2)] - i(b\sqrt{3}/2)$ where

$a = (2/49c) + c$, $b = -(2/49c) + c$, $c = \sqrt[3]{(-2/1715) + i(2/245)}$. The numerical values of these roots of the cubic polynomial are
$x_0 = 0.21234\,05382, x_1 = 0.59053\,31356, x_2 = 0.91141\,20405$. Using the method of undetermined coefficients, we need to solve the system $1/2 = A_0 + A_1 + A_2$, $1/3 = A_0x_0 + A_1x_1 + A_2x_2$, $1/4 = A_0x_0^2 + A_1x_1^2 + A_2x_2^2$, $1/5 = A_0x_0^3 + A_1x_1^3 + A_2x_2^3$, $1/6 = A_0x_0^4 + A_1x_1^4 + A_2x_2^4$, $1/7 = A_0x_0^5 + A_1x_1^5 + A_2x_2^5$. The solution is $A_0 = 0.06982\,69799, A_1 = 0.22924\,11064, A_2 = 0.20093\,19137$. One can show numerically that this formula is exact for polynomials of degree ≤ 5.

Note: Here we use the values given in Abramowitz-Stegan [1965, Table 25.8, page 921].

9. $\int_{-1}^1 f(x)dx \approx c[f(x_0) + f(x_1) + f(x_2)]$. $f(x) = 1 : LHS = 2,\ RHS = c(3) \Rightarrow c = 2/3$.
$f(x) = x : LHS = 0,\ RHS = (2/3)[x_0 + x_1 + x_2]$.
$f(x) = x^2 : LHS = 2/3,\ RHS = (2/3)[x_0^2 + x_1^2 + x_2^2]$. Let $x_1 = 0$, $x_0 = -x_2 = -1/\sqrt{2}$.

11. $f(x) = 1 : LHS = 2,\ RHS = 2$. $f(x) = x : LHS = 2,\ RHS = 2$.
$f(x) = x^2 : LHS = 8/3,\ RHS = 2\alpha^2 - 4\alpha + 4$.
$f(x) = x^3 : LHS = 4, RHS = 6\alpha^2 - 12\alpha + 8$. Solving, we have $3\alpha^2 - 6\alpha + 2 = 0$ or $\alpha = 1 \pm 1/\sqrt{3}$.

Section 7.4

6. (a) $a = 1, b = 3, f(x) = 1/x : R(0,0) = (1/2)(b-a)[f(a) + f(b)] = 4/3$,
$R(1,0) = (1/2)R(0,0) + (1/2)(b-a)[f(a + (b-a)/2)] = 7/6$,
$R(2,0) = (1/2)R(1,0) + 14(b-a)[f(a + (b-a)/4) + f(a + 3(b-a)/4)] = 67/60$,
$R(1,1) = (4/3)R(1,0) - (1/3)R(0,0) = 10/9$,
$R(2,1) = (4/3)R(2,0) - (1/3)R(1,0) = 11/10$,
$R(2,2) = (16/15)R(2,1) - (1/15)R(1,1) = 742/678$. From code:

$$R(0,0) = 1.33333$$
$$R(1,0) = 1.16667 \quad R(1,1) = 1.11111$$
$$R(2,0) = 1.11667 \quad R(2,1) = 1.10000 \quad R(2,2) = 1.09926$$

exact value $= \ln 3 \approx 1.09861$

(b) $a = 0, b = \pi/2, f(x) = (x/\pi)^2$: $R(0,0) = \pi/16$, $R(1,0) = 3\pi/64$, $R(2,0) = 11\pi/256$, $R(1,1) = \pi/24$, $R(2,1) = \pi/24$, $R(2,2) = \pi/24$. From code:

$$R(0,0) = 0.196350$$
$$R(1,0) = 0.147262 \quad R(1,1) = 0.130900$$
$$R(2,1) = 0.134990 \quad R(2,1) = 0.130900 \quad R(2,2) = 0.130900$$

exact value $= \pi/24 \approx 0.13090$.

9. $I = T(f,h) + c_1 h + c_2 h^2 = \cdots$ and $I = T(f,h/2) + c_1(h/2) + c_2(h/2)^2 + \cdots$. Combining, we have $I = 2T(f,h/2) - T(f,h) + (1/2-1)c_2h^2 + \cdots$. Let $R(1,1) = 2R(1,0) - R(0,0)$. In general, $R(n,1) = 2R(n,0) - R(n-1,0)$. Now, $I = R(1,1) + b_2h^2 + b_3h^3 + \cdots$ and $I + R(2,1) + b_2(h/2)^2 + b_3(h/2)^3 + \cdots$. Combining, we have $I = (4/3)R(2,1) - (1/3)R(1,1) + (1/2-1)b_3h^3 + \cdots$. Let $R(2,2) = (4/3)R(2,1) - (1/3)R(1,1)$. In general, $R(n,2) = (4/3)R(n,1) - (1/3)R(n-1,1)$. Now $I = R(2,2) + c_3h^3 + c_4h^4 + \cdots$ and $I = R(3,2) + c_3(h/2)^3 + c_4(h/2)^4 + \cdots$. Combining, we have $I = (8/7)R(3,2) - (1/7)R(2,2) + (1/2-1)c_4h^4 + \cdots$. Let $R(3,3) = (8/7)R(3,2) - (1/7)R(2,2)$. New Eq. (5) is $R(n,m) = R(n,m-1) + (1/2^m - 1)[R(n,m-1) - R(n-1,m-1)]$.

Section 8.2

3. $x(0.1) = 1.98$

7. $x' = \cos(tx)$, $x'' = -\sin(tx)(x + tx')$, $x''' = -\cos(tx)(x+tx')^2 - \sin(tx)(2x' + tx'')$, $x^{(4)} = \sin(tx)(x+tx')^3 - 3\cos(tx)(x+tx')(2x'+tx'') - \sin(tx)(3x'' + tx''')$.

Section 8.3

1. $x(t+h) = x + (1/4)hf + (3/4)hf(t + (2/3)h, x + (2/3)hf)$.

5. From the Taylor expansion and using the solution to Problem 8.2.8: $x(t+h) = x + hf + (h^2/2)(f_t + ff_x) + (h^3/3!)$ $[f_{tt} + f_{tx}f + (f_t + f_xf)f_x + f(f_{xt} + f_{xx}f)] + O(h^4)$, where all the functions are evaluated at $(t, x(t))$. We want $x(t+h) = x + w_1F_1 + w_2F_2 + w_3F_3$ where $F_1 = hf(t,x)$, $F_2 = hf(t + \alpha_1 h, x + \beta_1 hF_1)$, $F_3 = hf(t + \alpha_2 h, x + \beta_2 hF_2)$. We obtain the equations $w_1 + w_2 + w_3 = 1$, $w_2\alpha_1 + w_3\alpha_2 = 1/2$, $w_2\beta_1 + w_3\beta_2 = 1/2$, $(w_1\alpha_1^2/2) + (w_3\alpha_2^2/2) = 1/6$, $(w_2\beta_1^2/2) + \beta_2 w_3 = 1/6$, $w_2\beta_2\alpha_1 = 1/6$, $w_2\alpha_1\beta_1 = 1/6$, $w_3\beta_1\beta_2 = 1/6$, $w_3\beta_1\alpha_1 = 1/6$. Solving, this yields $w_1 = 2/9$, $w_2 = 3/9$, $w_3 = 4/9$, $\alpha_1 = 1/2$, $\alpha_2 = 3/4$, $\beta_1 = 1/2$, $\beta_2 = 3/4$. For the equation $x' = x + t$, we have $x'' = x''' = x + t + 1$ and the Runge-Kutta method gives $x(t+h) = x + h(x+t) + (h^2/2)(1 + x + t) + (h^3/6)(1 + x + t)$. The Taylor series method yields the same result.

6. Using $f(t,x) = \lambda x$ in Eq. (8), we have $F_1 = h\lambda x$, $F_2 = h\lambda x + (h^2/2)\lambda^2 x$, $F_3 = h\lambda x + (h^2/2)\lambda^2 x + (h^3/4)\lambda^3 x$, and $F_4 = h\lambda x + h^2\lambda^2 x + (h^3/2)\lambda^3 x + (h^4/4)\lambda^4 x$. Substituting, we obtain $x(t+h) = x + h\lambda x + (h^2/2)\lambda^2 x + (h^3/3)\lambda^3 x + (h^4/24)\lambda^4 x$.

Section 8.4

1. From the k-order formula given in the text, Adams-Moulton 1 step would be $a_1 x_n - a_0 x_{n-1} = h[b_1 f_n + b_0 f_{n-1}]$. It is an implicit method so $a_1 = 1$ and $a_0 = 1$. Therefore, $x_n = x_{n-1} + h[b_1 f_n + b_0 f_{n-1}]$. Want $\int_{t_{n-1}}^{t_n} f(t, x(t))\, dt \approx h[b_1 f_n + b_0 f_{n-1}]$. It should be correct when $f(t, x(t))$ is a first degree polynomial. This forces $b_1 = b_0 = 1/2$ and gives the trapezoid rule. In detail, WLOG consider $\int_{-1}^{0} p(t)\, dt = Ap(-1) + Bp(0)$. Using $p_0(t) = 1$, we obtain $1 = A + B$ and $p_1(t) = t$ gives $1/2 = A$. Hence, $A = B = 1/2$. So $x_{n+1} = x_n + (h/2)[f_{n+1} + f_n]$. Obviously, this is equivalent to the trapezoid rule for the integral.

2. Straightforward calculations.

3. Take the basis $\{1, t, t(t+1), t(t+1)(t+2), t(t+1)(t+2)(t+3)\}$ for Π_4 and impose the formula $\int_0^1 p\, dt \approx Ap(1) + Bp(0) + Cp(-1) + Dp(-2) + Ep(-3)$ to be exact for $p \in \Pi_4$. This yields the system of equations: $A + B + C + D + E = 1$, $A - C - 2D - 3E = 1/2$; $2A + 2D + 6E = 5/6$, $6A - 6E = 9/4$, $24A = 251/30$. By back substitution, we obtain $A = 251/720$, $B = 646/720$, $C = -264/720$, $D = 106/720$, $E = -19/720$.