POINTS ON HYPERBOLAS AT RATIONAL DISTANCE

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Richard Guy asked for the largest set of points which can be placed in the plane so that their pairwise distances are rational numbers. In this article, we consider such a set of rational points restricted to a given hyperbola. To be precise, for rational numbers $a, b, c,$ and $d$ such that the quantity $D = (ad - bc)/(2a^2)$ is defined and non-zero, we consider rational distance sets on the conic section $axy + bx + cy + d = 0$. We show that, if the elliptic curve $Y^2 = X^3 - D^2X$ has infinitely many rational points, then there are infinitely many sets consisting of four rational points on the hyperbola such that their pairwise distances are rational numbers. We also show that any rational distance set of three such points can always be extended to a rational distance set of four such points.

Keywords: Rational distance sets; elliptic curves; hyperbolas.

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1. Introduction

In Richard Guy’s classic text [2] there is a discussion of the largest set $S$ of points $P_t$ which can be placed in the plane $\mathbb{A}^2(\mathbb{R})$ so that their pairwise distances $\|P_s - P_t\|$ are rational numbers. One can find infinitely many such points if one restricts to a line. Indeed, this is the case for the set $S = \{\ldots, P_1, \ldots\}$ consisting of those points $P_t = (x_t : y_t : 1)$ on the line $L : ax + by = c$, all in terms of the rational numbers

\[ x_t = ac - bt, \quad a = \frac{2r}{r^2 + 1}, \quad \|P_s - P_t\| = |s - t|. \]

Similarly, one can find infinitely many such points if one restricts to a circle. Indeed, this is the case for the set $S = \{\ldots, P_t, \ldots\}$ consisting of those points
For rational numbers $a$, $b$, $c$, and $d$ such that the quantity $D = (ad - bc)/(2a^2)$ is defined and non-zero, consider the conic section

$$C: axy + bx + cy + d = 0.$$  
If the elliptic curve $E(D): Y^2 = X^3 - D^2X$ has infinitely many rational points $(X: Y : 1)$, then there are infinitely many rational distance sets $S = \{P_1, P_2, P_3, P_4\}$ consisting of four points on $C$ such that their pairwise distances $\|P_s - P_t\|$ are rational numbers.

In Theorem 6 we show explicitly that we may choose the rational points

$$P_1 = \left(-ac + (ad - bc)\frac{Y_1X_2X_3}{X_1Y_2Y_3} : -ab - a^2\frac{X_1Y_2Y_3}{X_1Y_2X_3} : a^2\right),$$  
$$P_2 = \left(-ac + (ad - bc)\frac{X_1Y_2X_3}{Y_1X_2Y_3} : -ab - a^2\frac{Y_1X_2Y_3}{X_1Y_2X_3} : a^2\right),$$  
$$P_3 = \left(-ac + (ad - bc)\frac{X_1X_2Y_3}{Y_1X_2Y_3} : -ab - a^2\frac{X_1Y_2X_3}{X_1X_2Y_3} : a^2\right),$$  
$$P_4 = \left(-ac + \frac{ad - bc}{4D^2}\frac{Y_1Y_2X_3}{X_1X_2X_3} : -ab - 4D^2a^2\frac{X_1X_2X_3}{Y_1Y_2Y_3} : a^2\right).$$

As a consequence of eliminating the $X_i$’s and $Y_i$’s from these formulas, we find that every rational distance set of three points can be extended to a rational distance set of four points.
Corollary 2. Say that \( \{P_1, P_2, P_3\} \) is a rational distance set of three points \( P_i = (x_i : y_i : 1) \) on the hyperbola \( C \). If we choose the point

\[
P_4 = \left( c + \frac{(ay_1 + b)(ay_2 + b)(ay_3 + b)}{ad - bc} : b + \frac{(ax_1 + c)(ax_2 + c)(ax_3 + c)}{ad - bc} : -a \right),
\]

then \( S = \{P_1, P_2, P_3, P_4\} \) is a rational distance set of four points on the hyperbola.

This work generalizes results found by Ly, Tsostie and Uresta in [3] during the Mathematical Sciences Research Institute’s Undergraduate Program.

2. Main Results

For the sequel, we fix rational numbers \( a, b, c, \) and \( d \) such that the quantity \( a(ad - bc) \neq 0 \). We also fix the notation

\[
\mathcal{C} = \{ (x : y : z) \in \mathbb{P}^2 \mid axy + bxz + cyz + d z^2 = 0 \},
\]

\[
D = \frac{ad - bc}{2a^2},
\]

\[
E^{(D)} = \{ (X : Y : Z) \in \mathbb{P}^2 \mid Y^2 Z = X^3 - D^2 X Z^2 \}.
\]

Let \( S \) be rational distance set on \( C \), that is, let \( S \) be a collection of affine rational points \( P_i = (x_i : y_i : 1) \) on the hyperbola \( C \) such that their pairwise distances \( ||P_i - P_j|| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \) are rational numbers. We give a way to generate such a set \( S \).

Proposition 3. Fix an integer \( n \geq 3 \). Consider a collection of \( (n - 1)/2 \) rational non-torsion points \( (X_{ij} : Y_{ij} : 1) \) on the elliptic curve \( E^{(D)} \) satisfying the \( n(n - 3)/2 \) compatibility relations

\[
\frac{Y_{ij}}{X_{ij}} = \frac{Y_{st}}{X_{st}} \quad \text{for all distinct indices } i, j, s, \text{ and } t.
\]

Then the \( n \) rational points

\[
P_t = \left( -ac + (ad - bc) \frac{Y_{ij}X_{st}X_{it}}{X_{ij}Y_{st}Y_{jt}} : -ab - a^2 \frac{X_{ij}Y_{st}Y_{jt}}{Y_{ij}X_{st}X_{jt}} : a^2 \right)
\]

for any distinct \( i, j \neq t \) form a rational distance \( S \) on the hyperbola \( C \).

Proof. We explain how to derive this construction. Let \( S = \{\ldots, P_1, \ldots\} \) be any rational distance set on \( C \). For each pair \( \{P_i, P_j\} \) of distinct points in \( S \), define the rational numbers

\[
X_{ij} = \frac{D(x_i - x_j)}{(y_i - y_j) + \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}}.
\]
Upon solving for $y$ in the equation $axy + bx + cy + d = 0$, we find the slope

$$\frac{X^2_i - D^2}{2DX_{ij}} = \frac{y_i - y_j}{x_i - x_j} = \frac{ad - bc}{(ax_i + c)(ax_j + c)}.$$ 

If $|S| = n$, there are $\binom{n}{2} = n(n-1)/2$ such quadratic relations among the $x_t$, so we try and solve for the $\binom{n}{2}$ $n$ variables $x_t$ in terms of the $X_{ij}$. We have the identity

$$\frac{ad - bc}{(ax_t + c)^2} = \left[ \frac{X^2_{it} - D^2}{2DX_{it}} \right] \left[ \frac{X^2_{jt} - D^2}{2DX_{jt}} \right]^{-1} \frac{X^2_{ij} - D^2}{2DX_{ij}}$$

for distinct $i, j \neq t$.

It makes sense to define the numbers

$$n_{ij} = \frac{\sqrt{X^3_{ij} - D^2X_{ij}}}{X_{ij}} = \frac{ad - bc}{\sqrt{(ax_i + c)(ax_j + c)}}$$

which must satisfy the $\binom{n}{2} - \binom{n}{1} = n(n-3)/2$ compatibility conditions

$$n_{ij}n_{st} = \frac{(ad - bc)^2}{\sqrt{(ax_i + c)(ax_j + c)(ax_s + c)(ax_t + c)}} = n_{is}n_{jt}$$

for all distinct indices $i, j, s$, and $t$. Even though these numbers are not rational in general, we still find the expression

$$P_t = (x_t : y_t : 1) = \left( -ac + (ad - bc) \frac{n_{ij}}{n_{is}n_{jt}} : -ab - a^2 \frac{n_{it}n_{jt}}{n_{ij}} : a^2 \right)$$

for distinct $i, j \neq t$.

We now prove the result above. Say that we are given a collection of $n(n-1)/2$ rational non-torsion points $(X_{ij} : Y_{ij} : 1)$ on the elliptic curve $E^{(D)}$ satisfying the $n(n-3)/2$ compatibility relations above. Then the rational numbers $n_{ij} = Y_{ij}/X_{ij}$ are non-zero, so that the $P_t$ as above are on the conic section. Since we have the identity

$$|P_s - P_t| = \left| x_s - x_t \sqrt{1 + \frac{y_s - y_t}{x_s - x_t}} \right|^2 = \left| \frac{n_{ij}}{n_{is}n_{js}} - \frac{n_{ij}}{n_{it}n_{jt}} \right| \left| \frac{X^2_{ij} + D^2}{X_{st}} \right|$$

we see that $S = \{\ldots, P_t, \ldots\}$ is indeed a rational distance set.

This result motivates study of the set

$$E^{(D)}_n = \left\{ \prod_{i,j}(X_{ij} : Y_{ij} : Z_{ij}) \in (\mathbb{P}^2)^{\binom{n}{2}} : \begin{cases} Y_{ij}^2 Z_{ij} = X_{ij}^3 - D^2X_{ij}Z_{ij} \\ X_{is}X_{jt}Y_{ij}Y_{st} = X_{ij}X_{st}Y_{is}Y_{jt} \end{cases} \right\}$$

which is a projective variety of dimension $n(n-1)/2$. The elliptic curve has involutions $E^{(D)} \to E^{(D)}$ defined by

$$(X' : Y' : Z') = \begin{cases} [±1](X : Y : Z) \oplus (0 : 0 : 1) \\ (-D^2XZ : ±D^2YZ : X^2) \end{cases} \Rightarrow \frac{Y'}{X'} = ±\frac{Y}{X}.$$
so the projective variety has involutions $E^{(D)}_{n} \to E^{(D)}_{n}$ as well. This corresponds to the involution $n_{ij} \mapsto \pm n_{ij}$ in the formulas above.

We give some examples of this projective variety.

**Case** $n = 3$. Then \( \binom{n}{2} - \binom{n}{1} = 0 \), so that $E^{(D)}_{3} = E^{(D)} \times E^{(D)} \times E^{(D)}$ is that projective variety of dimension 3 which is the product of elliptic curves.

![Diagram](image)

In other words, the formulae in Proposition 3 uniquely determine $P_1$, $P_2$, and $P_3$. We use this to generate rational distance sets $S$ on the hyperbola $C$.

**Corollary 4.** If $(X_1 : Y_1 : 1)$ are three rational points on $E^{(D)}$ which are not points of finite order, then $S = \{ P_1, P_2, P_3 \}$ is a rational distance set on $C : axy + bx + cy + d = 0$ when we choose

$$
P_1 = \left( -ac + (ad - bc) \frac{Y_1X_2X_3}{X_1Y_2Y_3} : -ab - a^2 \frac{X_1Y_2Y_3}{X_1X_2X_3} : a^2 \right),
$$

$$
P_2 = \left( -ac + (ad - bc) \frac{X_1Y_2X_3}{Y_1X_2Y_3} : -ab - a^2 \frac{Y_1X_2Y_3}{X_1Y_2X_3} : a^2 \right),
$$

$$
P_3 = \left( -ac + (ad - bc) \frac{X_1X_2Y_3}{Y_1Y_2X_3} : -ab - a^2 \frac{Y_1X_2Y_3}{X_1Y_2X_3} : a^2 \right).
$$

In particular, when $E^{(D)}$ has positive rank, there are infinitely many rational distance sets $S$ of three points on the hyperbola $C$. 
Proof. The desired rational points

\[ P_1 = \left( -ac + (ad - bc) \frac{Y_{23}X_{12}X_{13}}{X_{23}Y_{12}Y_{13}} : -ab - a^3 \frac{X_{23}Y_{12}Y_{13}}{Y_{23}X_{12}X_{13}} : a^2 \right) \]

\[ P_2 = \left( -ac + (ad - bc) \frac{Y_{14}X_{12}X_{23}}{X_{13}Y_{12}Y_{23}} : -ab - a^3 \frac{X_{14}Y_{12}Y_{23}}{Y_{13}X_{12}X_{23}} : a^2 \right) \]

\[ P_3 = \left( -ac + (ad - bc) \frac{Y_{12}X_{13}X_{23}}{X_{12}Y_{13}Y_{23}} : -ab - a^3 \frac{X_{12}Y_{13}Y_{23}}{Y_{12}X_{13}X_{23}} : a^2 \right) \]

come about by relabeling the three points on \( E^{(D)} \) as

\[ (X_1 : Y_1 : Z_1) = (X_{23} : Y_{23} : Z_{23}), \]

\[ (X_2 : Y_2 : Z_2) = (X_{13} : Y_{13} : Z_{13}), \]

\[ (X_3 : Y_3 : Z_3) = (X_{12} : Y_{12} : Z_{12}). \]

The result follows directly from Proposition 3.

As a specific example, consider the conic section \( C : xy + 12 = 0 \). Then \( D = 6 \), and the elliptic curve \( E^{(6)} \) has the three rational points \((12 : 36 : 1), (50 : 35 : 8), \) and \((3774 : 2065932 : 12167)\); which correspond to \( n_1 = 3, n_2 = 7/10, \) and \( n_3 = 4653/851, \) respectively. We find the affine rational points

\[
\begin{align*}
P_1 &= \left( \frac{34040}{3619} : -\frac{10857}{8510} : 1 \right) \Rightarrow \|P_1 - P_2\| = \frac{1555297}{65142}, \\
P_2 &= \left( \frac{11914}{23265} : -\frac{139590}{5957} : 1 \right) \Rightarrow \|P_1 - P_3\| = \frac{2555297}{65142}, \\
P_3 &= \left( \frac{186120}{5957} : -\frac{5957}{15510} : 1 \right) \Rightarrow \|P_2 - P_3\| = \frac{2555297}{65142}
\end{align*}
\]

lying on the hyperbola, so that \( S = \{P_1, P_2, P_3\} \) is a rational distance set on \( C \). A plot of this configuration can be found in Fig. 1.

Case \( n = 4 \). Then \( \binom{n}{2} - \binom{n}{4} = 2 \), so that \( n_{12}n_{34} = n_{13}n_{24} = n_{14}n_{23} \) are the two compatibility conditions which define the projective variety \( E^{(D)}_4 \) of dimension 6. We have a result that is similar to \( E^{(D)}_3 = E^{(D)} \times E^{(D)} \times E^{(D)} \).

Proposition 5. \( E^{(D)}_4 \) is the 3-fold fiber product of the two-dimensional variety

\[
E^{(D)} = \left\{ (u : v : w), T \in \mathbb{P}^2 \times \mathbb{P}^1 \left| v^4w^3 = u(u + D^2w) \times (u + D^2T^4w)(u^2 - D^4T^4w^2)^2 \right) \right\}
\]

corresponding to the projection \( E^{(D)} \to \mathbb{P}^1 \) which sends \((u : v : w), T) \mapsto T \).
Fig. 1. Rational distance set on $xy + 12 = 0$.

**Proof.** We will explain the diagram:

- $E(x)$
- $E(x) \times E(x) \times E(x)$
- $p_1$
- $(u_1 : v_1 : w_1, T)$
- $(u_2 : v_2 : w_2, T)$
- $(u_3 : v_3 : w_3, T)$
- $T$
A rational point on the projective variety $E^{(D)}$ is in the form

$$P = \left( (X_1 : Y_1 : Z_1), (X_2 : Y_2 : Z_2), (X_3 : Y_3 : Z_3), \right)$$

where $(X_1 : Y_1 : Z_1)$ and $(X'_1 : Y'_1 : Z'_1)$ are six rational points on the elliptic curve $E^{(D)}$, that is,

$$Y_1^2 Z_1 = X_1^3 - D^2 X_1 Z_1^2,$$

$$Y_2^2 Z_2 = X_2^3 - D^2 X_2 Z_2^2,$$

$$Y_3^2 Z_3 = X_3^3 - D^2 X_3 Z_3^2,$$

and we have the two compatibility relations

$$\frac{Y_1}{X_1} = \frac{Y_2}{X_2} = \frac{Y_3}{X_3}.$$

Given such a rational point $P$ on $E_4^{(D)}$, define the three rational points $((u_t : v_t : w_t), T)$ on $E^{(D)}$ in terms of

$$u_t = -4D^3 X_t^2 X_t' Y_t' Z_t' (X_t^2 - D^2 Z_t^2)(X_t'^2 - D^2 Z_t'^2),$$

$$v_t = 8D^5 X_t^3 X_t'^2 Z_t [D^2 Z_t'^2(X_t' + DZ_t')^2 - X_t'^2(X_t' - DZ_t')^2],$$

$$w_t = Y_t'[X_t^2 - D^2 Z_t'^2(X_t' - DZ_t')(X_t' + DZ_t')^3],$$

$$T = 2D \frac{X_t' Y_t'}{Y_t},$$

where $T$ is independent of $t$. Hence the expression

$$Q = ((u_1 : v_1 : w_1), (u_2 : v_2 : w_2), (u_3 : v_3 : w_3), T)$$

represents a point on the 3-fold fiber product of $E^{(D)}$ with respect to the projection $E^{(D)} \to \mathbb{P}^1$ which sends $((u : v : w), T) \mapsto T$.

Conversely, given such a rational point $Q$ on the 3-fold fiber product, define the rational points six rational points $(X_t : Y_t : Z_t)$ and $(X'_t : Y'_t : Z'_t)$ on the elliptic curve $E^{(D)}$ in terms of

$$X_t = T u_t^3 w_t,$$

$$Y_t = v_t(u_t^2 - D^4 T^4 w_t^2),$$

$$Z_t = T^3 (u_t + D^2 w_t)(u_t^2 - D^4 T^4 w_t^2),$$

$$X'_t = D w_t^2(u_t^2 - D^4 T^4 w_t^2)$$

$$\times [2D v_t^2 w_t^2 - (u_t^2 - D^4 T^4 w_t^2)(u_t^2 + 2D^2 u_t w_t + D^4 T^4 w_t^2)],$$

$$Y'_t = 2D^2 v_t w_t^3 [2D v_t^2 w_t^2 - (u_t^2 - D^4 T^4 w_t^2)(u_t^2 + 2D^2 u_t w_t + D^4 T^4 w_t^2)],$$

$$Z'_t = w_t^2(u_t^2 - D^4 T^4 w_t^2)^3.$$
It is easy to verify that the two compatibility relations hold:

$$\frac{Y_t}{X_t} = \frac{u_t^2 - D^4 T^4 w_t^2}{T v_t w_t}, \quad \frac{Y'_t}{X'_t} = \frac{2 D v_t w_t}{u_t^2 - D^4 T^4 w_t^2} \Rightarrow \frac{Y_t}{X_t} \cdot \frac{Y'_t}{X'_t} = \frac{2 D}{T}$$

so that we have a rational point \( P \) on the projective variety \( E^{(D)}_4 \). \hfill \Box

While the projective variety \( E^{(D)}_4 \) is somewhat cumbersome, the surface \( E^{(D)} \) gives us some insight on how to generate a family of rational distance sets \( S \) consisting of four rational points \( P_t \) on the hyperbola \( C \).

**Theorem 6.** Continue notation as in Proposition 5.

1. There exists a non-trivial maps such that the composition

$$E^{(D)} \longrightarrow E^{(D)} \longrightarrow \mathbb{P}^1$$

is a constant map.

2. If \( (X_t : Y_t : 1) \) are three rational points on \( E^{(D)} \) which are not points of finite order, then \( S = \{ P_1, P_2, P_3, P_4 \} \) is a rational distance set on \( C : axy + bx + cy + d = 0 \) when we choose

\[
P_1 = \left( -ac + (ad - bc) \frac{Y_2 X_2 X_3}{X_1 Y_2 Y_3}; -ab - a^2 \frac{Y_1 Y_2 Y_3}{X_1 X_2 X_3}; a^2 \right),
\]

\[
P_2 = \left( -ac + (ad - bc) \frac{X_1 Y_2 X_3}{Y_1 X_2 Y_3}; -ab - a^2 \frac{X_1 Y_2 Y_3}{X_1 X_2 X_3}; a^2 \right),
\]

\[
P_3 = \left( -ac + (ad - bc) \frac{X_1 X_2 Y_3}{Y_1 X_2 Y_3}; -ab - a^2 \frac{X_1 Y_2 Y_3}{X_1 X_2 X_3}; a^2 \right),
\]

\[
P_4 = \left( -ac + \frac{ad - bc}{4D^2} \frac{Y_1 Y_2 Y_3}{X_1 X_2 X_3}; -ab - 4D^2 a^2 \frac{X_1 X_2 Y_3}{Y_1 Y_2 Y_3}; a^2 \right).
\]

3. When \( E^{(D)} \) has positive rank, there are infinitely many rational distance sets \( S \) of four points on the hyperbola \( C \).

**Proof.** Define the map \( E^{(D)} \to E^{(D)} \) by

\[
(X : Y : Z) \mapsto ((X^2 Z : (X^2 + D^2 Z^2) Y : Z^3), 1).
\]

Clearly this is a non-trivial map whose composition with the projection \( E^{(D)} \to \mathbb{P}^1 \) which sends \((u : v : w), T) \mapsto T \) is a constant map.

Say that \( (X_t : Y_t : 1) \) are three rational points on \( E^{(D)} \) which are not points of finite order. Then we have three points \((u_t : v_t : z_t), 1\) on the surface \( E^{(D)} \) which corresponds to a rational point on the 3-fold fiber product. Using the expressions
in Eq. (*), we find six rational points on \( E^{(D)} \) given by

\[
\begin{align*}
(X_{14} : Y_{14} : Z_{14}) &= (X'_1 : Y'_1 : Z'_1) \\
&= (D(X_1^2 - D^2Z_1^2) : 2D^2Y_1Z_1 : -(X_1 + DZ_1)^2), \\
(X_{24} : Y_{24} : Z_{24}) &= (X'_2 : Y'_2 : Z'_2) \\
&= (D(X_2^2 - D^2Z_2^2) : 2D^2Y_2Z_2 : -(X_2 + DZ_2)^2), \\
(X_{34} : Y_{34} : Z_{34}) &= (X'_3 : Y'_3 : Z'_3) \\
&= (D(X_3^2 - D^2Z_3^2) : 2D^2Y_3Z_3 : -(X_3 + DZ_3)^2)
\end{align*}
\]

with the first three as labeled in Corollary 4. We have chosen our maps so that \((Y_{ij}/X_{ij})(Y_{kt}/X_{kt}) = 2D\) for all distinct \(i, j, s,\) and \(t\), so using the formulas in Proposition 3 we have the four rational points as above, with the last being

\[
P_4 = \left(-ac + (ad - bc) \frac{Y_{12}X_{14}X_{24}}{X_{12}Y_{14}Y_{24}} : \frac{-ab - a^2}{a} \frac{X_{12}Y_{14}Y_{24}}{X_{12}Y_{14}X_{24}} : a^2 \right),
\]

\[
= \left(-ac + (ad - bc) \frac{Y_{1}X_{1}'X_{2}'}{X_{1}'Y_{1}'Y_{2}'} : \frac{-ab - a^2}{a} \frac{X_{1}'Y_{1}'Y_{2}'}{X_{1}'Y_{1}'X_{2}'} : a^2 \right),
\]

\[
= \left(-ac + \frac{ad - bc}{4D^2} \frac{Y_{1}X_{2}X_{3}}{X_{1}X_{2}X_{3}} : \frac{-ab - 4D^2a^2}{a^2} \frac{X_{1}X_{2}X_{3}}{Y_{1}Y_{2}Y_{3}} : a^2 \right).
\]

Using Corollary 4, we find precisely what is listed above. \(\square\)

As a consequence, we find that we can always extend a rational distance set \(S\) of three points on a hyperbola to a set of four points.

**Corollary 2.** Say that \(\{P_1, P_2, P_3\}\) is a rational distance set of three points \(P_t = (x_t : y_t : 1)\) on the hyperbola \(C: axy + bx + cy + d = 0\). If we choose the point

\[
P_4 = \left(c + \frac{(ay_1 + b)(ay_2 + b)(ay_3 + b)}{ad - bc} : b \right.
\]

\[
+ \left. \frac{(ax_1 + c)(ax_2 + c)(ax_3 + c)}{ad - bc} : -a \right)
\]

then \(S = \{P_1, P_2, P_3, P_4\}\) is a rational distance set of four points on the hyperbola.

**Proof.** Following the proof of Proposition 3, we can express the points \(P_t\) in the form

\[
P_1 = \left(-ac + (ad - bc) \frac{Y_{1}X_{2}X_{3}}{X_{1}Y_{2}Y_{3}} : \frac{-ab - a^2}{a} \frac{X_{1}Y_{2}Y_{3}}{X_{1}X_{2}X_{3}} : a^2 \right),
\]

\[
P_2 = \left(-ac + (ad - bc) \frac{X_{2}X_{3}}{Y_{1}Y_{2}Y_{3}} : \frac{-ab - a^2}{a} \frac{Y_{1}X_{2}X_{3}}{Y_{1}Y_{2}X_{3}} : a^2 \right).
\]
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\[ P_3 = \left( -ac + (ad - bc) \frac{X_1X_3Y_3}{Y_1Y_2X_3}, -ab - a^2 \frac{Y_1Y_2X_3}{X_1X_2Y_3} : a^2 \right), \]

where we have chosen the not necessarily rational numbers

\[
\begin{align*}
Y_1 &= \frac{ad - bc}{a} \sqrt{\frac{1}{(ax_2 + c)(ax_3 + c)}} = \frac{(ay_2 + b)(ay_3 + b)}{a}, \\
Y_2 &= \frac{ad - bc}{a} \sqrt{\frac{1}{(ax_1 + c)(ax_3 + c)}} = \frac{(ay_1 + b)(ay_3 + b)}{a}, \\
Y_3 &= \frac{ad - bc}{a} \sqrt{\frac{1}{(ax_1 + c)(ax_2 + c)}} = \frac{(ay_1 + b)(ay_2 + b)}{a}.
\end{align*}
\]

Following Theorem 6, we choose

\[ P_4 = \left( -ac + \frac{ad - bc}{4D^2} \frac{Y_1Y_2Y_3}{X_1X_2Y_3} : -ab - 4D^2a^2 \frac{X_1X_2X_3}{Y_1Y_2Y_3} : a^2 \right) \]

which is seen to be a rational point on \( C \).

As a specific example, consider again the conic section \( C : xy + 12 = 0 \). We have seen that \( D = 6 \), and the elliptic curve \( E(6) \) has the three rational points \((12 : 36 : 1), (50 : 35 : 8), \text{ and } (377844 : 2065932 : 12167)\). These correspond to the rational points \((144 : 6480 : 1), (5000 : 42035 : 128), \text{ and } (3283620031728 : 578364811524720 : 3404825447)\), respectively, on the surface \( E^{(6)} \). We find the

![Fig. 2. Rational distance set on xy + 12 = 0.](image)
affine rational points

\[ P_1 = \left( \frac{34040}{3619}, \frac{10857}{8510}, 1 \right) \]
\[ P_2 = \left( \frac{11914}{23265}, \frac{139590}{5957}, 1 \right) \]
\[ P_3 = \left( \frac{186120}{5957}, -\frac{5957}{15510}, 1 \right) \]
\[ P_4 = \left( \frac{32571}{34040}, -\frac{136160}{10857}, 1 \right) \]

\[ \|P_1 - P_2\| = \frac{1555297}{65142}, \]
\[ \|P_1 - P_3\| = \frac{28848020}{1319901}, \]
\[ \|P_2 - P_3\| = \frac{2129555051}{55435842}, \]
\[ \|P_1 - P_4\| = \frac{1040847151}{73914456}, \]
\[ \|P_2 - P_4\| = \frac{1726556399}{158388120}, \]
\[ \|P_3 - P_4\| = \frac{1555297}{47656} \]

lying on the hyperbola, so that \( S = \{P_1, P_2, P_3, P_4\} \) is a rational distance set on \( C \). A plot of this configuration can be found in Fig. 2.

References

