4-Covering Maps on Elliptic Curves with Torsion Subgroup $Z_2 \times Z_8$

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Abstract

We consider elliptic curves over $\mathbb{Q}$ with the torsion subgroup $Z_2 \times Z_8$. In particular, we discuss how to determine the rank of the curve $E : y^2 = (1-x^2)(1-k^2\,x^2)$ where $k = (t^4-6t^2+1)/(t^2+1)^2$ and $t = 9/296$.

We use a 4-covering map $\widehat{C}'_{d_2} \to \widehat{C}_{d_2} \to E$ in terms of homogeneous spaces for $d_2 \in \{-1, 6477590, 2, 7, 37\}$. We show that the Mordell-Weil group is

$$E(\mathbb{Q}) \simeq Z_2 \times Z_8 \times \mathbb{Z}^3$$

thereby proving a conjecture of Flores-Jones-Rollick-Weigandt and Rathbun.

Introduction

Given an elliptic curve E, we denote the set of $\mathbb{Q}$-rational points as $E(\mathbb{Q})$. Since $E(\mathbb{Q})$ is finitely generated, there exists a finite group $E(\mathbb{Q})_{\text{tors}}$ and a nonnegative integer r such that $E(\mathbb{Q}) \simeq E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^r$, where r is called the (Mordell-Weil) rank. Those elliptic curves with $E(\mathbb{Q})_{\text{tors}} \simeq Z_2 \times Z_8$ are birationally equivalent to

$$y^2 = (1-x^2)(1-k^2\,x^2) \quad \text{where} \quad k = \frac{t^4-6t^2+1}{(t^2+1)^2} \quad \text{for some } t \in \mathbb{Q}.$$

The highest known rank of curves with this torsion subgroup is $r = 3$. It is conjectured in (3) that this bound is obtained when

$$t = \frac{9}{296}.$$

One knows that $r \geq 2$ for this particular t. We focus on determining the exact rank of this particular elliptic curve.

Acknowledgements

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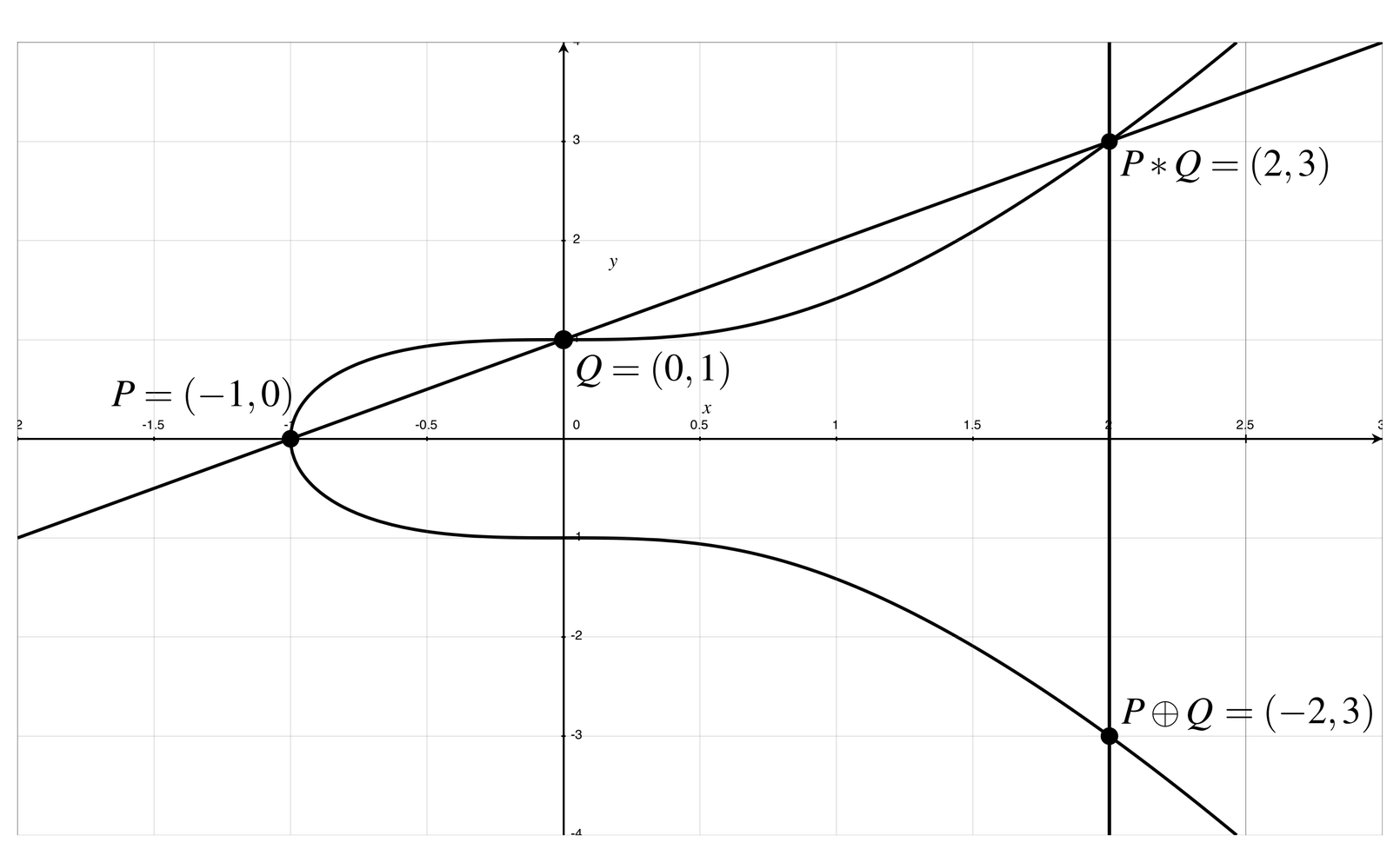
The Group Law for Elliptic Curves

An elliptic curve E is a set of points which satisifies an equation of the form $Y^2 = X^3 + AX + B$, where A and B are rational numbers such that the discriminant $-16(4A^3+27B^2) \neq 0$. In particular, an elliptic curve is a type of nonsingular cubic curve.

Define $E(\mathbb{Q})$ as the set of $\mathbb{Q}$-rational points, where we append a "point at infinity" $\mathcal{O}$.



The Group Law on an Elliptic Curve

We may use geometry to turn $E(\mathbb{Q})$ into a group. For $P, Q \in E(\mathbb{Q})$, draw a line through P and Q. (If $P = Q$, we choose the line tangent to the curve at P.) This line will intersect the curve at a third $\mathbb{Q}$-rational point $P * Q$ – which may possibly be $\mathcal{O}$. By reflecting this point about the x-axis, we obtain another $\mathbb{Q}$-rational point, denoted by $P \oplus Q$. For a graphical represention, see the figure above. Formally define

$$P \oplus Q = (P * Q) * \mathcal{O}.$$

Under this operation, the set $E(\mathbb{Q})$ forms an abelian group with the identity element $\mathcal{O}$ and inverse $[-1]P = P * \mathcal{O}$. For more information, see (8).

Poincaré's Conjecture

In 1901, Henri Poincaré conjectured (7) that this abelian group is finitely generated. This was proved by Louis Mordell in 1922.

Theorem 1 (Mordell, (6)). If E is an elliptic curve over $\mathbb{Q}$, then the abelian group $E(\mathbb{Q})$ is finitely generated. Furthermore, there exists a finite group $E(\mathbb{Q})_{\text{tors}}$ and a nonnegative integer r such that

$$E(\mathbb{Q}) \simeq E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^r.$$

Mordell-Weil Group

The set $E(\mathbb{Q})_{\text{tors}}$ is the collection of points with finite order; it is called the **torsion subgroup** of E. This implies that the quotient group $E(\mathbb{Q})/E(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}^r$ is a torsion-free group.

The nonnegative integer r is called the **Mordell-Weil rank** of E; it signifies the number of independent generators having infinite order.

Mazur's Theorem

While the rank r is mysterious, the torsion subgroup is well-understood. In 1977, Barry Mazur completely classified the structure of torsion subgroups of elliptic curves.

Theorem 2 (Mazur, (5)). If E is an elliptic curve over $\mathbb{Q}$, then $E(\mathbb{Q})_{\text{tors}}$ is isomorphic to one of the following 15 groups:

(i) Z_n, with $1 \leq n \leq 10$ or $n = 12$;

(ii) $Z_2 \times Z_{2m}$, with $1 \leq m \leq 4$.

Here we denote Z_n as the cyclic group of order n. We will focus more specifically on those curves with torsion subgroup $Z_2 \times Z_8$.

Example

$$E: \quad \begin{aligned} &Y^2 + X\,Y = X^3 - 7181359868024838434108428477109624412 0\,X \\ &\quad + 2342384302041141813702521859646228641128533374139589904 00\end{aligned}$$

has torsion subgroup $E(\mathbb{Q})_{\text{tors}} \simeq Z_2 \times Z_8$ generated by

$P_1 = (4892533141966211376 : -2446266570983105688 : 1)$;

$P_2 = (6793371071343566640 : 773920780858934092533330468 0 : 1)$.

Rank Records with Prescribed Torsion

In contrast to the torsion subgroup, less is known about the rank r of an elliptic curve. As of 2006, the largest known rank satisfies $r \geq 28$. Andrej Dujella (2) has a listing of the largest known ranks among families of elliptic curves with prescribed torsion. The table below contains this information.

Note that the highest known rank for elliptic curves E with $E(\mathbb{Q})_{\text{tors}} \simeq Z_2 \times Z_8$ is $r = 3$.

Torsion Subgroup	Highest Known Rank	Author(s)
Z_1	28	Elkies (2006)
Z_2	18	Elkies (2006)
Z_3	13	Eroshkin (2007, 2008)
Z_4	12	Elkies (2006)
Z_5	6	Dujella - Lecacheux (2001)
Z_6	8	Eroshkin (2008) Dujella - Eroshkin (2008) Elkies (2008) Dujella (2008)
Z_7	5	Dujella - Kulesz (2001) Elkies (2006)
Z_8	6	Elkies (2006)
Z_9	3	Dujella (2001) MacLeod (2004) Eroshkin (2006) Eroshkin - Dujella (2007)
Z_{10}	4	Dujella (2005) Elkies (2006)
Z_{12}	3	Dujella (2001, 2005, 2006) Rathbun (2003, 2006)
$Z_2 \times Z_2$	14	Elkies (2005)
$Z_2 \times Z_4$	8	Elkies (2005) Eroshkin (2008) Dujella - Eroshkin (2008)
$Z_2 \times Z_6$	6	Elkies (2006)
$Z_2 \times Z_8$	3	Connell (2000) Dujella (2000, 2001, 2006) Campbell - Goins (2003) Rathbun (2003, 2006) Flores - Jones - Rollick - Weigandt (2007)

m-Isogenies and Quotient Groups

Mordell's proof (6) of Poincaré's conjecture began by considering the quotient group, $E(\mathbb{Q})/2E(\mathbb{Q})$. Although $E(\mathbb{Q})$ in general is an infinite group, the quotient group is always finite. In fact,

$$E(\mathbb{Q})_{\text{tors}} \simeq Z_2 \times Z_{2m} \qquad \Longrightarrow \qquad \frac{E(\mathbb{Q})}{2\,E(\mathbb{Q})} \simeq Z_2^{r+2},$$

where r is the Mordell-Weil rank. Hence, the basic idea to compute r is to calculate $|E(\mathbb{Q})/2E(\mathbb{Q})|$.

To this end, we will compute two smaller, yet related, quotient groups. One constructs quotient groups for elliptic curves by considering a special type of group homomorphism. For instance, let E and E' be two elliptic curves over $\mathbb{Q}$. We say that a group homomorphism $\phi : E \to E'$ is an **m-isogeny** if

(i) the coordinates of ϕ involve rational functions with $\mathbb{Q}$-rational coefficients, and

(ii) there are exactly m points in the kernel $E(\mathbb{Q})[\phi]$ of the map.

We will be interested in the case where $m = 2$.

Counting Cosets to Determine the Rank

Proposition 3. Let E and E' be elliptic curves over $\mathbb{Q}$. Let $\phi : E \to E'$ and $\hat{\phi} : E' \to E$ be 2-isogenies such that $\hat{\phi} \circ \phi = [2]$ is the map which sends $P \mapsto P \oplus P$. Assume that $E'(\mathbb{Q})[\hat{\phi}] = \phi(E(\mathbb{Q})[2])$. Then,

$$\left|\frac{E(\mathbb{Q})}{2\,E(\mathbb{Q})}\right| = \left|\frac{E'(\mathbb{Q})}{\phi(E(\mathbb{Q}))}\right| \left|\frac{E(\mathbb{Q})}{\hat{\phi}(E'(\mathbb{Q}))}\right|.$$

Proof: Using the isogeny $\hat{\phi}$, we have the exact sequence:

$$\{\mathcal{O}\} \longrightarrow \frac{E'[\hat{\phi}]}{\phi(E[2])} \longrightarrow \frac{E'(\mathbb{Q})}{\phi(E(\mathbb{Q}))} \xrightarrow{\hat{\phi}} \frac{E(\mathbb{Q})}{2\,E(\mathbb{Q})} \longrightarrow \frac{E(\mathbb{Q})}{\hat{\phi}(E'(\mathbb{Q}))} \longrightarrow \{\mathcal{O}\}.$$

Using the assumption above, Lagrange's Theorem and the First Isomorphism Theorem imply the equalities:

$$\left|\frac{E(\mathbb{Q})}{2E(\mathbb{Q})}\right| = \left|\frac{E'(\mathbb{Q})}{\phi(E(\mathbb{Q}))}\right| \left|\frac{E(\mathbb{Q})}{2E(\mathbb{Q})} \Big/ \frac{E'(\mathbb{Q})}{\phi(E(\mathbb{Q}))}\right| = \left|\frac{E'(\mathbb{Q})}{\phi(E(\mathbb{Q}))}\right| \left|\frac{E(\mathbb{Q})}{\hat{\phi}(E'(\mathbb{Q}))}\right|.$$

□

If $E(\mathbb{Q})_{\text{tors}} \simeq Z_2 \times Z_{2m}$, then $E'(\mathbb{Q})[\hat{\phi}] = \phi(E(\mathbb{Q})[2])$ holds.

2-Isogenies for Quartic Models

Theorem 4 (Goins, (4)). Say that E is an elliptic curve over $\mathbb{Q}$ with torsion subgroup $Z_2 \times Z_8$.

(i) There exists a rational number t such that E is birationally equivalent to the curve

$$y^2 = (1-x^2)(1-k^2\,x^2) \quad \text{where} \quad k = \frac{t^4-6t^2+1}{(t^2+1)^2}.$$

(ii) There exists a 2-isogeny $\phi : E \to E'$ in terms of the curve

$$E' : \quad y^2 = (1+x^2)(1+\kappa^2 x^2) \quad \text{where} \quad \kappa = \left(\frac{2t}{t^2-1}\right)^2.$$

Moreover, E' has torsion subgroup $Z_2 \times Z_4$.

(iii) There exists a 2-isogeny $\phi' : E' \to E''$, and hence a 4-isogeny $\phi' \circ \phi : E \to E''$, in terms of the curve

$$E'' : \quad y^2 = (1+x^2)(1+k'^2 x^2) \quad \text{where} \quad k' = \frac{4(t^3-t)}{(t^2+1)^2}.$$

Moreover, E'' has torsion subgroup $Z_2 \times Z_2$.

Example

We focus on the case where $t = 9/296$, which was first considered in (3). Weierstrass models are as follows:

$$E: \quad \begin{aligned} &Y^2 + X\,Y = X^3 - 7181359868024838434108428477109624412 0\,X \\ &\quad + 2342384302041141813702521859646228641128533374139589904 00;\end{aligned}$$

$$E': \quad \begin{aligned} &Y^2 + X\,Y = X^3 - 7182838410586195768223026686032504412 0\,X \\ &\quad + 2341371525751308852524074565174235775172724198311084304 00;\end{aligned}$$

$$E'': \quad \begin{aligned} &Y^2 + X\,Y = X^3 - 8791041401157870064556943644005177212 0\,X \\ &\quad + 1215293335280977803803190858716511058206567605433869568 00.\end{aligned}$$

2-Covering Maps

Theorem 5. Using $\phi : E \to E'$ and $\hat{\phi} : E' \to E$, define the **connecting homomorphisms**:

$$\delta : \quad \frac{E'(\mathbb{Q})}{\phi(E(\mathbb{Q}))} \longrightarrow \frac{\mathbb{Q}^\times}{(\mathbb{Q}^\times)^2}, \qquad (X:Y:1) \longmapsto 4X + 39141876845580405121;$$

$$\widehat{\delta} : \quad \frac{E(\mathbb{Q})}{\hat{\phi}(E'(\mathbb{Q}))} \longrightarrow \frac{\mathbb{Q}^\times}{(\mathbb{Q}^\times)^2}, \qquad (X:Y:1) \longmapsto X - 4892734605697550640.$$

Both δ and $\widehat{\delta}$ are injective group homomorphisms with finite images. To be more precise, let $\Sigma(k) = \{82207, 87697, 92863\}$ and $\Sigma(\kappa) = \{2,3,5,7,37,41,61\}$ be the set of primes dividing k and κ, respectively, as in Theorem 4. Then

$$\delta\left(\frac{E'(\mathbb{Q})}{\phi(E(\mathbb{Q}))}\right) = \left\{ d_1 \equiv \pm \prod_{\ell \in \Sigma(k)} \ell^{e(\ell)} \;\middle|\; \begin{array}{l} C_{d_1} :\ d_1\,w^2 = (1-d_1\,z^2)\,(1-d_1\,k^2\,z^2) \\ \text{has a } \mathbb{Q}\text{-rational point } (z,w) \end{array} \right\};$$

$$\widehat{\delta}\left(\frac{E(\mathbb{Q})}{\hat{\phi}(E'(\mathbb{Q}))}\right) = \left\{ d_2 \equiv \pm \prod_{\ell \in \Sigma(\kappa)} \ell^{e(\ell)} \;\middle|\; \begin{array}{l} \widehat{C}_{d_2} :\ d_2\,w^2 = (1+d_2\,z^2)\,(1+d_2\,\kappa^2\,z^2) \\ \text{has a } \mathbb{Q}\text{-rational point } (z,w) \end{array} \right\}.$$

Here, $e(\ell)$ is either 0 or 1.

The various curves introduced in this theorem fit together in the diagrams:

We consider these diagrams to be "2-covers" because the diagonal maps involve quadratic polynomials.

4-Covering Maps

Theorem 6. Using $\phi' : E' \to E''$ and $\hat{\phi}' : E'' \to E'$, define the **connecting homomorphisms**:

$$\delta' : \quad \frac{E''(\mathbb{Q})}{\phi'(E'(\mathbb{Q}))} \longrightarrow \frac{\mathbb{Q}^\times}{(\mathbb{Q}^\times)^2}, \qquad (X:Y:1) \longmapsto 4X + 40011942240487566721;$$

$$\widehat{\delta}' : \quad \frac{E'(\mathbb{Q})}{\hat{\phi}'(E''(\mathbb{Q}))} \longrightarrow \frac{\mathbb{Q}^\times}{(\mathbb{Q}^\times)^2}, \qquad (X:Y:1) \longmapsto X - 5001492780060945840.$$

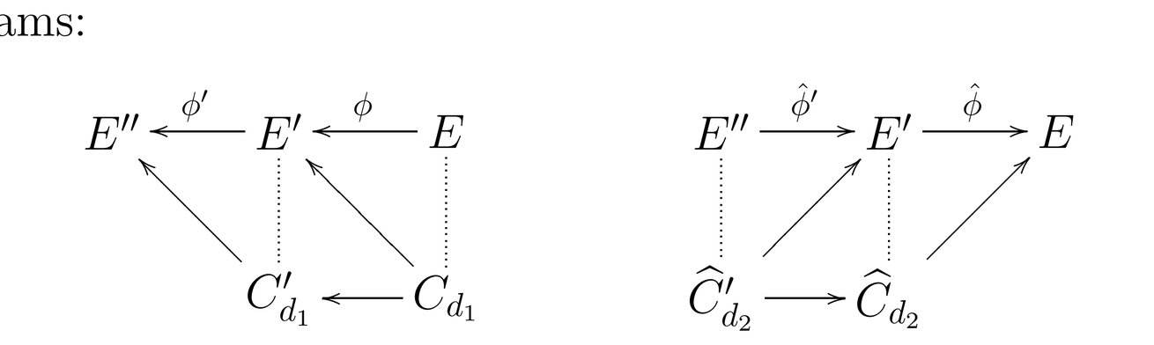
Both δ' and $\widehat{\delta}'$ are injective group homomorphisms with finite images. To be precise, let $\Sigma(\kappa') = \{82207, 92863\}$ and $\Sigma(k') = \{2,3,5,7,37,41,61,87697\}$ be the set of primes dividing $\kappa' = (1-k')/(1+k')$ and k', respectively, as in Theorem 4. Then

$$\delta'\left(\frac{E''(\mathbb{Q})}{\phi'(E'(\mathbb{Q}))}\right) = \left\{ d_1 \equiv \pm \prod_{\ell \in \Sigma(\kappa')} \ell^{e(\ell)} \;\middle|\; \begin{array}{l} C'_{d_1} :\ d_1\,w^2 = (1-d_1\,z^2)\,(1-d_1\,\kappa'^2\,z^2) \\ \text{has a } \mathbb{Q}\text{-rational point } (z,w) \end{array} \right\};$$

$$\widehat{\delta}'\left(\frac{E'(\mathbb{Q})}{\hat{\phi}'(E''(\mathbb{Q}))}\right) = \left\{ d_2 \equiv \pm \prod_{\ell \in \Sigma(k')} \ell^{e(\ell)} \;\middle|\; \begin{array}{l} \widehat{C}'_{d_2} :\ d_2\,w^2 = (1+d_2\,z^2)\,(1+d_2\,k'^2\,z^2) \\ \text{has a } \mathbb{Q}\text{-rational point } (z,w) \end{array} \right\}.$$

Here, $e(\ell)$ is either 0 or 1.

The various curves introduced in this theorem fit together in the diagrams:



We consider these diagrams to be "4-covers" because they both contain pairs of diagonal maps, each involving quadratic polynomials. In the diagram on the right, a $\mathbb{Q}$-rational point on the elliptic curve E (E', respectively) will correspond to a $\mathbb{Q}$-rational point on the **homogeneous space** $\widehat{C}_{d_2}$, ($\widehat{C}'_{d_2}$, respectively) having half as many digits.

Proof of Main Result

By Theorem 4, the Mordell-Weil group of the elliptic curve

$$E: \quad \begin{aligned} &Y^2 + X\,Y = X^3 - 7181359868024838434108428477109624412 0\,X \\ &\quad + 2342384302041141813702521859646228641128533374139589904 00\end{aligned}$$

is $Z_2 \times Z_8 \times \mathbb{Z}^r$ for some r. It suffices to show that the rank $r = 3$. Using Cremona's `mwrank` (1) we find that

$$\delta\left(\frac{E'(\mathbb{Q})}{\phi(E(\mathbb{Q}))}\right) = \{1\}, \quad \widehat{\delta}\left(\frac{E(\mathbb{Q})}{\hat{\phi}(E'(\mathbb{Q}))}\right) \subseteq \langle -1, 6477590, 2, 7, 37 \rangle;$$

so by Proposition 3 and Theorem 5,

$$2^{r+2} = \left|\frac{E(\mathbb{Q})}{2\,E(\mathbb{Q})}\right| = \left|\delta\left(\frac{E'(\mathbb{Q})}{\phi(E(\mathbb{Q}))}\right)\right| \left|\widehat{\delta}\left(\frac{E(\mathbb{Q})}{\hat{\phi}(E'(\mathbb{Q}))}\right)\right| \leq 2^5.$$

The authors in (3) found the following two $\mathbb{Q}$-rational points:

$$P_3 = \left(\frac{125691121567490117748583092936844134429 0}{16027875241^2} : \frac{786698395807855729596742215337306893212760400533639780}{16027875241^3} : 1\right);$$

$$P_4 = \left(\frac{4191463551901344114152227396505816107691618 40}{9255646526131^2} : \frac{105211386192778469849488967231903854157073005646773612879981880}{9255646526131^3} : 1\right).$$

The torsion subgroup is generated by P_1 and P_2, and the points P_3 and P_4 have infinite order. Hence $2 \leq r \leq 3$.

P on E	$d_2 = \widehat{\delta}(P)$	Point (z,w) on $\widehat{C}_{d_2}$
P_1	-1	$(1, 0)$
P_2	6477590	$\left(\frac{305}{7992}, \frac{8143806511}{200779379640}\right)$
P_3	2	$\left(\frac{116263507795895}{683172154272384}, \frac{119018475593848690746927861139}{163644731958920474581067710080}\right)$
P_4	7	$\left(\frac{9477908247062185}{147942254073677904}, \frac{280293077744848430200683737737110507 1}{731156866637839791234914733446622024 0}\right)$

P' on E'	$d_2 = \widehat{\delta}'(P')$	Point (z,w) on $\widehat{C}'_{d_2}$
$\phi(P_1)$	-1	$\left(\frac{7690763809}{932772960}, 0\right)$
$\phi(P_2)$	6477590	$\left(\frac{87697}{77731080}, \frac{8623536769}{6816782522760}\right)$
$\phi(P_3)$	2	$\left(\frac{40272144553979320937196 7}{166898987428842244395 68}, -\frac{546790296971729700371447998389073244144667329763}{531973821646899800424840471186998835301787518 4}\right)$
$\phi(P_4)$	7	$\left(\frac{1434298275377041049916550061461}{783098675276557692464875877 61}, -\frac{45852135158706046452821064687371285272034083270789515130064924}{41982527953301741489751898677947978759737801679261659817777 7}\right)$

Using our "2-cover", we seek a point P_5 such that $d_2 = \widehat{\delta}(P_5) = 37$. To be more precise, we seek a $\mathbb{Q}$-rational point (z,w) on the curve

$$\widehat{C}_{37} : \quad 37\,w^2 = (1+37\,z^2)\,(1+37\,\kappa^2\,z^2) \quad \text{where} \quad \kappa = \tfrac{28387584}{7662376225}.$$

Alternatively, using our "4-cover", we seek a point $P'_5 = \phi(P_5)$ such that $d_2 = \widehat{\delta}'(P'_5) = \widehat{\delta}(P_5) = 37$. To be more precise, we seek a $\mathbb{Q}$-rational point (z,w) on the curve

$$\widehat{C}'_{37} : \quad 37\,w^2 = (1+37\,z^2)\,(1+37\,k'^2\,z^2) \quad \text{where} \quad k' = \tfrac{932772960}{7690763809}.$$

We used John Cremona's algorithm `QuarticMinimise()` to find "minimal" integral models for $\widehat{C}_{37}$ and $\widehat{C}'_{37}$ then Michael Stoll's program `ratpoints` (9) to find rational points. We are running these on both Miami University's 128-node high-powered computing cluster, RedHawk, and Purdue University's Euler. □

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