Modern number theory has translated questions of rational numbers satisfying various conditions into questions of points on group schemes. For example, the infinitude of primes in arithmetic progressions may be shown using representations associated to cyclic groups; and even Fermat’s Last Theorem may be shown using representations associated to elliptic curves. Surprisingly, many branches of math come into play: the topology of Galois groups, the combinatorics of counting points over finite fields, the linear algebra of Tate modules, and the analysis of the convergence of $L$-series.

In this talk we give a tour of the mathematics used to answer questions from number theory and discuss some open problems from the Langlands Program.

(Abstract Number: 1067-11-2267)
**Dudley Weldon Woodard** (October 3, 1881 - July 1, 1965) had already been on the faculty for nine years at Howard University when he became the second African American to earn a doctorate in mathematics when he graduated from the University of Pennsylvania in 1928. In 1929, he established the graduate program in mathematics at Howard University.

**William Waldron Schieffelin Claytor** (January 4, 1908 - 1967), the most promising student in the inaugural year the new graduate program at Howard University, was recommended by Woodard for admission to the University of Pennsylvania’s Graduate School. Claytor became the third African American to earn a doctorate in mathematics when he graduated from the University of Pennsylvania in 1933.

http://www.math.upenn.edu/History/bh/text99.html
Dirichlet $L$-Series

For a positive integer $N$, choose a homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. Extend by writing

$$\mathbb{Z} \longrightarrow \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$$

Then $\chi(n_1 n_2) = \chi(n_1) \chi(n_2)$. The $L$-Series associated to $\chi$ is the sum

$$L(\chi, s) = \chi(1) + \frac{\chi(2)}{2^s} + \frac{\chi(3)}{3^s} + \cdots = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$ 

This is a generalization of the Riemann Zeta function:

$$\zeta(s) = L(1, s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$ 

Motivating Questions

- For which region is this series a well-defined function?
- Can this be continued analytically to the entire complex plane?
A sequence $S = \{n_1, n_2, \ldots, n_m, \ldots\}$ of integers $n_m$ is an arithmetic progression if there exist integers $N$ and $a$ such that $n_m = Nm + a$.

**Theorem (Johann Peter Gustav Lejeune Dirichlet, 1837)**

Assuming $\gcd(a, N) = 1$, there are infinitely many $k$ such that $n_m \in S$ is a prime number. That is, $S$ has infinitely many primes.

The proof is in three main steps for characters $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$.

1. Establish an “Euler product”: $L(\chi, s) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$.

2. Establish “orthogonality” relations:

   $$\sum \overline{\chi(a)} \log L(\chi, s) = \phi(N) \cdot \sum_{p \in S} \frac{1}{p^s} + R_{\chi, a}(s)$$

3. Establish nonvanishing, i.e., $L(\chi, 1) \neq 0$:

   $$\sum_{p \in S} \frac{1}{p} = \lim_{s \to 1} \sum \overline{\chi(a)} \phi(N) \log L(\chi, s) = \lim_{s \to 1} \frac{1}{\phi(N)} \log \zeta(s) = \infty.$$
### Examples

#### Arithmetic Progression $S$

<table>
<thead>
<tr>
<th>$S$</th>
<th>First 10 Primes $p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2m + 1$</td>
<td>3, 5, 7, 11, 13, 17, 19, 23, 29, 31</td>
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<td>$4m + 1$</td>
<td>5, 13, 17, 29, 37, 41, 53, 61, 73, 89</td>
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<td>3, 7, 11, 19, 23, 31, 43, 47, 59, 67</td>
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<td>$6m + 1$</td>
<td>7, 13, 19, 31, 37, 43, 61, 67, 73, 79</td>
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<td>5, 11, 17, 23, 29, 41, 47, 53, 59, 71</td>
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<td>17, 41, 73, 89, 97, 113, 137, 193, 233, 241</td>
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<td>$8m + 7$</td>
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<td>11, 31, 41, 61, 71, 101, 131, 151, 181, 191</td>
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</tr>
<tr>
<td>$10m + 9$</td>
<td>19, 29, 59, 79, 89, 109, 139, 149, 179, 199</td>
</tr>
</tbody>
</table>

Are there other arithmetic properties which can be encoded via a series?
Climbing Staircases

Question (Mohammad K. Azarian, 2004)
Let $S$ be a subset of the positive integers; this will denote the “sizes” of steps allowed. Let $M$ be a positive integer; this will denote the maximum multiplicity of each step-size $s \in S$ taken. What are the possible ways to climb a staircase containing $n$ stairs taking step-sizes $S$ and multiplicities at most $M$?

Proposition (EHG and Talitha Washington, 2009)
Let $p_{S}^{(M)}(n)$ denote the number of ways to climb a staircase containing $n$ stairs using step-sizes $S$ and multiplicities at most $M$.

$$
\sum_{n=0}^{\infty} p_{S}^{(M)}(n) x^n = \prod_{s \in S} \frac{1 - x^{(M+1)s}}{1 - x^s} \quad \text{on the interval} \quad |x| < 1.
$$

Say $M = \infty$, $S = \mathbb{Z}_{>0}$. Then $p_{S}^{(M)}(n) = p(n)$, the partition function.

$$
\sum_{n=0}^{\infty} p(n) x^n = \prod_{s=1}^{\infty} \frac{1}{1 - x^s} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + \cdots.
$$
Does analysis appear elsewhere in number theory?
Melvin Robert Currie and Edray Herber Goins, Summer of 1995
Motivating Question

Consider positive integers $n$ and $k \leq n$. Denoting $\{x\} = x - \lfloor x \rfloor$ as the fractional part of a real number $x$, find the average of $\{k/n\}$ and $\{n/k\}$.

<table>
<thead>
<tr>
<th>$(k, n)$</th>
<th>$k/n$</th>
<th>${k/n}$</th>
<th>$n/k$</th>
<th>${n/k}$</th>
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<td>1/1</td>
<td>0</td>
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<tr>
<td>(1, 2)</td>
<td>1/2</td>
<td>1/2</td>
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<tr>
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<td>1/5</td>
<td>5/2</td>
<td>1/2</td>
</tr>
<tr>
<td>(2, 5)</td>
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<td>1/5</td>
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</tr>
<tr>
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<td>2/5</td>
<td>2/5</td>
<td>5/3</td>
<td>1/3</td>
</tr>
<tr>
<td>(4, 5)</td>
<td>1/5</td>
<td>4/5</td>
<td>5/4</td>
<td>1/4</td>
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<tr>
<td>(5, 5)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$(k, n)$</th>
<th>$k/n$</th>
<th>${k/n}$</th>
<th>$n/k$</th>
<th>${n/k}$</th>
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<tbody>
<tr>
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<td>1/4</td>
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<tr>
<td>(2, 4)</td>
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<td>2</td>
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<tr>
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<tr>
<td>(6, 6)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Conjecture (Donald J. Newman, 1995)

Both averages are $1/2$. 
Proposition (Riemann Summation)

- The average of \( \{ k/n \} \) is 1/2.
- More generally, for a continuous function \( f : [0, 1] \to \mathbb{C} \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f \left( \left\{ \frac{k}{n} \right\} \right) = \int_{0}^{1} f(x) \, dx.
\]

Proposition (Melvin R. Currie and EHG, 1995)

- The average of \( \{ n/k \} \) is \( 1 - \gamma = 0.422784 \ldots \).
- More generally, for a continuous function \( f : [0, 1] \to \mathbb{C} \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f \left( \left\{ \frac{n}{k} \right\} \right) = \int_{0}^{1} f(x) \, dF(x)
\]

in terms of \( F(x) = \frac{d}{dx} \log(x!) \). Recall \( F(0) = \gamma = 0.577216 \ldots \).
Galois Representations and $L$-Series
Theorem (Frank K. Kenter, 1999)

The Euler-Mascheroni constant $\gamma = 0.577216 \ldots$ is equal to the product

\[
\left(1 \ 1/2 \ 1/3 \ 1/4 \ \ldots \ 1/n \ \ldots\right)
\left(\begin{array}{cccc}
1 & & & \\
1/2 & 1 & & \\
1/3 & 1/2 & 1 & \\
1/4 & 1/3 & 1/2 & 1 \\
\vdots & \vdots & \vdots & \vdots \\
1/n & 1/(n-1) & 1/(n-2) & 1/(n-3) & \ldots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}\right)
\left(-1\right)^n
\left(\begin{array}{c}
1/2 \\
1/3 \\
1/4 \\
1/5 \\
\vdots \\
1/(n+1) \\
\vdots
\end{array}\right)
\]
Theorem (EHG and Asamoah Nkwanta, 2011)

Let \( a(x), b(x), \text{ and } c(x) \) be power series and \( e \) be an exponent. Then

\[
\begin{pmatrix}
  \vdots \\
  b_n & b_{n-1} & b_{n-2} & b_{n-3} & \cdots & b_0 \\
  \vdots \\
  a_0 & a_1 & a_2 & a_3 & \cdots & a_n & \cdots
\end{pmatrix}
\]

is equal to the residue

\[
\text{Res}_{z=0} \left[ \frac{a(z) b(z^{-1})^e c(z^{-1})}{z} \right] = \frac{1}{2\pi i} \oint_{|z|=1} a(z) b(z^{-1})^e c(z^{-1}) \frac{dz}{z}
\]
Are there other groups to generalize \((\mathbb{Z}/N\mathbb{Z})^\times?\)
What is an Elliptic Curve?

Definition

Let $A$ and $B$ be rational numbers such that $4A^3 + 27B^2 \neq 0$. An elliptic curve $E$ is the set of all $(x, y)$ satisfying the equation

$$y^2 = x^3 + Ax + B.$$

We will also include the “point at infinity” $O$.

Example: $y^2 = x^3 - 36x$ is an elliptic curve.

Non-Example: $y^2 = x^3 - 3x + 2$ is not an elliptic curve.
Formally, an elliptic curve $E$ over $\mathbb{Q}$ is a nonsingular projective curve of genus 1 possessing a $\mathbb{Q}$-rational point $O$.

Such a curve is birationally equivalent over $\mathbb{Q}$ to a cubic equation in Weierstrass form:

$$E : \quad y^2 = x^3 + Ax + B;$$

with rational coefficients $A$ and $B$, and nonzero discriminant $\Delta(E) = -16 \left(4A^3 + 27B^2\right)$. For any field $K$, define

$$E(K) = \left\{ (x_1 : x_2 : x_0) \in \mathbb{P}^2(K) \mid x_2^2 x_0 = x_1^3 + Ax_1 x_0^2 + Bx_0^3 \right\};$$

where $O = (0 : 1 : 0)$ is on the projective line at infinity $x_0 = 0$.

**Remark:** In practice we choose either $K = \mathbb{Q}$ or $\mathbb{F}_p$. 
Chord-Tangent Method

Given two rational points on an elliptic curve $E$, we explain how to construct more.

1. Start with two rational points $P$ and $Q$.

2. Draw a line through $P$ and $Q$.

3. The intersection, denoted by $P \ast Q$, is another rational point on $E$. 
Definition

Let $E$ be an elliptic curve defined over a field $K$, and denote $E(K)$ as the set of $K$-rational points on $E$. Define the operation $\oplus$ as

$$P \oplus Q = (P \ast Q) \ast O.$$
Poincaré’s Theorem

Theorem (Henri Poincaré, 1901)

Let $E$ be an elliptic curve defined over a field $K$. Then $E(K)$ is an abelian group under $\oplus$.

Recall that to be an abelian group, the following five axioms must be satisfied:

- **Closure**: If $P, Q \in E(K)$ then $P \oplus Q \in E(K)$.
- **Associativity**: $(P \oplus Q) \oplus R = P \oplus (Q \oplus R)$.
- **Commutativity**: $P \oplus Q = Q \oplus P$.
- **Identity**: $P \oplus \mathcal{O} = P$ for all $P$.
- **Inverses**: $[-1]P = P \star \mathcal{O}$ satisfies $P \oplus [-1]P = \mathcal{O}$. 
Conjecture (Henri Poincaré, 1901)

Let $E$ be an elliptic curve. Then $E(\mathbb{Q})$ is finitely generated.

Recall that an abelian group $G$ is said to be finitely generated if there exists a finite generating set

$$\{a_1, a_2, \ldots, a_n\}$$

such that, for each given $g \in G$, there are integers $n_1, n_2, \ldots, n_m$ such that

$$g = [n_1]a_1 \circ [n_2]a_2 \circ \cdots \circ [n_m]a_m.$$

Example: $G = \mathbb{Z}$ is a finitely generated abelian group because all integers are generated by $a_1 = 1$. 
**Theorem (Louis Mordell, 1922)**

Let $E$ be an elliptic curve. Then $E(\mathbb{Q})$ is finitely generated.

That is, there exists a finite group $E(\mathbb{Q})_{\text{tors}}$ and a nonnegative integer $r$ such that

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^r.$$ 

- The set $E(\mathbb{Q})$ is called the **Mordell-Weil group** of $E$.
- The finite set $E(\mathbb{Q})_{\text{tors}}$ is called the **torsion subgroup** of $E$. It contains all of the points of finite order, i.e., those $P \in E(\mathbb{Q})$ such that $[m]P = \mathcal{O}$ for some positive integer $m$.
- The nonnegative integer $r$ is called the **Mordell-Weil rank** of $E$. 

What are some applications with elliptic curves?
Hello!

Davin Maddox
Heron’s Formula

The area of the triangle is

\[ n = \sqrt{s(s-a)(s-b)(s-c)} \]

where

\[ s = \frac{a+b+c}{2}. \]
Theorem (EHG and Davin Maddox, 2003)

Fix a positive integer \( n \).

1. There are infinitely many rational triangles with area \( n \).

2. \( n \) is a congruent number, that is, the area of a rational right triangle if and only if \( y^2 = x^3 - n^2 x \) has a nontrivial rational point.

3. \( n \) is the area of a rational isosceles triangle if and only if \( v^2 = u^4 + n^2 \) has a nontrivial rational point.

4. Say that \( n \) is the area of a rational triangle with an angle \( \theta = A, B, C \). When not isosceles, there are infinitely many rational triangles with area \( n \) possessing this fixed angle.

To see why, set \( \tau = \tan(\theta/2) \) and define \( y^2 = x(x - n\tau)(x + n/\tau) \).

\[
\begin{align*}
\tau &= 4n/((a + b)^2 - c^2) \\
x &= ((a + c)^2 - b^2)/4 \\
y &= a((a + c)^2 - b^2)/4 \\
\end{align*}
\]

\[
\begin{align*}
a &= y/x \\
b &= n(\tau + 1/\tau)x/y \\
c &= (x^2 + n^2)/y \\
\end{align*}
\]
Edray Goins and Garikai Campbell
Question (Diophantus of Alexandria, 250 BC)

Find four numbers such that the product of any two of them increased by unity is a perfect square.

Diophantus presented the set \( \left\{ \frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16} \right\} \) as a solution, because

\[
\frac{1}{16} \cdot \frac{33}{16} + 1 = \left( \frac{17}{16} \right)^2 \quad \quad \quad \quad \frac{33}{16} \cdot \frac{17}{4} + 1 = \left( \frac{25}{8} \right)^2 \\
\frac{1}{16} \cdot \frac{17}{4} + 1 = \left( \frac{9}{8} \right)^2 \quad \quad \quad \quad \frac{33}{16} \cdot \frac{105}{16} + 1 = \left( \frac{61}{16} \right)^2 \\
\frac{1}{16} \cdot \frac{105}{16} + 1 = \left( \frac{19}{16} \right)^2 \quad \quad \quad \quad \frac{17}{4} \cdot \frac{105}{16} + 1 = \left( \frac{43}{8} \right)^2
\]

Remark: Around 1650, Pierre de Fermat found \( \{1, 3, 8, 120\} \).
A set $S = \{n_1, n_2, \ldots, n_m\}$ of $m$ rational numbers is called a rational Diophantine $m$-tuple if $n_i n_j + 1$ is the square of a rational number for $i \neq j$. More information can be found at the website

http://web.math.hr/~duje/dtuples.html

**Remarks:** When the word “rational” is omitted, it is assumed that each $n_i$ is an integer. We will focus on distinct, nonzero rational numbers.

**Motivating Questions**

- **Existence Problem:** Given a positive integer $m$, does there exist a rational Diophantine $m$-tuple?

- **Extension Problem:** For which $M$ can a given a rational Diophantine $m$-tuple $\{n_1, n_2, \ldots, n_m\}$ be extended to a rational Diophantine $M$-tuple $\{n_1, n_2, \ldots, n_m, \ldots, n_M\}$?
Theorem (Garikai Campbell and EHG, 2007)

There exist infinitely many rational Diophantine 3-tuples.

We show existence by classifying all 3-tuples \( \{n_1, n_2, n_3\} \). Consider the substitution

\[
\begin{align*}
   t_1 &= \frac{n_1}{1 + \sqrt{n_1 n_2 + 1}} \\
   t_2 &= \frac{n_2}{1 + \sqrt{n_2 n_3 + 1}} \\
   t_3 &= \frac{n_3}{1 + \sqrt{n_3 n_1 + 1}}
\end{align*}

\begin{align*}
   n_1 &= \frac{2 t_1 (1 + t_1 t_2 + t_1 t_2^2 t_3)}{1 - t_1^2 t_2^2 t_3^2} \\
   n_2 &= \frac{2 t_2 (1 + t_2 t_3 + t_2 t_3^2 t_1)}{1 - t_1^2 t_2^2 t_3^2} \\
   n_3 &= \frac{2 t_3 (1 + t_3 t_1 + t_3 t_1^2 t_2)}{1 - t_1^2 t_2^2 t_3^2}
\end{align*}

Hence each \( \{n_1, n_2, n_3\} \) corresponds to \((t_1, t_2, t_3) \in \mathbb{A}^3(\mathbb{Q})\).
Theorem (Andrej Dujella, 2001)

Let \( \{n_1, n_2, n_3\} \) be a nontrivial rational Diophantine 3-tuple. Fix the elliptic curve

\[
E : \quad Y^2 = (n_1 X + 1)(n_2 X + 1)(n_3 X + 1)
\]

define \( n_{ij} = \sqrt{n_i n_j + 1} \), and consider the two rational points

\[
Q_1 = (0, 1),
\]
\[
Q_2 = \left( \frac{n_{12} n_{13} + n_{12} n_{23} + n_{13} n_{23} + 1}{n_1 n_2 n_3}, \frac{(n_{12} + n_{13})(n_{12} + n_{23})(n_{13} + n_{23})}{n_1 n_2 n_3} \right).
\]

- \( \{n_1, n_2, n_3, n_4\} \) is a rational Diophantine 4-tuple if and only if \( n_4 \) is the X-coordinate of \( Q_1 \oplus [2]P \) for some \( P \in E(\mathbb{Q}) \).

- Choose a point \( P \in E(\mathbb{Q}) \). Let \( n_4 \) and \( n_5 \) be the X-coordinates of \( Q_1 \oplus [2]P \) and \( (Q_1 \oplus [2]Q_2) \oplus [2]P \). Then \( \{n_1, n_2, n_3, n_4, n_5\} \) is a rational Diophantine 5-tuple.
Elliptic Curves with Large Rank

Observations

For each nontrivial rational Diophantine 3-tuple \( \{n_1, n_2, n_3\} \), the elliptic curve \( E : Y^2 = (n_1 X + 1)(n_2 X + 1)(n_3 X + 1) \) has torsion subgroup containing \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). It appears that \( Q_1 \) and \( Q_2 \) are in general two independent points of infinite order.

This suggests that certain 3-tuples will yield curves with large rank:

<table>
<thead>
<tr>
<th>Rat'l Diophantine 3-tuple</th>
<th>Torsion</th>
<th>Rank</th>
<th>Largest Known</th>
</tr>
</thead>
<tbody>
<tr>
<td>( {\frac{1270}{2323}, \frac{5916}{2323}, \frac{664593861324}{12535672267} } )</td>
<td>( \mathbb{Z}_2 \times \mathbb{Z}_2 )</td>
<td>9</td>
<td>14 (Elkies, 2005)</td>
</tr>
<tr>
<td>( {-\frac{22552}{5129}, \frac{5129}{22552}, -\frac{52463190}{14458651} } )</td>
<td>( \mathbb{Z}_2 \times \mathbb{Z}_4 )</td>
<td>7</td>
<td>8 (Elkies, 2005)</td>
</tr>
<tr>
<td>( {\frac{39123}{96976}, \frac{12947200}{418209}, \frac{42427}{1104} } )</td>
<td>( \mathbb{Z}_2 \times \mathbb{Z}_6 )</td>
<td>4</td>
<td>6 (Elkies, 2006)</td>
</tr>
<tr>
<td>( {\frac{145}{408}, -\frac{408}{145}, -\frac{145439}{59160} } )</td>
<td>( \mathbb{Z}_2 \times \mathbb{Z}_8 )</td>
<td>3</td>
<td>3 (G, 2003)</td>
</tr>
</tbody>
</table>

Motivating Question

Which rational Diophantine 3-tuples \( \{n_1, n_2, n_3\} \) yield large rank?
Number Theory is Everywhere!