Artin's Conjecture and Elliptic Curves

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Abstract. Artin conjectured that certain Galois representations should give rise to entire $L$-series. We give some history on the conjecture and motivation of why it should be true by discussing the one-dimensional case. The first known example to verify the conjecture in the icosahedral case did not surface until Buhler's work in 1977. We explain how this icosahedral representation is attached to a modular elliptic curve isogenous to its Galois conjugates, and then explain how it is associated to a cusp form of weight 5 with level prime to 5.

1. Introduction

In 1917, Erich Hecke [10] proved a series of results about certain characters which are now commonly referred to as Hecke characters; one corollary states that one-dimensional complex Galois representations give rise to entire $L$-series. He revealed, through a series of lectures [9] at Princeton's Institute for Advanced Study in the years that followed, the relationship between such representations as generalizations of Dirichlet characters and modular forms as the eigenfunctions of a set of commuting self-adjoint operators. Many mathematicians were inspired by his ground-breaking insight and novel proof of the analytic continuation of the $L$-series.

In the 1930's, Emil Artin [1] conjectured that a generalization of such a result should be true; that is, irreducible complex projective representations of finite Galois groups should also give rise to entire $L$-series. He came to this conclusion after proving himself that both 3-dimensional and 4-dimensional representations of the simple group of order 60, the alternating group on five letters, might give rise to $L$-series with singularities. It is known, due to the insight of Robert Langlands [16] in the 1970's relating Hecke characters with Representation Theory, that in order to prove the conjecture it suffices to prove that such representations are associated to cusp forms. This conjecture has been the motivation for much study in both Algebraic and Analytic Number Theory ever since.

In this paper, we present an elementary approach to Artin's Conjecture by considering the problem over $\mathbb{Q}$. We consider Dirichlet's theorem which preceded Hecke's results, and sketch a proof by introducing theta series. We then introduce Langland's program to exhibit cusp forms. We conclude by studying a specific example which is associated to an elliptic curve. We assume in the final sections that the reader is somewhat familiar with the basic properties of elliptic curves.

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2. One-Dimensional Representations

We begin with some classical definitions and theorems. We are motivated by the expositions in [11] and [4].

2.1. Reimann Zeta Function and Dirichlet L-Series. Let $N$ be a fixed positive integer, and $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a group homomorphism. We extend $\chi : \mathbb{Z} \to \mathbb{C}$ to the entire ring of integers by defining 1) $\chi(n) = \chi(n \mod N)$ on the residue class modulo $N$; and 2) $\chi(n) = 0$ if $n$ and $N$ have a factor in common. One easily checks that this extended definition still yields a multiplicative map i.e. $\chi(n_1 n_2) = \chi(n_1) \chi(n_2)$.

Fix a complex number $s \in \mathbb{C}$ and associate the $L$-series to $\chi$ as the sum

$$L(\chi, s) = \chi(1) + \frac{\chi(2)}{2^s} + \frac{\chi(3)}{3^s} + \cdots = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

Quite naturally, two questions arise:

For which region is this series a well-defined function?

Can that function be continued analytically to the entire complex plane?

This series is reminiscent of the Reimann zeta function, defined as

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

We may express this sum in a slightly different fashion. It is easy to check that\footnote{In the literature, it is standard to define
$$\Gamma(s) = \int_0^\infty e^{-y} y^{s-1} \, dy = (s-1)!, \quad \text{Re}(s) \geq 1$$
as the Generalized Factorial or Gamma function.}

$$\int_0^\infty e^{-ny} y^{s-1} \, dy = \frac{(s-1)!}{n^s}$$

so that the sum becomes

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{(s-1)!} \int_0^\infty e^{-ny} y^{s-1} \, dy = \frac{1}{(s-1)!} \int_0^\infty \frac{y^{s-1}}{e^y - 1} \, dy$$

The function $e^y$ grows faster than any power of $y$, so the integrand converges for all complex $s - 1$ with positive real part. However, we must be careful when $y = 0$; the denominator of the integral vanishes, so we need the numerator to vanish as well. This is not the case if $s = 1$. Hence, we find the

**Theorem 2.1 (Dirichlet [11]).** $L(\chi, s)$ is a convergent series if $\text{Re}(s) > 1$. If $\chi$ is not the trivial character $\chi_0 = 1$ (i.e. $N \neq 1$) then $L(\chi, s)$ has analytic continuation everywhere. On the other hand, $L(\chi_0, s) = \zeta(s)$ has a pole at $s = 1$. In either case, the $L$-series has the product expansion

$$L(\chi, s) = \prod_{\text{primes } p} \left( 1 - \frac{\chi(p)}{p^s} \right)^{-1}$$

(2.5)
PROOF. We present a sketch of proof when \( N = 5 \), although the method works in general. Consider the character

\[
(5 \mod n) = \begin{cases} 
+1 & \text{if } n \equiv 1, 4 \mod 5; \\
-1 & \text{if } n \equiv 2, 3 \mod 5; \\
0 & \text{if } n \text{ is divisible by 5.}
\end{cases}
\]

The L-series in this case may be expressed as the integral

\[
L \left( \left( \frac{5}{*} \right), s \right) = \frac{1}{(s-1)!} \int_0^\infty \left[ \sum_{n=1}^\infty \left( \frac{5}{n} \right) e^{-ny} \right] y^{s-1} dy
\]

The infinite sum inside the integrand can be evaluated by considering different cases: When \( n \) is divisible by 5, the character vanishes. Otherwise write \( n = 5k + 1, \ldots, 5k + 4 \) in the other cases so that the sum becomes

\[
\sum_{n=1}^\infty \left( \frac{5}{n} \right) e^{-ny} = \sum_{k=0}^\infty \left[ +e^{-(5k+1)y} - e^{-(5k+2)y} - e^{-(5k+3)y} + e^{-(5k+4)y} \right]
\]

\[
e^{-y} - e^{-2y} - e^{-3y} + e^{-4y} = \frac{e^{3y} - e^y}{1 + e^y + e^{2y} + e^{3y} + e^{4y}}
\]

which gives the L-series as

\[
L \left( \left( \frac{5}{*} \right), s \right) = \frac{1}{(s-1)!} \int_0^\infty \frac{e^{3y} - e^y}{1 + e^y + e^{2y} + e^{3y} + e^{4y}} y^{s-1} dy
\]

Hence the integrand is analytic at \( y = 0 \), so the L-series does not have a pole at \( s = 1 \). \( \square \)

2.2. Galois Representations. Fix \( q(x) \) as an irreducible polynomial of degree \( d \) with leading coefficient 1 and integer coefficients, and set \( K/\mathbb{Q} \) as its splitting field. The group of permutations of the roots \( \text{Gal}(K/\mathbb{Q}) \) has a canonical representation as \( d \times d \) matrices. To see why, write the \( d \) roots \( q_k \) of \( q(x) \) as \( d \)-dimensional unit vectors:

\[
q_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \quad q_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}; \quad \ldots; \quad q_d = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.
\]

Any permutation \( \sigma \) on these roots may be represented as a \( d \times d \) matrix \( \varrho(\sigma) \). One easily checks that \( \varrho \) is a multiplicative map i.e. \( \varrho(\sigma_1 \sigma_2) = \varrho(\sigma_1) \varrho(\sigma_2) \). As an example, consider \( x^2 + x - 1 \). Then \( K = \mathbb{Q}(\sqrt{5}) \), and the only permutation of interest is

\[
\sigma: \frac{-1 + \sqrt{5}}{2} \mapsto \frac{-1 - \sqrt{5}}{2} \quad \Rightarrow \quad \varrho(\sigma) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

Denote \( G_\mathbb{Q} \), the absolute Galois group, as the union of each of the \( \text{Gal}(K/\mathbb{Q}) \) for such polynomials \( q(x) \) with integer coefficients. This larger group still permutes
the roots of a specific polynomial \( q(x) \); we consider it because it is a universal object independent of \( q(x) \). We view the permutation representation as a group homomorphism \( \rho : G_Q \to GL_d(\mathbb{C}) \).

The finite collection of matrices \( \{ \varrho(\sigma) | \sigma \in G_Q \} \) acts on the \( d \)-dimensional complex vector space \( \mathbb{C}^d \), so we are concerned with lines which are invariant under the action of all of the \( \varrho(\sigma) \). That is, if we attempt to simultaneously diagonalize all of the \( \varrho(\sigma) \) then we want to consider one-dimensional invariant subspaces. For example, for \( x^2 + x - 1 \) we may instead choose the basis

\[
(2.12) \quad q_1' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad q_2' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \implies \quad \varrho'(\sigma) = \begin{pmatrix} 1 \\ -1 \end{pmatrix};
\]

so that we have the more intuitive representation \( \rho(\sigma) = -1 \) defined on the eigenvalues. In general, we do not wish to consider the \( d \times d \) matrix representation \( \varrho \), but rather the scalar \( \rho : G_Q \to \mathbb{C}^\times \).

In order to define an \( L \)-series, we use the product expansion found in 2.1 above. To this end, choose a prime number \( p \) and factor \( q(x) \) modulo \( p \), say in the form

\[
(2.13) \quad q(x) \equiv (x^{f_1} + \ldots)_{e_1} (x^{f_2} + \ldots)_{e_2} \ldots (x^{f_r} + \ldots)_{e_r} \mod p
\]

If each \( e_j = 1 \), we say \( p \) is unramified (and ramified otherwise). In this case, there is a universal automorphism \( \text{Frob}_p \in G_Q \) which yields surjective maps \( G_Q \to \mathbb{Z}/f_j \mathbb{Z} \) for \( j = 1, \ldots, r \). This automorphism is canonically defined by the congruence

\[
(2.14) \quad \text{Frob}_p(q_k) \equiv q_k^p \mod p \quad \text{on the roots } q_k \text{ of } q(x).
\]

Unfortunately, when \( p \) is ramified there is not a canonical choice of Frobenius element because there are repeated roots. We will denote \( \Sigma \) as a finite set containing the ramified primes.

If \( \rho \) is a map \( G_Q \to \mathbb{C}^\times \), the Frobenius element induces a map \( G_Q \to \mathbb{Z}/f_j \mathbb{Z} \), and Dirichlet characters are maps \( \mathbb{Z}/f_j \mathbb{Z} \to \mathbb{C} \), it should follow that Dirichlet characters \( \chi \) are closely related to such maps \( \rho \). This is indeed the case.

**Theorem 2.2 (Artin Reciprocity [11]).** Fix \( q(x) \) and \( \rho : G_Q \to \mathbb{C}^\times \) be as described above. Define \( \chi_\rho : \mathbb{Z} \to \mathbb{C} \) on primes by the identification

\[
(2.15) \quad \chi_\rho(p) = \begin{cases} \rho(\text{Frob}_p) & \text{when unramified,} \\ 0 & \text{when ramified;} \end{cases}
\]

and extend \( \chi_\rho \) to all of \( \mathbb{Z} \) by multiplication. Then there exists a positive integer \( N = N(\rho) \), called the conductor of \( \rho \), divisible only by the primes which ramify such that \( \chi_\rho \) is a Dirichlet character modulo \( N \).

As an example, \( x^2 + x - 1 \) factors modulo the first few primes as

\[
(2.16) \quad x^2 + x - 1 \equiv \begin{cases} x^2 + x + 1 \mod 2 \\ x^2 + x + 2 \mod 3 \\ (x + 3)^2 \mod 5 \\ x^2 + x + 6 \mod 7 \\ (x + 4)(x + 8) \mod 11 \end{cases}
\]
Then $\text{Frob}_2$, $\text{Frob}_3$, and $\text{Frob}_7$ are each nontrivial automorphisms, while $\text{Frob}_{11}$ is the identity. The only ramified prime is 5 because of the repeated roots. We map

$$\rho(\text{Frob}_2) = \rho(\text{Frob}_3) = \rho(\text{Frob}_7) = -1; \quad \rho(\text{Frob}_{11}) = +1.$$  

The associated Dirichlet character $\chi_\rho = \left( \frac{\cdot}{5} \right)$ is just the character modulo $N = 5$ we considered above.

### 2.3. Artin L-Series: 1-Dimensional Case

We are now in a position to define and study the L-series associated to Galois representations.

**Corollary 2.3 (Artin, Dirichlet).** Given a map $\rho : G_\mathbb{Q} \to \mathbb{C}^\times$ as defined above with $\Sigma$ a finite set containing the ramified primes, define the Artin L-series as

$$L_\Sigma(\rho, s) = \sum_{n=1}^{\infty} \frac{\chi_\rho(n)}{n^s} = \prod_{p \notin \Sigma} \left(1 - \frac{\rho(Frob_p)}{p^s}\right)^{-1}$$

Then $L(\rho, s)$ converges if $\Re(s) > 1$. If $\rho$ is not the trivial map $\rho_0 = 1$ then $L_\Sigma(\rho, s)$ has analytic continuation everywhere. On the other hand,

$$L_\Sigma(\rho_0, s) = \zeta(s) \cdot \prod_{p \in \Sigma} (1 - p^s)$$

has a pole at $s = 1$.

**Proof.** We sketch the proof. First, we express the L-series in terms of the integral of a function which dies exponentially fast. This will guarantee that the L-series converges for some right-half plane. For $\tau = x + iy$ in the upper-half plane (i.e. $y > 0$) define

$$\theta_\rho(\tau) = \sum_{n=1}^{\infty} \chi_\rho(n) n^\epsilon e^{\pi i n^2 \tau}$$

where $\epsilon = \begin{cases} 0 & \text{if } \chi_\rho(-1) = +1, \\ 1 & \text{if } \chi_\rho(-1) = -1; \end{cases}$

so that the L-series may be expressed as the integral

$$L(\rho, s) = \frac{\pi \frac{\epsilon + 1}{2}!}{(s + \epsilon - 2)!} \int_0^\infty \theta_\rho(iy) y^{\frac{s + \epsilon - 2}{2}} dy$$

The integrand may have a pole at $y = 0$, so we perform our second trick. Break the integral up into two regions $0 < y < 1$ and $1 < y$, and then use the functional equation

$$\theta_{\overline{\rho}} \left(-\frac{1}{N^2 \tau}\right) = w(\rho) N^{\epsilon + \frac{1}{2}} \tau^{\epsilon + \frac{1}{2}} \theta_\rho(\tau)$$

— where $w(\rho)$ is a complex number of absolute value 1, $N = N(\rho)$ is the conductor, and $\overline{\rho}$ is the complex conjugate — to express the integral solely in terms of values $1/N < y$:
\[ \int_0^\infty \theta_\rho(iy) y^{s+\frac{1}{2}} \, dy = \int_{1/N}^\infty \left[ \theta_\rho(iy) y^{s+\frac{1}{2}} + w(\bar{\rho}) N^{-\frac{s}{2}+\frac{1}{2}} \theta_\rho(iy) y^{-s+\frac{1}{2}} \right] \, dy \]

Hence the integral converges for all complex \( s \). \( \square \)

3. Artin's Conjecture

3.1. Artin L-Series: General Case. Fix \( q(x) \) as an irreducible polynomial of degree \( d \) with leading coefficient 1 and integer coefficients, let \( \Sigma \) be a finite set containing the primes which ramify, and set \( G_Q \) as the absolute Galois group as before. We found that there is a canonical permutation \( \varrho : G_Q \to GL_d(\mathbb{C}) \) induced by the action on the roots. Consider the ring

\[
V_\varrho = \left\{ \sum_{\sigma \in G_Q} \lambda_\sigma \varrho(\sigma) \in GL_d(\mathbb{C}) \, \middle| \, \lambda_\sigma \in \mathbb{C} \right\}
\]

generated by the linear combinations of matrices in the image of \( \varrho \). We view this as a complex vector space, which is acted upon by the linear transformations \( \varrho(\sigma) \) quite naturally by matrix multiplication. As with any complex vector space, we may decompose it into invariant subspaces. Before, we considered only one-dimensional spaces, but now we generalize to an arbitrary invariant irreducible subspace \( V \subseteq V_\varrho \). We restrict \( \varrho \) such that the action is faithful on this subspace. That is,

\[
\rho = \varrho|_V : G_Q \to GL(V) \text{ is irreducible.}
\]

Define the L-series associated to \( \rho \) as the product

\[
L_\Sigma(\rho, s) = \prod_{p \in \Sigma} \det \left( 1 - \frac{\rho(Frob_p)}{p^s} \right)^{-1} = \sum_{n=1}^\infty \frac{a_\rho(n)}{n^s}
\]

As before, there is a corresponding \( N = N(\rho) \), called the conductor, which is divisible by the primes \( p \in \Sigma \). By considering the determinant, we find a Dirichlet character \( \epsilon_\rho \), called the nebentype, which is associated to the one-dimensional Galois representation \( \det \varrho \). The coefficients \( a_\rho(n) \) are closely related to the Frobenius element:

\[
a_\rho(p) = \begin{cases} 
\text{tr} \rho(Frob_p) & \text{when unramified}, \\
0 & \text{when ramified}.
\end{cases}
\]

Quite naturally, two questions arise once again:

For which complex numbers \( s \) is this series a well-defined function?

Can that function be continued analytically to the entire complex plane?

The first question has an answer which is consistent with the theme so far.

Proposition 3.1 (Artin [1]). \( L_\Sigma(\rho, s) \) converges if \( \text{Re}(s) > 1 \). The L-series associated to the trivial map \( L_\Sigma(\rho_0, s) \) has a pole at \( s = 1 \).

However, the second question remains an open problem.
Conjecture 3.2 (Artin). If $\rho$ is irreducible, not trivial, and is unramified outside a finite set of primes $\Sigma$, then $L_\Sigma(\rho, s)$ has analytic continuation everywhere.

The case of one-dimensional Galois representations (i.e. $V \simeq \mathbb{C}$) was proved in full generality with the advent of Class Field Theory. Indeed, any one-dimensional Galois representation must necessarily be abelian, so that by Artin Reciprocity the representation can be associated with a character defined on the idele group. When working over $\mathbb{Q}$, this amounts to saying that every one-dimensional representation may be associated with a Dirichlet character.

Many mathematicians, inspired by this result, began work on the irreducible two-dimensional representations (i.e. $V \simeq \mathbb{C}^2$). Felix Klein [12] had showed that the only finite projective images in the complex general linear group correspond to the Platonic Solids; that is, they are the rotations of the regular polygons and regular polyhedra. This is because we have the injective map

$$SO_3(\mathbb{R}) = \{ \gamma \in \text{Mat}_3(\mathbb{R}) | \det \gamma = 1, \gamma^t = \gamma^{-1} \} \rightarrow PGL_2(\mathbb{C})$$

which relates rotations of three-dimensional symmetric objects with $2 \times 2$ matrices modulo scalars. Hence, it suffices to consider irreducible two-dimensional projective representations with these images in order to prove Artin's Conjecture in this case.

Most of these cases of the conjecture have been answered in the affirmative. Irreducible cyclic and dihedral representations (that is, representations whose image in $PGL_2(\mathbb{C})$ is isomorphic to $\mathbb{Z}_n$ or $D_n$, respectively) may be interpreted as representations induced from abelian ones, so that the proof of analytic continuation may be reduced to one using Dirichlet characters. Irreducible tetrahedral and some octahedral representations (i.e. projective image isomorphic to $A_4$ or $S_4$, respectively) were proved to give entire $L$-series due to work by Robert Langlands [15] in the 1970's on base change for $GL(2)$. The remaining cases for irreducible octahedral representations were proved shortly thereafter by Jerrold Tunnell [19]. Such methods worked because they exploited the existence proper nontrivial normal subgroups. Unfortunately, the simple group of order 60 has none, so it is still not known whether the irreducible icosahedral representations (i.e. projective image isomorphic to $A_5$) have analytic continuation. The first known example to verify Artin's conjecture in this case did not surface until Joe Buhler's work [3] in 1977. It is this example we wish to consider in detail.

For general finite dimensional representations $\rho : G_\mathbb{Q} \rightarrow GL(V)$, not much is known. It is easy to show that the $L$-series is analytic in a right-half of the complex plane. In 1947, Richard Brauer [2] proved that the characters associated to representations of finite groups are a finite linear combination of one-dimensional characters, and so the corresponding $L$-series have meromorphic continuation; that is, the functions have at worst poles at a finite number of places. Brauer's proof does not guarantee that the integral coefficients of such a linear combination are positive; it can be shown that in many cases the coefficients are negative so that the proof of continuation to the entire complex plane may be reduced to showing that the poles of the $L$-series are cancelled by the zeroes.

3.2. Maass Forms and the Langlands Program. Robert Langlands completed a circle of ideas which related $L$-series, complex representations, and automorphic representations. While the deep significance of these ideas is far beyond the scope of this paper, we will content ourselves with a simplified consequence.
THEOREM 3.3 (Langlands [15]). In order to prove Artin's Conjecture for two-dimensional representations \( \rho \) unramified outside a finite set of primes, it suffices to prove that

\[
f_\rho(\tau) = \sum_{n=0}^{\infty} a_\rho(n) \frac{1}{n!} e^{2\pi i n \tau} \quad (\tau = x + iy)
\]

is a Maass cusp form of weight \( k = 1 \), level \( N = N(\rho) \), and nebentype \( \epsilon_\rho = \det \rho \).

The idea is to express the L-series as the integral of a function which dies exponentially fast. Indeed, we have the relation

\[
L_\Sigma(\rho, s) = \frac{2^s \pi^s}{(s-1)!} \int_{0}^{\infty} f_\rho(iz) y^{s-\frac{1}{2}} dy
\]

and, by definition, the function dies off exponentially fast as \( y \to \infty \).

In general, a smooth function \( f : \{x + iy \mid y > 0\} \to \mathbb{C} \) is called a Maass form of weight \( k \in \mathbb{Z} \), level \( N \in \mathbb{Z} \), and nebentype \( \epsilon : \mathbb{Z} \to \mathbb{C} \) if it satisfies the following properties:

1. **Eigenfunction of the non-Euclidean Laplacian.** \( f(x + iy) \) satisfies the differential equation

\[
\left\{-y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \frac{\partial}{\partial x} \right\} f(\tau) = \frac{k}{2} \left( 1 - \frac{k}{2} \right) f(\tau)
\]

2. **Exponential Decay.** For any complex \( s \) with \( \Re(s) > 1 \),

\[
\lim_{y \to \infty} f(x + iy) y^{s-1} = 0
\]

3. **Transformation Property.** For all matrices in the group

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}_2(\mathbb{Z}) \bigg| ad - bc = 1 \text{ and } c \text{ is divisible by } N \right\}
\]

we have the identity

\[
f \left( \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} \right) = \epsilon(d) \left( \frac{c\tau + d}{c\tau + d} \right)^{k/2} f(\tau)
\]

4. If in addition we have

\[
\int_{0}^{1} f(\tau + x) \, dx = 0 \quad \text{for all } \tau \in \{x + iy \mid y > 0\};
\]

we say that \( f \) is a cusp form. Otherwise, we call \( f \) an Eisenstein series.

For a given irreducible complex representation \( \rho \) which is ramified outside of a finite number of primes \( \Sigma \), the series in (3.6) always satisfies the first two conditions. The condition in (3.12) is equivalent to \( a_\rho(0) = 0 \), which happens if and only if \( \rho \) is not the trivial representation. Hence, in order to invoke Theorem 3.3 it suffices to prove the transformation property. Langlands succeeded in proving this in many cases by proving a generalization of the Selberg Trace Formula. Unfortunately, there does not appear to be a way to use these ideas in the icosahedral case.
4. Constructing Examples of Icosahedral Representations

Not many examples satisfying Artin's Conjecture in the icosahedral case are known. Those that are may be generated by the following program, initiated by Joe Buhler [3] and furthered by Ian Kiming [7].

1. Construct an $A_5$-extension $K/\mathbb{Q}$ by considering quintics.
2. Consider the discriminant of $K$ in order find possible conductors. This will uniquely specify the representation.
3. Construct weight 2 cusp forms by considering dihedral representations.
4. Divide by an Eisenstein series to find a weight 1 form.

While this program is straightforward, there are two computational barriers. First, finding the discriminant of an extension can be a tedious procedure. Unfortunately, the method above relies on this step in order to pinpoint the representation and to generate the weight 2 cusp forms. Second, division by Eisenstein series is much more difficult that it sounds. One must find the zeroes of the weight 2 cusp forms, the zeroes of the Eisenstein series, and then show that they occur at the same places.

Motivated by these difficulties, we ask

Can we generate Buhler's/Kiming's examples by using elliptic curves?

We modify the program above with this question in mind.

1. Construct an $A_5$-extension $K/\mathbb{Q}$ by considering quintics. Use classical results due to Klein to associate elliptic curves.
2. Construct the $A_5$-representations by considering the 5-torsion.
3. Construct weight 2 cusp forms associated to the elliptic curve.
4. Multiply by an Eisenstein series to find a weight 5 form.

4.1. Step #1: Relating $A_5$-Extensions to Elliptic Curves. Felix Klein showed how to associate an elliptic curve to a certain class of polynomials.

**Theorem 4.1 (Klein [12]).** Fix $q(x) = x^5 + A x^2 + B x + C$ as a polynomial with rational coefficients, and assume that $q(x)$ has Galois group $A_5$. Once one solves for $j$ in the system

\[
A = -\frac{20}{j} \left[ 2 m^3 + 3 m^2 n + 432 \frac{6 m n^2 + n^3}{1728 - j} \right]
\]

\[
B = -\frac{5}{j} \left[ m^4 - 864 \frac{3 m^2 n^2 + 2 m n^3}{1728 - j} + 559872 \frac{n^4}{(1728 - j)^2} \right]
\]

\[
C = -\frac{1}{j} \left[ m^5 - 1440 \frac{m^3 n^2}{1728 - j} + 62208 \frac{15 m n^4 + 4 n^5}{(1728 - j)^2} \right]
\]

then every root of $q(x)$ can be expressed in terms of the 5-torsion on any elliptic curve $E$ with $j = j(E) \in \mathbb{Q}(\sqrt{5})$.

As an example, consider the quintic $x^5 + 10 x^3 - 10 x^2 + 35 x - 18$. This is not in the form of the principal quintic above, but after making the substitution

\[
x \mapsto \frac{(1 + \sqrt{5}) x + (10 - 30 \sqrt{5})}{2 x + (35 + 5 \sqrt{5})}
\]

the polynomial of interest is
(4.3) \[ x^5 - 125 \left( 185 + 39\sqrt{5} \right) x^2 - 6875 \left( 56 + 19\sqrt{5} \right) x - 625 \left( 10691 + 2225\sqrt{5} \right) \]

with ramified primes \( \Sigma = \{2, \sqrt{5}\} \). One solves the equations above to find the elliptic curve

(4.4) \[ E_0 : \quad y^2 = x^3 + (5 - \sqrt{5}) x^2 + \sqrt{5} x; \quad j(E_0) = 86048 - 38496 \sqrt{5}. \]

**4.2. Step #2: Constructing Icosahedral Representations.** Once the elliptic curve is found, one constructs the icosahedral representation by following a rather simple algorithm.

**Lemma 4.2** (Goins [8], Klute [13]). *Let \( E \) be an elliptic curve as constructed above, and \( \Sigma \) a set containing ramified primes. Then there exists an icosahedral representation \( \rho_E \) with L-series*

\[
L_{\Sigma}(\rho_E, s) = \prod_{p \notin \Sigma} \left( 1 - \frac{a_E(p)}{Np^s} + \frac{\omega_5(Np)}{Np^{2s}} \right)^{-1}
\]

*where \( \omega_5 \), a Dirichlet character modulo 5, is the nebentype; and \( a_E(p) \in \mathbb{Q}(i, \sqrt{5}) \) is the trace of Frobenius.*

We explain how this works in the simplest case, when the elliptic curve is defined over \( \mathbb{Q}(\sqrt{5}) \). Given a prime number \( p \), denote the prime ideal lying above \( p \) as

(4.6) \[ p = \left\{ a + \frac{-1 + \sqrt{5}}{2} b \in \mathbb{Q}(\sqrt{5}) \mid a, b \in \mathbb{Z}; \ p \text{ divides } a^2 - ab - b^2 \right\} \]

The nebentype may be expressed as

(4.7) \[ \omega_5(Np) = \left( \frac{5}{p} \right) \quad \text{where} \quad Np = \begin{cases} p & \text{if } p \equiv 1, 4 \mod 5; \\ p^2 & \text{if } p \equiv 2, 3 \mod 5. \end{cases} \]

To calculate the trace of Frobenius, one would factor the polynomial

(4.8) \[ (x + 3)^3 (x^2 + 11x + 64) - j(E) \mod p \]

consult the table

<table>
<thead>
<tr>
<th>Irred. Factors</th>
<th>Linear</th>
<th>Quadratics</th>
<th>Cubic</th>
<th>Quintic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_E(p)^2 )</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>( \frac{-1 \pm \sqrt{5}}{2} )^2</td>
</tr>
</tbody>
</table>

and finally decide upon which square root by the congruence

(4.9) \[ a_E(p) \equiv Np + 1 - |\tilde{E}(\mathbb{F}_p)| \mod \left(2 - i, \sqrt{5}\right) \]

where \( |\tilde{E}(\mathbb{F}_p)| \) is the number of points on the elliptic curve mod \( p \).

As an application, we take a closer look at the elliptic curve associated to the polynomial \( x^5 + 10x^3 - 10x^2 + 35x - 18 \).
Proposition 4.3. Let $E_0$ be the elliptic curve $y^2 = x^3 + (5 - \sqrt{5})x^2 + \sqrt{5}x$.

1. $E_0$ is isogeneous over $\mathbb{Q}(\sqrt{5}, \sqrt{-2})$ to each of its Galois conjugates. That is, $E_0$ is a $\mathbb{Q}$-curve.

2. There is a character $\chi_0$ such that $\chi \otimes \rho_{E_0}$ is the base change of an icosahedral representation $\rho$ with conductor $N(\rho) = 800$ and nebentype $\epsilon_{\rho} = (\frac{-1}{\cdot})$.

Specifically, $\rho$ is the icosahedral Galois representation studied in [3].

Proof. We sketch the ideas. The L-series of the twisted representation $\chi \otimes \rho_{E_0}$ is defined as

\begin{equation}
L_{\Sigma}(\rho_{E_0}, \chi, s) = \prod_{p \not\in \Sigma} \left( 1 - \frac{\chi(p) a_{E_0}(p)}{Np^s} + \frac{\chi(p)^2 \omega_5(Np)}{Np^{2s}} \right)^{-1}
\end{equation}

If we were to find a character $\chi$ such that 1) it is unramified outside of $\Sigma$; 2) $\chi(\sigma p) a_{E_0}(\sigma p) = \chi(p) a_{E_0}(p)$ for all $\sigma \in G_{\mathbb{Q}}$; and 3) $\chi(p)^2 \omega_5(Np) = \epsilon_\rho(Np)$; then the L-series would be in the form

\begin{equation}
L_{\Sigma}(\rho_{E_0}, \chi, s) = \prod_{p \not\in \Sigma} \left( 1 - \frac{\alpha(Np)}{Np^s} + \frac{\epsilon_\rho(Np)}{Np^{2s}} \right)^{-1} = L_{\Sigma}(\rho_{\mathbb{Q}(\sqrt{5})}, s)
\end{equation}

for some representation $\rho$ defined over $\mathbb{Q}$. Clearly $\rho$ has nebentype $\epsilon_{\rho}$ and is unramified outside of $\Sigma$, so that $\rho$ is the unique representation studied in [3]. It suffices to construct the character $\chi$.

The elliptic curve $E_0$ is isogeneous to its congugates, which means

\begin{equation}
a_{E_0}(\sigma p) = \left(\frac{-2}{Np}\right) a_{E_0}(p) \implies \chi(\sigma p) = \left(\frac{-2}{Np}\right) \chi(p) \implies \chi(p)^2 = \omega_5(Np)^{-1} \left(\frac{-1}{Np}\right)
\end{equation}

One constructs the character explicitly by considering the ideal of $\mathbb{Q}(\sqrt{5})$ lying above 40.

We have shown the existence of the icosahedral representation in [3] without the worry of computing the discriminant of the splitting field. Moreover, using the algorithm above the coefficients can be calculated explicitly.

4.3. Step #3: Constructing Weight 2 Cusp Forms. We will exploit the fact that $E_0$ is isogeneous to its Galois conjugates. While it is not necessary in general that $E$ be a $\mathbb{Q}$-curve in order to find an icosahedral representation using the steps outlined in the previous subsection, we do need this fact in this specific case to work with cusp forms. Indeed, using the formulas in the previous subsection one can show that there is always a character such that the twisted icosahedral representation comes from $\mathbb{Q}$, but a priori there seems to be little evidence that the elliptic curve will always be isogeneous to its conjugates.

Proposition 4.4. Let $E_0$ be the elliptic curve $y^2 = x^3 + (5 - \sqrt{5})x^2 + \sqrt{5}x$ and $\Sigma = \{2, \sqrt{5}\}$. There is a cusp form $f_0$ of weight 2, level 160 such that

\begin{equation}
L_{\Sigma}(\rho, s) \equiv L_{\Sigma}(f_0, s) \mod \left(2 - i, \sqrt{5}\right)
\end{equation}
Proof. We compute the discriminant of the elliptic curve to see that it has good reduction outside of $\Sigma$. It is straightforward to use the Modular Symbol Algorithm [5] to calculate coefficients and match a cusp form $f_0'$ over $\mathbb{Q}(\sqrt{5})$ which is also unramified outside $\Sigma$. By 4.3, the twist $\chi \otimes f_0'$ is Galois invariant so it must be the base change of a cusp form $f_0$ over $\mathbb{Q}$. We use the Modular Symbol Algorithm again to find that the level is 160.

4.4. Step #4: Constructing Weight 5 Cusp Forms. Using an idea of Ken Ribet [17], we may strip 5 altogether from the level as long as we increase the weight.

Theorem 4.5. Let $E_0$ and $\rho$ be as in 4.4. There is a cusp form $f_1$ of weight $5$, level 32 and nebentype $\epsilon_{\rho} = \left( \frac{-1}{\cdot} \right)$ such that

\begin{equation}
L(\rho, s) \equiv L(f_0, s) \equiv L(f_1, s) \mod \left( 2 - i, \sqrt{5} \right).
\end{equation}

Proof. As defined in [18], consider the $\ell$-adic Eisenstein series

\begin{equation}
E(\tau) = 1 + \sum_{n=1}^{\infty} a_n(E) q^n; \quad a_n(E) = 2 \frac{\sum_{d|n} \epsilon_\ell(d)^{-k} d^{k-1}}{L(\epsilon_\ell^{-k}, 1 - k)} \in \ell \mathbb{Z}_\ell.
\end{equation}

where $\epsilon_\ell$ is the cyclotomic character. By [18, Lemme 10], this has weight $k$, level $\ell$, nebentype $\epsilon_\ell^{-w}$, and satisfies $E \equiv 1 \mod (\ell)$. Setting $w = 3$ and $\ell = 5$, the product $f_0 \cdot E$ has weight 5, level 160, and nebentypus $\epsilon_{\rho}$ so by [6, Lemme 6.11], there is a bona fide eigenform $f_1 \equiv f_0 \cdot E$. Using the Modular Symbol Algorithm one more time we see that $f_1$ has level 32. \qed

References


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