ABC Triples in Families

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Abstract

Given three positive, relative prime integers $A$, $B$, and $C$ such that the first two sum to the third i.e. $A + B = C$, it is rare to have the product of the primes $p$ dividing them to be smaller than each of the three.

In 1985, David Masser and Joseph Osterlé made this precise by defining a “quality” $q(P)$ for such a triple of integers $P = (A, B, C)$; their celebrated “ABC Conjecture” asserts that it is rare for this quality $q(P)$ to be greater than 1 – even though there are infinitely many examples where this happens. In 1987, Gerhard Frey offered an approach to understanding this conjecture by introducing elliptic curves.

In this presentation, we introduce families of triples so that the Frey curve has nontrivial torsion subgroup, and explain how certain triples with large quality appear in these families. We also discuss how these families contain infinitely many examples where the quality $q(P)$ is greater than 1.
Mathematical Sciences Research Institute Undergraduate Program
MSRI-UP 2010
Outline of Talk

1 History of the Conjecture
   • The Mason-Stothers Theorem
   • $ABC$ Conjecture
   • Examples with Exceptional Quality

2 Relation with Elliptic Curves
   • Elliptic Curves
   • Torsion Subgroups
   • Quality by Torsion Subgroup

3 Transfer Maps
   • Transfer Maps of Degree 2
   • Transfer Maps of Degree 4
   • Transfer Maps of Degree 8
History of the $ABC$ Conjecture
Mason-Stothers Theorem

Theorem (W. W. Stothers, 1981; R. C. Mason, 1983)

Denote \( n(ABC) \) as the number distinct zeroes of the product of relatively prime polynomials \( A(t), B(t), \) and \( C(t) \) satisfying \( A + B = C \). Then

\[
\max\{\deg(A), \deg(B), \deg(C)\} \leq n(ABC) - 1.
\]

Proof: We follow Lang’s Algebra. Explicitly write

\[
A(t) = A_0 \prod_{i=1}^{a} (t - \alpha_i)^{p_i}
\]

\[
B(t) = B_0 \prod_{j=1}^{b} (t - \beta_j)^{q_j}
\]

\[
C(t) = C_0 \prod_{k=1}^{c} (t - \gamma_k)^{r_k}
\]

\[
\deg(A) = \sum_{i=1}^{a} p_i
\]

\[
\deg(B) = \sum_{j=1}^{b} q_j
\]

\[
\deg(C) = \sum_{k=1}^{c} r_k
\]

\[
\text{rad}(ABC)(t) = \prod_{i=1}^{a} (t - \alpha_i) \prod_{j=1}^{b} (t - \beta_j) \prod_{k=1}^{c} (t - \gamma_k)
\]

\[
n(ABC) = a + b + c
\]
\[
\begin{align*}
F(t) &= \frac{A(t)}{C(t)} \\
G(t) &= \frac{B(t)}{C(t)} \\
\end{align*}
\]

\[
\begin{align*}
\left\{ \frac{A(t)}{C(t)} \right\} \implies -\frac{B(t)}{A(t)} &= \frac{F(t)}{F'(t)} \\
\left\{ \frac{B(t)}{C(t)} \right\} &= \frac{G(t)}{G'(t)} = \frac{\sum_{i=1}^{a} \frac{p_i}{t - \alpha_i} - \sum_{i=k}^{c} \frac{r_k}{t - \gamma_k}}{\sum_{j=1}^{b} \frac{q_j}{t - \beta_j} - \sum_{k=1}^{c} \frac{r_k}{t - \gamma_k}}
\end{align*}
\]

Clearing, we find polynomials of degrees \(\max\{\deg(A), \deg(B)\} \leq n - 1:\)

\[
\begin{align*}
\text{rad}(ABC)(t) \cdot \frac{F'(t)}{F(t)} &= \sum_{i=1}^{a} p_i \prod_{e \neq i} (t - \alpha_e) \prod_{j=1}^{b} (t - \beta_j) \prod_{k=1}^{c} (t - \gamma_k) \\
&\quad - \sum_{k=1}^{c} r_k \prod_{i=1}^{a} (t - \alpha_i) \prod_{j=1}^{b} (t - \beta_j) \prod_{e \neq k}^{e} (t - \gamma_e)
\end{align*}
\]

\[
\begin{align*}
\text{rad}(ABC)(t) \cdot \frac{G'(t)}{G(t)} &= \sum_{j=1}^{b} q_j \prod_{i=1}^{a} (t - \alpha_i) \prod_{e \neq j}^{e} (t - \beta_e) \prod_{k=1}^{c} (t - \gamma_k) \\
&\quad - \sum_{k=1}^{c} r_k \prod_{i=1}^{a} (t - \alpha_i) \prod_{j=1}^{b} (t - \beta_j) \prod_{e \neq k}^{e} (t - \gamma_e)
\end{align*}
\]
The Mason-Stothers Theorem is sharp. Here is a list of relatively prime polynomials such that $A(t) + B(t) = C(t)$ and

$$\max\{\deg(A), \deg(B), \deg(C)\} = n(A B C) - 1.$$ 

<table>
<thead>
<tr>
<th>$A(t)$</th>
<th>$B(t)$</th>
<th>$C(t)$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2t)^2$</td>
<td>$(t^2 - 1)^2$</td>
<td>$(t^2 + 1)^2$</td>
<td>5</td>
</tr>
<tr>
<td>$8t (t^2 + 1)$</td>
<td>$(t - 1)^4$</td>
<td>$(t + 1)^4$</td>
<td></td>
</tr>
<tr>
<td>$16t$</td>
<td>$(t + 1)^3 (t - 3)$</td>
<td>$(t + 3) (t - 1)^3$</td>
<td>5</td>
</tr>
<tr>
<td>$16t^3$</td>
<td>$(t + 1) (t - 3)^3$</td>
<td>$(t + 3)^3 (t - 1)$</td>
<td></td>
</tr>
<tr>
<td>$(2t)^4$</td>
<td>$(t^4 - 6t^2 + 1) (t^2 + 1)^2$</td>
<td>$(t^2 - 1)^4$</td>
<td>9</td>
</tr>
<tr>
<td>$16t (t^2 - 1) (t^2 + 1)^2$</td>
<td>$(t^2 - 2t - 1)^4$</td>
<td>$(t^2 + 2t - 1)^4$</td>
<td></td>
</tr>
</tbody>
</table>
The polynomial ring \( \overline{\mathbb{Q}}[t] \) has an absolute value

\[
| \cdot : \overline{\mathbb{Q}}[t] \to \mathbb{Z}_{\geq 0} \text{ defined by } |A(t)| = \begin{cases} 0 & \text{if } A(t) \equiv 0, \\ 2^{\deg(A)} & \text{otherwise.} \end{cases}
\]

It has the following properties:

- **Multiplicativity:** \( |A \cdot B| = |A| \cdot |B| \)
- **Non-degeneracy:** \( |A| = 0 \text{ iff } A = 0; |A| = 1 \text{ iff } A(t) = A_0 \text{ is a unit.} \)
- **Ordering:** \( |A| \leq |B| \text{ iff } \deg(A) \leq \deg(B). \)

**Corollary**

For each \( \epsilon > 0 \) there exists a uniform \( C_\epsilon > 0 \) such that the following holds: For any relatively prime polynomials \( A, B, C \in \overline{\mathbb{Q}}[t] \) with \( A + B = C \),

\[
\max\{|A|, |B|, |C|\} \leq C_\epsilon \ |\text{rad}(A B C)|^{1+\epsilon}.
\]

**Proof:** Using Mason-Stothers, we may choose \( C_\epsilon = 1/2. \)
Multiplicative Dedekind-Hasse Norms

Let $R$ be a Principal Ideal Domain with quotient field $K$:
- $\mathbb{Q}[t]$ in $\overline{\mathbb{Q}}(t)$ with primes $t - \alpha$.
- $\mathbb{Z}$ in $\mathbb{Q}$ with primes $p$.

Define $\text{rad}(\alpha)$ of an ideal $\alpha$ is the intersection of primes $p$ containing it:
- $\text{rad}(A) = \prod_i (t - \alpha_i)$ for $A(t) = A_0 \prod_i (t - \alpha_i)^{e_i}$ in $\overline{\mathbb{Q}}[t]$.
- $\text{rad}(A) = \prod_i p_i$ for $A = \prod_i p_i^{e_i}$ in $\mathbb{Z}$.

Theorem (Richard Dedekind; Helmut Hasse)

$R$ is a Principal Ideal Domain if and only if there exists an absolute value $| \cdot | : R \rightarrow \mathbb{Z}_{\geq 0}$ with the following properties:
- **Multiplicativity:** $|A \cdot B| = |A| \cdot |B|$
- **Non-degeneracy:** $|A| = 0$ iff $A = 0$; $|A| = 1$ iff $A$ is a unit.
- **Ordering:** $|A| \leq |B|$ if $A$ divides $B$.

We may define $|\alpha| = |A|$ as $\alpha = AR$. Hence $|\text{rad}(\alpha)| \leq |\alpha|$ as $\text{rad}(\alpha) \subseteq \alpha$. 
Conjecture (David Masser, 1985; Joseph Oesterlé, 1985)

For each $\epsilon > 0$ there exists a uniform $C_\epsilon > 0$ such that the following holds: For any relatively prime integers $A, B, C \in \mathbb{Z}$ with $A + B = C$,

$$\max\{|A|, |B|, |C|\} \leq C_\epsilon \left| \text{rad}(A B C) \right|^{1+\epsilon}.$$ 

Lemma

The symmetric group on three letters acts on the set of $ABC$ Triples:

$$\sigma : \begin{bmatrix} A \\ B \\ C \end{bmatrix} \mapsto \begin{bmatrix} B \\ -C \\ -A \end{bmatrix}, \quad \tau : \begin{bmatrix} A \\ B \\ C \end{bmatrix} \mapsto \begin{bmatrix} B \\ A \\ C \end{bmatrix}$$

where

$$\sigma^3 = 1, \quad \tau^2 = 1, \quad \tau \circ \sigma \circ \tau = \sigma^2$$

In particular, we may assume $0 < A \leq B < C$. 
Corollary

If the ABC Conjecture holds, then \( \lim \sup q(A, B, C) \leq 1 \) for the quality

\[
q(A, B, C) = \frac{\max\{\ln |A|, \ln |B|, \ln |C|\}}{\ln |\text{rad}(A B C)|}.
\]

Proof: Say \( \epsilon = (\lim \sup q(P) - 1)/3 \) is positive. Choose a sequence \( P_k = (A_k, B_k, C_k) \) with \( q(P_k) \geq 1 + 2\epsilon \). But this must be finite because

\[
\max\{|A_k|, |B_k|, |C_k|\} \leq C\epsilon |\text{rad}(A_k B_k C_k)|^{1+\epsilon} \leq \exp \left[ \frac{q(P_k)}{q(P_k) - 1 - \epsilon} \ln C\epsilon \right].
\]

Question

For each \( \epsilon > 0 \), there are only finitely many ABC Triples \( P = (A, B, C) \) with \( q(P) \geq 1 + \epsilon \). What is the largest \( q(P) \) can be?
Proposition (Bart de Smit, 2010)
There are only 233 known ABC Triples $P = (A, B, C)$ with $q(P) \geq 1.4$.

<table>
<thead>
<tr>
<th>Rank</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$q(A, B, C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>$3^{10} \cdot 109$</td>
<td>$23^5$</td>
<td>1.6299</td>
</tr>
<tr>
<td>2</td>
<td>$11^2$</td>
<td>$3^2 \cdot 5^6 \cdot 7^3$</td>
<td>$2^{21} \cdot 23$</td>
<td>1.6260</td>
</tr>
<tr>
<td>3</td>
<td>$19 \cdot 1307$</td>
<td>$7 \cdot 29^2 \cdot 31^8$</td>
<td>$2^8 \cdot 3^{22} \cdot 5^4$</td>
<td>1.6235</td>
</tr>
<tr>
<td>4</td>
<td>283</td>
<td>$5^{11} \cdot 13^2$</td>
<td>$2^8 \cdot 3^8 \cdot 17^3$</td>
<td>1.5808</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>$2 \cdot 3^7$</td>
<td>$5^4 \cdot 7$</td>
<td>1.5679</td>
</tr>
<tr>
<td>6</td>
<td>$7^3$</td>
<td>$3^{10}$</td>
<td>$2^{11} \cdot 29$</td>
<td>1.5471</td>
</tr>
<tr>
<td>7</td>
<td>$7^2 \cdot 41^2 \cdot 311^3$</td>
<td>$11^{16} \cdot 13^2 \cdot 79$</td>
<td>$2 \cdot 3^3 \cdot 5^{23} \cdot 953$</td>
<td>1.5444</td>
</tr>
<tr>
<td>8</td>
<td>$5^3$</td>
<td>$2^9 \cdot 3^{17} \cdot 13^2$</td>
<td>$11^5 \cdot 17 \cdot 31^3 \cdot 137$</td>
<td>1.5367</td>
</tr>
<tr>
<td>9</td>
<td>$13 \cdot 19^6$</td>
<td>$2^{30} \cdot 5$</td>
<td>$3^{13} \cdot 11^2 \cdot 31$</td>
<td>1.5270</td>
</tr>
<tr>
<td>10</td>
<td>$3^{18} \cdot 23 \cdot 2269$</td>
<td>$17^3 \cdot 29 \cdot 31^8$</td>
<td>$2^{10} \cdot 5^2 \cdot 7^{15}$</td>
<td>1.5222</td>
</tr>
</tbody>
</table>

ABC Conjecture Home Page

http://www.math.unicaen.fr/~nitaj/abc.html
**History of the Conjecture**

**Relation with Elliptic Curves**

**Transfer Maps**

**The Mason-Stothers Theorem**

**ABC Conjecture**

**Examples with Exceptional Quality**

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### ABC at Home

ABC@home is an educational and non-profit distributed computing project finding abc-triples related to the **ABC conjecture**.

### What is ABC@home?

ABC@home is an educational and non-profit distributed computing project finding abc-triples related to the **ABC conjecture**.

#### Join ABC@home

1. Read our rules and policies
2. Download BOINC
3. When prompted, enter
   

#### What is the ABC conjecture?

The ABC conjecture involves abc-triples: positive integers $a, b, c$ such that $a + b = c$, $a < b < c$, $a, b, c$ have no common divisors and $c > \text{rad}(abc)$, the so-called radical of abc. The ABC conjecture says that there are only finitely many $a, b, c$ such that $\log(c)/\log(\text{rad}(abc)) > h$ for any real $h > 1$. The ABC conjecture is currently one of the greatest open problems in mathematics. If it is proven to be true, a lot of other open problems can be answered directly from it.

#### Why should I join?

The ABC conjecture is one of the greatest open mathematical questions, one of the holy grails of mathematics. It will teach us something about our very own numbers. Furthermore, the application of ABC@home is tiny, secure and stable, we like to keep things simple.

#### Who is involved?

The project is run by the [Mathematical Institute of Leiden University](http://www.math.leidenuniv.nl/) as part of [Reken mee met ABC](http://www.rekenmeemabet.nl/)

#### Minimum System requirements

- min. 256MB ram free
- 2 MB of free disk space
- Windows, Linux, Mac (recommended with a 64bit cpu)

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### User of the Day

bill 🎩

### News

#### 24 August 2010

Binaries for all platforms have now been updated. There seem to be some issues with Windows 2000 hosts, which we'll try and fix in the next few days. Please report any other problems!

#### 21 August 2010

We will be releasing new binaries over the next few days, with some speed improvements, fixes for some problematic regions, and as preparation for yet more updates. Please report any problems!

#### 18 August 2010

The old applications have been released with a new code signing key as 2.01.

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[...more](http://abcathome.com/)

News is available as an [RSS feed](http://abcathome.com/rss.xml).

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**Center for Communications Research**

**ABC Triples in Families**
The $ABC$ Conjecture and Elliptic Curves
Frey’s Observation

Theorem (Gerhard Frey, 1989)

Let \( P = (A, B, C) \) be an ABC Triple, that is, a triple of relatively prime integers such that \( A + B = C \). Then the corresponding curve

\[
E_{A,B,C} : \quad y^2 = x(x - A)(x + B)
\]

has “remarkable properties.”

Question

How do you explain this to undergraduates?

Answer: You don’t!
Elliptic Curves

Definition
An elliptic curve $E$ over a field $K$ is a nonsingular projective curve of genus 1 possessing a $K$-rational point $O$.

Proposition
An elliptic curve $E$ over $K$ is birationally equivalent to a cubic equation

$$E : \quad y^2 = x^3 - 3c_4x - 2c_6$$

having nonzero discriminant $\Delta(E) = 12^3 (c_6^2 - c_4^3)$.

Define the set of $K$-rational points on $E$ as the projective variety

$$E(K) = \left\{ (x_1 : x_2 : x_0) \in \mathbb{P}^2(K) \mid x_2^2 x_0 = x_1^3 - 3c_4 x_1 x_0^2 - 2c_6 x_0^3 \right\}$$

where $O = (0 : 1 : 0)$ is on the line $x_0 = 0$. In practice, $K = \mathbb{Q}(t)$ or $\mathbb{Q}$. 

**Group Law**

**Definition**

Let $E$ be an elliptic curve over a field $K$. Define the operation $⊕$ as

$$P ⊕ Q = (P ∗ Q) ∗ O.$$

![Graph of elliptic curve with points P, Q, P+Q, and P*Q marked.](image-url)
Theorem (Henri Poincaré, 1901)

Let $E$ be an elliptic curve defined over a field $K$. Then $E(K)$ is an abelian group under $\oplus$ with identity $\mathcal{O}$.

Theorem (Louis Mordell, 1922)

Let $E$ be an elliptic curve over $\mathbb{Q}$. Then $E(\mathbb{Q})$ is finitely generated: there exists a finite group $E(\mathbb{Q})_{\text{tors}}$ and a nonnegative integer $r$ such that

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^r.$$ 

Definition

$E(\mathbb{Q})$ is called the Mordell-Weil group of $E$. The finite set $E(\mathbb{Q})_{\text{tors}}$ is called the torsion subgroup of $E$, and the nonnegative integer $r$ is called the rank of $E$. 
Theorem (Barry Mazur, 1977)

Let $E$ be an elliptic curve over $\mathbb{Q}$. Then

$$E(\mathbb{Q})_{\text{tors}} \simeq \begin{cases} 
Z_N & \text{where } 1 \leq N \leq 10 \text{ or } N = 12; \\
Z_2 \times Z_{2N} & \text{where } 1 \leq N \leq 4.
\end{cases}$$

Corollary (Gerhard Frey, 1989)

For each $ABC$ Triple, the elliptic curve

$$E_{A,B,C} : y^2 = x(x - A)(x + B)$$

has discriminant $\Delta(E_{A,B,C}) = 16 A^2 B^2 C^2$ and $E_{A,B,C}(\mathbb{Q})_{\text{tors}} \simeq Z_2 \times Z_{2N}$.

Question

For each $ABC$ Triple, which torsion subgroups do occur?
Proposition (EHG and Jamie Weigandt, 2009)

All possible subgroups do occur – and infinitely often.

Proof: Choose relatively prime integers $m$ and $n$. We have the following $N$-isogeneous curves:

$$\begin{array}{|c|c|c|c|}
\hline
A & B & C & E_{A,B,C}(\mathbb{Q})_{\text{tors}} \\
\hline
(2mn)^2 & (m^2 - n^2)^2 & (m^2 + n^2)^2 & \mathbb{Z}_2 \times \mathbb{Z}_4 \\
8mn(m^2 + n^2) & (m - n)^4 & (m + n)^4 & \mathbb{Z}_2 \times \mathbb{Z}_2 \\
16mn^3 & (m + n)^3(m - 3n) & (m + 3n)(m - n)^3 & \mathbb{Z}_2 \times \mathbb{Z}_6 \\
16m^3n & (m + n)(m - 3n)^3 & (m + 3n)^3(m - n) & \mathbb{Z}_2 \times \mathbb{Z}_2 \\
(2mn)^4 & (m^4 - 6m^2n^2 + n^4)(m^2 + n^2)^2 & (m^2 - n^2)^4 & \mathbb{Z}_2 \times \mathbb{Z}_8 \\
16mn(m^2 - n^2)(m^2 + n^2)^2 & (m^2 - 2mn - n^2)^4 & (m^2 + 2mn - n^2)^4 & \mathbb{Z}_2 \times \mathbb{Z}_2 \\
\hline
\end{array}$$
### Examples

<table>
<thead>
<tr>
<th>Rank of Quality</th>
<th>$E_{A,B,C}(\mathbb{Q})_{\text{tors}}$</th>
<th>$m$</th>
<th>$n$</th>
<th>Quality $q(A, B, C)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>–</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_4$</td>
<td>1029</td>
<td>1028</td>
<td>1.2863664657, 1.3475851066</td>
</tr>
<tr>
<td>–</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>–</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_4$</td>
<td>4</td>
<td>3</td>
<td>1.2039689894, 1.4556731002</td>
</tr>
<tr>
<td>35</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>–</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_6$</td>
<td>5</td>
<td>1</td>
<td>1.0189752355, 1.4265653296</td>
</tr>
<tr>
<td>113</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_6$</td>
<td>729</td>
<td>7</td>
<td>1.4508584088, 1.3140518205</td>
</tr>
<tr>
<td>–</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_8$</td>
<td>3</td>
<td>1</td>
<td>1.0370424407, 1.4556731002</td>
</tr>
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<td>–</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>–</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_8$</td>
<td>577</td>
<td>239</td>
<td>1.2235280800, 1.2951909301</td>
</tr>
<tr>
<td>–</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Exceptional Quality Revisited

**Proposition**

There are infinitely many $ABC$ Triples $P = (A, B, C)$ with $q(P) > 1$.

*Proof:* For each positive integer $k$, define the relatively prime integers

$$A_k = 1, \quad B_k = 2^{k+2} (2^k - 1), \quad \text{and} \quad C_k = (2^{k+1} - 1)^2.$$ 

Then $P_k = (A_k, B_k, C_k)$ is an $ABC$ Triple. Moreover,

$$\text{rad}(A_k B_k C_k) = \text{rad}((2^{k+1} - 2)(2^{k+1} - 1)) \leq (2^{k+1} - 2)(2^{k+1} - 1) < C_k.$$ 

Hence

$$q(P_k) = \frac{\max\{\ln|A_k|, \ln|B_k|, \ln|C_k|\}}{\ln|\text{rad}(A_k B_k C_k)|} = \frac{\ln|C_k|}{\ln|\text{rad}(A_k B_k C_k)|} > 1.$$

**Corollary**

If the $ABC$ Conjecture holds, then $\limsup q(A, B, C) = 1$. 
Fix $N = 1, 2, 3, 4$. Let $\mathcal{F}(N)$ denote the those $ABC$ Triples $(A, B, C)$ such that $E_{A,B,C}(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2N}$. What can we say about 

$$\limsup_{(A,B,C) \in \mathcal{F}(N)} q(A, B, C)?$$

**Theorem (Alexander Barrios, Caleb Tillman and Charles Watts, 2010)**

Fix $N = 1, 2, 4$. There are infinitely many $ABC$ Triples with 

$$E_{A,B,C}(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2N} \quad \text{and} \quad q(A, B, C) > 1.$$ 

In particular, if the $ABC$ Conjecture holds, then $\limsup_{P \in \mathcal{F}(N)} q(P) = 1$.

**Proof:** Use the formulas above to create a dynamical system!
Can we use elliptic curves to find $ABC$ Triples with exceptional quality $q(A, B, C)$?
In what follows, we will substitute $A_k = m$, $B_k = n$, and $C_k = m + n$.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$E_{A, B, C}(\mathbb{Q})_{\text{tors}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2mn)^2$</td>
<td>$(m^2 - n^2)^2$</td>
<td>$(m^2 + n^2)^2$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_4$</td>
</tr>
<tr>
<td>$8mn(m^2 + n^2)$</td>
<td>$(m-n)^4$</td>
<td>$(m+n)^4$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$16m^3n$</td>
<td>$(m+n)^3(m-3n)$</td>
<td>$(m+3n)(m-n)^3$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_6$</td>
</tr>
<tr>
<td>$16m^3n$</td>
<td>$(m+n)(m-3n)^3$</td>
<td>$(m+3n)^3(m-n)$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$(2mn)^4$</td>
<td>$(m^4 - 6m^2n^2 + n^4)$</td>
<td>$(m^2 - n^2)^4$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_8$</td>
</tr>
<tr>
<td>$16mn(m^2-n^2)$</td>
<td>$(m^2 - 2mn - n^2)^4$</td>
<td>$(m^2 + 2mm - n^2)^4$</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td>
</tr>
</tbody>
</table>
Motivation

Consider a sequence \(\{P_0, \ldots, P_k, P_{k+1}, \ldots\}\) defined recursively by

\[
\begin{bmatrix}
A_{k+1} \\
B_{k+1} \\
C_{k+1}
\end{bmatrix} = \begin{bmatrix}
A_k^2 \\
B_k^2 - A_k^2 \\
B_k^2
\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
4 A_k B_k \\
(A_k - B_k)^2 \\
C_k^2
\end{bmatrix}.
\]

Proposition (EHG and Jamie Weigandt, 2009)

If the following properties hold for \(k = 0\), they hold for all \(k \geq 0\):

i. \(A_k, B_k,\) and \(C_k\) are relatively prime, positive integers.

ii. \(A_k + B_k = C_k.\)

iii. \(A_k \equiv 0 \pmod{16}\) and \(C_k \equiv 1 \pmod{4}.\)

Corollary

- For \(\epsilon > 0\), there exists \(\delta\) such that \(\max\{\ln |A_k|, \ln |B_k|, \ln |C_k|\} > \epsilon\) when \(k \geq \delta\). Hence \(q(P_k) > 1\) for all \(k \geq 0\) if and only if \(q(P_0) > 1.\)

- There exists an infinite sequence with \(q(P_k) > 1.\)
Consider a sequence \( \{P_0, \ldots, P_k, P_{k+1}, \ldots\} \) defined recursively by

\[
\begin{bmatrix}
A_{k+1} \\
B_{k+1} \\
C_{k+1}
\end{bmatrix} = \begin{bmatrix}
8 A_k B_k (A_k^2 + B_k^2) \\
8 A_k B_k (A_k^2 - B_k^2)^4 \\
C_k^4
\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
(2 A_k B_k)^2 \\
(2 A_k B_k)^2 \\
(2 A_k B_k)^2 \\
(2 A_k B_k)^2
\end{bmatrix}.
\]

**Proposition (Alexander Barrios, Caleb Tillman and Charles Watts, 2010)**

If the following properties hold for \( k = 0 \), they hold for all \( k \geq 0 \):

i. \( A_k, B_k, \) and \( C_k \) are relatively prime, positive integers.

ii. \( A_k + B_k = C_k \).

iii. \( A_k \equiv 0 \pmod{16} \) and \( C_k \equiv 1 \pmod{4} \).

**Corollary**

- For \( \epsilon > 0 \), there exists \( \delta \) such that \( \max\{\ln|A_k|, \ln|B_k|, \ln|C_k|\} > \epsilon \) when \( k \geq \delta \). Hence \( q(P_k) > 1 \) for all \( k \geq 0 \) if and only if \( q(P_0) > 1 \).

- There exist infinitely \( E_{P_k}(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 N \) for \( N = 1, 2; \ q(P_k) > 1 \).
Consider a sequence \( \{P_0, \ldots, P_k, P_{k+1}, \ldots\} \) defined recursively by

\[
\begin{bmatrix}
A_{k+1} \\
B_{k+1} \\
C_{k+1}
\end{bmatrix} = \begin{bmatrix}
16 A_k B_k^3 \\
(A_k + B_k)^3 (A_k - 3 B_k) \\
(A_k + 3 B_k) (A_k - B_k)^3
\end{bmatrix}.
\]

**Proposition (Alexander Barrios, Caleb Tillman and Charles Watts, 2010)**

If the following properties hold for \( k = 0 \), they hold for all \( k \geq 0 \):

i. \( A_k, B_k, \) and \( C_k \) are relatively prime integers.

ii. \( A_k + B_k = C_k \).

iii. \( A_k \equiv 0 \pmod{16} \) and \( C_k \equiv 1 \pmod{4} \).

**Question**

What condition do we need to guarantee that these are positive integers?
We sketch why perhaps $3.214 B_k > A_k$. Define the rational number

$$x_k = \frac{A_k}{B_k} \quad \Rightarrow \quad \begin{cases} \frac{1}{x_{k+1}} - \frac{1}{x_k} = \frac{(A_k + B_k)^3 (A_k - 3 B_k)}{16 A_k B_k^3} - \frac{B_k}{A_k} \\ = \frac{x_k^4 - 6 x_k^2 - 8 x_k - 19}{16 x_k}. \end{cases}$$

The largest root is $x_0 = 3.2138386$, so $0 < x_{k+1} < x_k < x_0$ is decreasing.
Consider a sequence \( \{P_0, \ldots, P_k, P_{k+1}, \ldots\} \) defined recursively by

\[
\begin{bmatrix}
A_{k+1} \\
B_{k+1} \\
C_{k+1}
\end{bmatrix}
= \begin{bmatrix}
(2A_k B_k)^4 \\
(A_k^4 - 6A_k^2 B_k^2 + B_k^4)(A_k^2 + B_k^2)^2 \\
(A_k^2 - B_k^2)^4
\end{bmatrix}.
\]

Proposition (Alexander Barrios, Caleb Tillman and Charles Watts, 2010)

If the following properties hold for \( k = 0 \), they hold for all \( k \geq 0 \):

i. \( A_k, B_k, \) and \( C_k \) are relatively prime, positive integers.

ii. \( A_k + B_k = C_k \) and \( B_k > 3.174 A_k \).

iii. \( A_k \equiv 0 \pmod{16} \) and \( C_k \equiv 1 \pmod{4} \).

Corollary

- For \( \epsilon > 0 \), there exists \( \delta \) such that \( \max\{\ln|A_k|, \ln|B_k|, \ln|C_k|\} > \epsilon \) when \( k \geq \delta \). Hence \( q(P_k) > 1 \) for all \( k \geq 0 \) if and only if \( q(P_0) > 1 \).
- There exists a sequence \( E_{P_k}(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}_2 \times \mathbb{Z}_8 \) and \( q(P_k) > 1 \).
We sketch why $B_k > 3.174 A_k$. Define the rational number

\[ x_k = \frac{B_k}{A_k} \implies \begin{cases} 
 x_{k+1} - x_k = \frac{\left( A_k^4 - 6 A_k^2 B_k^2 + B_k^4 \right) \left( A_k^2 + B_k^2 \right)^2}{\left( 2 A_k B_k \right)^4} - \frac{B_k}{A_k} \\
 = \frac{x_k^8 - 4 x_k^6 - 16 x_k^5 - 10 x_k^4 - 4 x_k^2 + 1}{16 x_k^4}.
\end{cases} \]

The largest root is $x_0 = 3.1737378$, so $x_0 < x_k < x_{k+1}$ is increasing.
Example

We can generate many examples of ABC Triples $P = (A, B, C)$ with

$$E_{A,B,C}(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}_2 \times \mathbb{Z}_8$$
and

$$q(A, B, C) > 1.$$  

We consider the recursive sequence defined by

$$
\begin{bmatrix}
A_{k+1} \\
B_{k+1} \\
C_{k+1}
\end{bmatrix} =
\begin{bmatrix}
(2A_k B_k)^4 \\
(A_k^4 - 6A_k^2 B_k^2 + B_k^4)(A_k^2 + B_k^2)^2 \\
(A_k^2 - B_k^2)^4
\end{bmatrix}.
$$

Initialize with $P_0 = (16^2, 63^2, 65^2)$ so that we have

i. $A_k$, $B_k$, and $C_k$ are relatively prime, positive integers.

ii. $A_k + B_k = C_k$ and $B_k > 3.174 A_k$.

iii. $A_k \equiv 0 \pmod{16}$ and $C_k \equiv 1 \pmod{4}$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$A_k$</th>
<th>$B_k$</th>
<th>$C_k$</th>
<th>$q(P_k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2^8$</td>
<td>$3^4 \cdot 7^2$</td>
<td>$5^2 \cdot 13^2$</td>
<td>1.05520</td>
</tr>
<tr>
<td>1</td>
<td>$2^{36} \cdot 3^{16} \cdot 7^8$</td>
<td>$41^2 \cdot 881 \cdot 20113 \cdot 385817^2 \cdot 13655297$</td>
<td>$5^8 \cdot 13^8 \cdot 47^4 \cdot 79^4$</td>
<td>1.00676</td>
</tr>
</tbody>
</table>
Questions?