The Area of a Surface Generated by Revolving a Graph About any Line

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The Area of a Surface Generated by Revolving a Graph About any Line

Edray Herber Goins and Talitha M. Washington

Abstract: We discuss a general formula for the area of the surface that is generated by a graph $[t_0, t_1] \rightarrow \mathbb{R}^2$ sending $t \mapsto (x(t), y(t))$ revolved around a general line $L : Ax + By = C$. As a corollary, we obtain a formula for the area of the surface formed by revolving $y = f(x)$ around the line $y = mx + k$.

Keywords: Surfaces, area, rotation.

1. INTRODUCTION

You spin me right round, baby, right round!
– Dead or Alive

Many calculus courses teach how to compute the area of a surface of revolution. Perhaps the best known example goes something like this: For functions $y = f(x)$ differentiable on the interval $a \leq x \leq b$, the area of the surface of revolution of its graph about the $x$-axis is given by the integral

$$2\pi \int_a^b |y| \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = 2\pi \int_a^b |f(x)| \sqrt{1 + [f'(x)]^2} \, dx.$$ (1)

An example of this type of rotation can be found in Figure 1.

There is a similar formula for computing the area of the surface of revolution about the $y$-axis, but one must work with the inverse function $x = g(y)$ and assume that it is a function differentiable on the interval $c \leq y \leq d$,
Figure 1. Surface of revolution of $x = y^3 - 12y^2 + 444y + 62$, (a) graph of function and (b) revolved around the $y$-axis (color figure available online).

\[
2\pi \int_c^d |x| \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = 2\pi \int_c^d |g(y)| \sqrt{1 + [g'(y)]^2} \, dy. \tag{2}
\]

As above, an example of this type of rotation can be found in Figure 2.

These are special cases of graphs which can be parametrized by continuous functions, say $x = x(t)$ and $y = y(t)$, which are both differentiable on the interval $t_0 \leq t \leq t_1$. The area of the surface of revolution of a parametrized graph is given by the integral.

Figure 2. Surface of revolution of $x = y^3 - 12y^2 + 444y + 62$, (a) graph of function and (b) revolved around the $y$-axis (color figure available online).
Area of a Surface

\[2\pi \int_{t_0}^{t_1} |y(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \] when revolved about the x-axis,

\[2\pi \int_{t_0}^{t_1} |x(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \] when revolved about the y-axis.

Such formulas can be found in Stewart’s *Calculus* [2, Section 8.2, Section 10.2]. A general discussion of integrals over surfaces can be found in Apostol’s *Calculus* [1, Chapter 12].

2. GENERALIZED FORMULAE

These seemingly unrelated formulas in equation (3) may be intimidating for a learner of calculus. Which one should be used to solve problems? Why are the formulas so different? And where do these formulas come from in the first place? In this note, we hope to ease any fear as we will provide a relatively simple proof of the following generalization.

**Proposition 1.** Consider the graph parametrized by functions \(x = x(t)\) and \(y = y(t)\), both differentiable on the interval \(t_0 \leq t \leq t_1\). The area of the surface of revolution of this graph, when revolved about a line \(L: Ax + By = C\), is given by the integral

\[2\pi \int_{t_0}^{t_1} r(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = 2\pi \int_{t_0}^{t_1} r(t) \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt,\]

in terms of the non-negative function

\[r(t) = \frac{|Ax(t) + By(t) - C|}{\sqrt{A^2 + B^2}}.\]

Before we present the proof, we explain how to use this generalization to derive equations (1) and (2). For the former, the graph of \(y = f(x)\) can be parametrized by the functions \(x(t) = t\) and \(y(t) = f(t)\) on the interval \(a \leq t \leq b\). The x-axis is simply the line \(L: y = 0\), so we may choose \((A, B, C) = (0, 1, 0)\). Then \(r(t) = |f(t)|\), and we find equation (1). For the latter, the graph of \(x = g(y)\) can be parametrized by the functions \(x(t) = g(t)\) and \(y(t) = t\) on the interval \(C \leq t \leq d\). The y-axis is simply the line \(L: x = 0\), so we may choose \((A, B, C) = (1, 0, 0)\). Then \(r(t) = |g(t)|\), and we find equation (2).
3. APPLICATION

On pages 551 and 552 of Stewart’s Calculus [2], there is a Discovery Project entitled “Rotations on a Slant,” where a differentiable function \( y = f(x) \) is rotated about the line \( y = mx + k \) expressed in point-slope form. As a solution to Exercise 5 in the aforementioned Discovery Project, we recover the following result.

**Corollary 1.** Consider a function \( y = f(x) \) which is differentiable on the interval \( a \leq x \leq b \). The area of the surface of revolution of its graph about the line \( y = mx + k \) is given by the integral

\[
2\pi \int_a^b |y - mx - k| \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
\]

\[
= \frac{2\pi}{\sqrt{1 + m^2}} \int_a^b |f(x) - mx - k| \sqrt{1 + [f'(x)]^2} \, dx.
\]

We explain why this corollary is true. As above, write \( y = f(x) \) in terms of the functions \( x(t) = t \) and \( y(t) = f(t) \) on the interval \( a \leq t \leq b \). The line \( L: y = mx + k \) can be expressed in the form \( Ax + By = C \) upon choosing \((A, B, C) = (-m, 1, k)\), so that we have the non-negative function

\[
r(t) = \frac{|Ax(t) + By(t) - C|}{\sqrt{A^2 + B^2}} = \frac{|f(t) - mt - k|}{\sqrt{1 + m^2}}.
\]

Proposition 1 implies that we have area

\[
2\pi \int_{t_0}^{t_1} r(t) \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt
\]

\[
= \frac{2\pi}{\sqrt{1 + m^2}} \int_a^b |f(x) - mx - k| \sqrt{1 + [f'(x)]^2} \, dx.
\]

Some examples of this result can be found in Figure 3.

4. PROOF OF MAIN RESULT

The idea for the proof will be to create a new coordinate system using the line \( L \) as an axis. To this end, let us define the following three points in the plane
Figure 3. Revolutions of $y = x^2 - 3x + 12$ around lines: (a) $y = 0$; (b) $3x + 4y = 0$; (c) $x = 0$; and (d) $3x - 4y = 0$ (color figure available online).

$$
O = \left( \frac{AC}{A^2 + B^2}, \frac{BC}{A^2 + B^2} \right);
$$
$$
u = \left( -\frac{B}{\sqrt{A^2 + B^2}}, \frac{A}{\sqrt{A^2 + B^2}} \right),
$$
$$
v = \left( \frac{A}{\sqrt{A^2 + B^2}}, \frac{B}{\sqrt{A^2 + B^2}} \right).
$$

Each of these points has a geometric interpretation. The first point $O$ will act as our “origin”; it lies on the line $L$. The second point $u$ lies along the direction of $L$, while the third point $v$ lies perpendicular to $L$. A diagram of these points with respect to $L$ can be found in Figure 4.

In order to prove Proposition 1, we will prove a series of results.

**Lemma 1.** $\{u, v\}$ forms an orthonormal basis for $\mathbb{R}^2$.

Recall that an orthonormal basis $\{u, v\}$ is a set of two vectors which are orthogonal (that is, $u \times v = 0$) and normal (that is, $\|u\| = \|v\| = 1$). This is indeed the case for the vectors above because we have the identities
Figure 4. Plot of 0, u, and v on the line $L : Ax + By = C$ (color figure available online).

\[
\mathbf{u} \times \mathbf{v} = -\frac{B}{\sqrt{A^2 + B^2}} \frac{A}{\sqrt{A^2 + B^2}} + \frac{A}{\sqrt{A^2 + B^2}} \frac{B}{\sqrt{A^2 + B^2}}
\]

\[
= -\frac{BA + AB}{A^2 + B^2} = 0,
\]

\[
\|\mathbf{u}\| = \sqrt{\left(-\frac{B}{\sqrt{A^2 + B^2}}\right)^2 + \left(\frac{A}{\sqrt{A^2 + B^2}}\right)^2} = \sqrt{\frac{B^2 + A^2}{A^2 + B^2}} = 1,
\]

\[
\|\mathbf{v}\| = \sqrt{\left(\frac{A}{\sqrt{A^2 + B^2}}\right)^2 + \left(-\frac{B}{\sqrt{A^2 + B^2}}\right)^2} = \sqrt{\frac{A^2 + B^2}{A^2 + B^2}} = 1.
\]

Next, we will use these three points $O$, $u$, and $v$ to decompose points $(x, y)$ in the plane along the axes defined previously.

**Lemma 2.** Every point in the plane can be decomposed into a part that lies on the line $L$ and a part that lies perpendicular to the line $L$. Explicitly

\[
(x, y) = \left[ O + \frac{-Bx + Ay}{\sqrt{A^2 + B^2}} \mathbf{u} \right] + \left[ \frac{Ax + By - C}{\sqrt{A^2 + B^2}} \mathbf{v} \right].
\]

In order to prove this, we must perform some algebra. The decomposition holds because we have the following which sum as claimed.
\[ \mathbf{O} = \left( \frac{AC}{A^2 + B^2}, \frac{BC}{A^2 + B^2} \right) \]

\[ \frac{-Bx + Ay}{\sqrt{A^2 + B^2}} \mathbf{u} = \left( \frac{B^2 x - AB y}{A^2 + B^2}, \frac{-AB x + A^2 y}{A^2 + B^2} \right) \]

\[ \frac{Ax + By - C}{\sqrt{A^2 + B^2}} \mathbf{v} = \left( \frac{A^2 x + AB y - AC}{A^2 + B^2}, \frac{AB x + B^2 y - BC}{A^2 + B^2} \right) \]

\[
\text{Sum of Points} = \left( \frac{A^2 + B^2}{A^2 + B^2} x, \frac{A^2 + B^2}{A^2 + B^2} y \right).
\]

Moreover, the sum of the first two points

\[ \mathbf{O} + \frac{-Bx + Ay}{\sqrt{A^2 + B^2}} \mathbf{u} = \left( \frac{B^2 x - AB y + AC}{A^2 + B^2}, \frac{-AB x + A^2 y + BC}{A^2 + B^2} \right), \]

lies on the line \( L \)

\[
A \left( \frac{B^2 x - AB y + AC}{A^2 + B^2} \right) + B \left( \frac{-AB x + A^2 y + BC}{A^2 + B^2} \right)
= \frac{(AB^2 x - A^2 By + A^2 C) + (-AB^2 x + A^2 By + B^2 C)}{A^2 + B^2} = C.
\]

The third point is perpendicular to the line because \( \mathbf{v} \) is orthogonal to \( \mathbf{u} \).

Next, we put these results together to compute a useful formula.

**Lemma 3.** The distance from a point \((x, y)\) to the line \( L \) is given by

\[ r = \frac{|Ax + By - C|}{\sqrt{A^2 + B^2}}. \]

To prove this, we essentially use the Pythagorean theorem: we project any point \((x, y)\) in the plane into two directions, one along the line \( L \) and the other perpendicular to it. The distance from this point to the line is simply the length of the perpendicular direction, that is, the length of the vector along \( \mathbf{v} \)

\[ r = \frac{||Ax + By - C||}{\sqrt{A^2 + B^2}} = \frac{|Ax + By - C|}{\sqrt{A^2 + B^2}} \cdot ||\mathbf{v}|| = \frac{|Ax + By - C|}{\sqrt{A^2 + B^2}}. \]

**Proof of Proposition 1.** Say that we have two functions \( x = x(t) \) and \( y = y(t) \), both differentiable on the interval \( t_0 \leq t \leq t_1 \), as well as a line \( L : Ax + By = C \). According to Lemma 3, the distance from a point \((x, y) = (x(t), y(t))\) to the line \( L \) is given by the function
This will act as the radius of rotation around the line. A small arc length along the graph at this point is given by the differential
\[
\begin{align*}
    ds &= \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{[x'(t)]^2 + [y'(t)]^2} dt,
\end{align*}
\]
This will act as the width of the cylindrical shell. (See Figure 5 for an illustration.) Putting these together, the area differential of the surface of revolution is the circumference \(2\pi r(t)\) times the arc length differential \(ds\):
\[
\begin{align*}
    d\text{Area} &= 2\pi r(t) \, ds \\
    &= 2\pi \frac{|Ax(t) + By(t) - C|}{\sqrt{A^2 + B^2}} \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt,
\end{align*}
\]
so that the desired area of revolution is given by
\[
\begin{align*}
    \text{Area} &= 2\pi \int_{t_0}^{t_1} r(t) \, ds \\
    &= 2\pi \int_{t_0}^{t_1} \frac{|Ax(t) + By(t) - C|}{\sqrt{A^2 + B^2}} \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt.
\end{align*}
\]
\[\text{Figure 5. Cylindrical shell of radius } r(t) \text{ and length } ds \text{ (color figure available online).}\]
5. WORKED EXAMPLE

We illustrate these ideas with an example. Say that we wish to rotate the circle \( x^2 + y^2 = r^2 \) of radius \( r \) around the line \( Ax + By = C \) – but we’ll assume for simplicity that

\[
R = \frac{C}{\sqrt{A^2 + B^2}} > r.
\]

As shown in Figure 6, this is just a torus. We will show that the area of this surface is \( 4\pi^2 Rr \).

To begin, we’ll plot the circle using the functions \( x(t) = r \cos t \) and \( y(t) = r \sin t \) for \( t_0 \leq t \leq 2\pi + t_0 \), where \( t_0 \) is an angle such that

\[
\cos t_0 = \frac{A}{\sqrt{A^2 + B^2}} \quad \text{and} \quad \sin t_0 = \frac{B}{\sqrt{A^2 + B^2}}.
\]

The distance from a point \((x(t), y(t))\) to the line is given by the function

\[
r(t) = \frac{|Ax(t) + By(t) - C|}{\sqrt{A^2 + B^2}} = |\cos t_0 \times r \cos t + \sin t_0 \times r \sin t - R| = R - r \cos (t - t_0).
\]

(We have used the angle difference formula \( \cos(t - t_0) = \cos t \cos t_0 + \sin t \sin t_0 \). Recall that \( R > r \) by assumption.) We have the arc length differential

\[
dL = \sqrt{(x'(t))^2 + (y'(t))^2} dt = \sqrt{r^2 \cos^2 t + r^2 \sin^2 t} dt = r dt.
\]

Figure 6. Circle \( x^2 + y^2 = r^2 \) revolved around \( Ax + By = C \) (color figure available online).
\[ ds = \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \sqrt{(-r \sin t)^2 + (r \cos t)^2} \, dt = r \, dt. \]

The desired area of revolution is given by the integral

\[ 2\pi \int_{t_0}^{t_1} r(t) \, ds = 2\pi \int_{t_0}^{2\pi + t_0} (R - r \cos(t - t_0)) \, dt = 4\pi^2 Rr. \]

**ACKNOWLEDGEMENTS**

This exposition is dedicated to the inquisitive students in advanced calculus courses who enjoy seeing mathematics beyond those topics covered in a standard textbook on the subject.

**APPENDIX**

All of the graphs shown in this article were generated using *Mathematica*. For the convenience of the reader, we include the code which generates these graphs. The input is a line \( Ax + By = C \), entered as \( \{A, B, C\} \); two functions \( x = x(t) \) and \( y = y(t) \), entered as \( \{x, y\} \); and an interval \( t_0 \leq t \leq t_1 \), entered as \( \{t, t_0, t_1\} \).

```mathematica
Rotation[{A_, B_, C_}, {x_, y_}, {t_, t0_, t1_}] := Module[{a, b, c, X, Y, origin, axisPerpendicular, axisParallel, line, graph, surface},

(* Normalize the coefficients *)
{a, b, c} = {A, B, C}/Sqrt[A^2 + B^2];

(* Set up the coordinate axes *)
origin = Graphics3D [{Green, Sphere[{a*c, b*c, 0}, 1]}];
axisParallel = Graphics3D [{Red, Arrow[Tube[{-25*(-b, a, 0), 25*(-b, a, 0)}]]}];
axisPerpendicular = Graphics3D [{Blue, Arrow[Tube[{-25*(a, b, 0), 25*(a, b, 0)}]]}];

(* Here is the line about which we rotate *)
line = Graphics3D [{Black, Draw[Line[{{-25*(-b, a, 0), 25*(-b, a, 0)}, {-25*(a, b, 0), 25*(a, b, 0)}}], Thickness[0.007]]}];

(* Add the rotation *)
graph = Graphics3D [{Black, Rotate[line, axisParallel, origin], Rotate[line, axisPerpendicular, origin], Rotate[line, axisParallel, origin]}, Boxed -> False, ViewPoint -> {1.3, -2.4, 2.}, ViewVertical -> {0, 0, -1}];

(* Output the graphics *)
Show[graph];
```
Area of a Surface

\[
\text{Line[\{\{a*c, b*c, 0\} - 50*\{-b, a, 0\}, \{a*c, b*c, 0\} + 50*\{-b, a, 0\}\}\}\];}
\]

("Here is the non-rotated plot of the function in the xy-plane")
graph = Graphics3D [{Purple, Tube[
  Table[{x, y, 0} /. t -> time, {time, 1.1*t0 - 0.1*t1, 1.1*t1 - 0.1 t0, 0.01*(t1 - t0)}, 0.5]}];

("Here is the rotated plot in 3-dimensions")
surface = ParametricPlot3D [
  {a*c - 5*b*time, b*c + 5*a*time, 0},
  {b*X + a*Y*Cos[angle] + a*c, -a*X + b*Y*Cos[angle] + b*c, Y*Sin[angle]} /. {X -> b*(x - a*c) - a*(y - b*c),
    Y -> a*(x - a*c) + b*(y - b*c)}/. t -> time
  ],
  {angle, 0, 2 Pi},
  {time, t0, t1},
  Axes -> None, Mesh -> False];

("Show everything on one graph")
Return[Show[{origin, axisParallel, axisPerpendicular, line, graph, surface}]];

REFERENCES


BIOGRAPHICAL SKETCHES

Dr. Edray Herber Goins works in the field of number theory, as it pertains to the intersection of representation theory and algebraic geometry. He grew up in South Central Los Angeles where he taught himself calculus. Even though there was no class at his high school that offered the Calculus B
advanced placement exam, he passed it with a perfect score. When he’s not using Mathematica to solve equations, he can be found on the beaches of California playing volleyball.

Dr. Talitha M. Washington’s current fields of interest include applying ordinary and partial differential equations to problems in biology and engineering. Her work on Dr. Elbert Frank Cox, the first Black in the world to earn a PhD in mathematics, has been shared on radio and television stations. She and Dr. Cox grew up in the same neighborhood in Evansville, Indiana and they both taught at Howard University.