Visualizing Dessins d'Enfants on the Torus

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Motivation

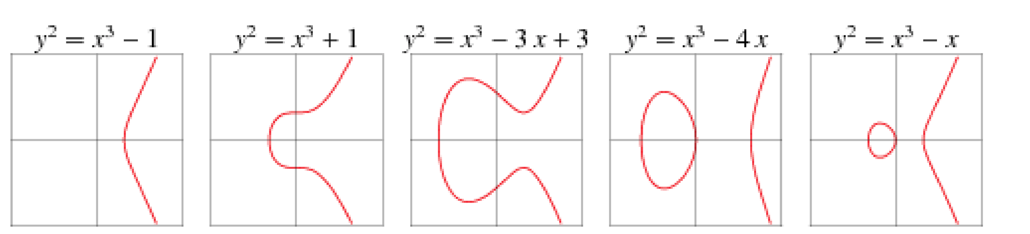
With given Belyĭ maps and their corresponding elliptic curves, we can give a general description of their dessins d'enfants in 2 dimensions. We don't know, however, what these dessins will look like when embedded on the torus, in 3 dimensions. Our goal is to create a program that will allow us to visualize these dessins on the torus.

Background

- **Elliptic Curves** An elliptic curve E is a set

$$E(\mathbb{C}) = \left\{ (x:y:z) \in \mathbb{P}^2(\mathbb{C}) \;\middle|\; \begin{aligned} y^2 z + a_1 x\,y\,z + a_3 y\,z^2 \\ = x^3 + a_2 x^2 z \\ + a_4 x z^2 + a_6 z^3 \end{aligned} \right\}$$

for complex numbers a_1, a_3, a_2, a_4, a_6.



Examples of elliptic curves

- **Belyĭ Map** A Belyĭ Map is a rational function $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ with at most 3 critical values, which we assume to be $\{0, 1, \infty\}$. Here $\mathbb{P}^1(\mathbb{C})$ is the Complex Projective Line.

Some examples include:

$$\beta(x,y) = \frac{y+1}{2} \quad \text{for} \quad E: y^2 = x^3 + 1$$
$$\beta(x,y) = \frac{(y - x^2 - 17x)^3}{2^{14} y} \quad \text{for} \quad E: y^2 + 15xy + 128y = x^3$$
$$\beta(x,y) = \frac{(x-5)y + 16}{32} \quad \text{for} \quad E: y^2 = x^3 + 5x + 10$$

- **Dessins d'Enfant** A bipartite graph is a graph whose vertices will be composed of 2 disjoint sets, in this case represented by 2 different colors: Black and Red. Given a Belyĭ map, its corresponding dessin d'enfant is a bipartite graph of red and black vertices given by:
  - $\beta^{-1}(0)$ = Red Vertices
  - $\beta^{-1}(1)$ = Black Vertices
  - $\beta^{-1}([0,1])$ = Edges.

Objectives

Given an Elliptic Curve $E(\mathbb{C})$ and a Belyĭ map $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, we want to compute the image

$$\beta^{-1}([0,1]) \subseteq E(\mathbb{C}) \simeq \mathbb{C}/\mathbb{Z}[\omega_1, \omega_2] \simeq T^2(\mathbb{R}).$$

Simply put,

Input: A Belyĭ map β and its corresponding Elliptic curve.

Output: The dessin d'enfant plotted in 2 and in 3 dimensions on the torus

Algorithm

We are initially given an elliptic curve E and a Belyĭ map β.

Step 1: Given a "large" integer N, compute

$$\left\{ (x,y) \in \mathbb{A}^2(\mathbb{C}) \;\middle|\; \begin{aligned} y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \\ \beta(x,y) = \frac{k}{N} \text{ for } k = 0, 1, 2 \ldots, N \end{aligned} \right\}$$

This will result in a list of points on the elliptic curve approximating $\beta^{-1}([0,1])$ that we will use to calculate the dessin d'enfant.

Step 2: Compute the Map

$$E(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/\mathbb{Z}[\omega_1, \omega_2]$$

The result will be the estimated elliptic logarithm.

Step 3: For each point $P = (x_0, y_0)$ from the list of points obtained in step 1, and $z = \log_E(P)$ from step 2, compute

1. real numbers m and n where $z = m\omega_1 + n\omega_2$ such that $0 \le m < 1$ and $0 \le n < 1$.
2. numbers (u, v, w) where

$$\begin{aligned} u &= (R + r\cos(2\pi m))\cos(2\pi n) \\ v &= (R + r\cos(2\pi m))\sin(2\pi n) \\ w &= r\sin(2\pi m) \end{aligned}$$

Step 4: Plot the points (m, n) onto $\mathbb{A}^2(\mathbb{R})$ and the points (u, v, w) onto $\mathbb{A}^3(\mathbb{R})$.

Computational packages such as Sage and Mathematica had trouble computing the integral necessary to calculate the elliptic logarithm. An alternate method that could bypass the integral was needed to calculate the elliptic logarithm. Such a variation is offered in a paper by Cremona and Thongjunthug [2].

Cremona and Thongjunthug Variation

This algorithm computes the elliptic logarithm using Arithmetic-Geometric Means (AGM).

Step 2a: Calculate the roots

The roots e_1, e_2, and e_3 of E can be calculated from

$$\begin{aligned} 4(x^3 + a_2 x^2 + a_4 x + a6) + (a_1 x + a_3)^2 \\ = 4(x - e_1)(x - e_2)(x - e_3). \end{aligned}$$

Step 2b: Calculate the periods Using these roots, for a chosen integer N, iterate for $p \in (0, N)$

$$\begin{aligned} A_0 &= \sqrt{e_1 - e_3} & A_{p+1} &= \frac{A_p + B_p}{2} \\ B_0 &= \sqrt{e_1 - e_2} & B_{p+1} &= \sqrt{A_p B_p} \\ C_0 &= \sqrt{e_2 - e_3} & C_{p+1} &= \frac{C_p + D_p}{2} \\ D_0 &= \sqrt{e_2 - e_1} & D_{p+1} &= \sqrt{C_p D_p} \end{aligned}$$

A_N converges to the AGM(A_0,B_0) and C_N converges to the AGM(C_0,D_0). The periods are calculated from these numbers A_N and C_N as $\omega_1 = \pi/A_N$ and $\omega_2 = \pi/C_N$.

Step 2c: Calculate the elliptic logarithm Given a point $P = (x, y)$ from the list of points in step 1 of the original algorithm, iterate $p \in (1, N)$ calculate the following values

$$\begin{aligned} I_1 &= \sqrt{\frac{x - e_1}{x - e_2}} & I_{p+1} &= \sqrt{\frac{A_p(I_p + 1)}{B_{p-1} I_p + A_{p-1}}} \\ J_1 &= \frac{-(2y + a_1 x + a_3)}{2I_1(x - e_2)} & J_{p+1} &= I_{p+1} J_p \end{aligned}$$

Then the elliptic logarithm can be calculated as

$$z = \log_E(P) = \frac{1}{A_N} \arctan \frac{A_N}{J_N}$$

Future Projects

These examples are all plotted on surfaces of genus 1, we now look to see the plots of dessins d'enfants on genus g>1, or g-holed torii.

References

[1] Antoine D. Coste, Gareth A. Jones, Manfred Streit, and Jürgen Wolfart, "Generalised Fermat Hypermaps and Galois Orbits". Glasgow Math Journal, Vol. 51 (2): 289-99. 2009.
[2] John E. Cremona and Thotsaphon Thongjunthug, "The complex AGM, periods of elliptic curves over $\mathbb{C}$ and complex elliptic logarithms". `arXiv.org`, November 2010.
[3] Noam Elkies, "Elliptic Curves in Nature". `http://www.math.harvard.edu/~elkies/nature.html`
[4] Lily S. Khadjavi and Victor Scharaschkin, "Belyĭ Maps and Elliptic Curves". Preprint. `http://myweb.lmu.edu/lkhadjavi/BelyiElliptic.pdf`
[5] Leonardo Zapponi, "On the Belyĭ Degree(s) of a Curve Defined Over a Number Field." `arXiv.org`, April 6, 2009.

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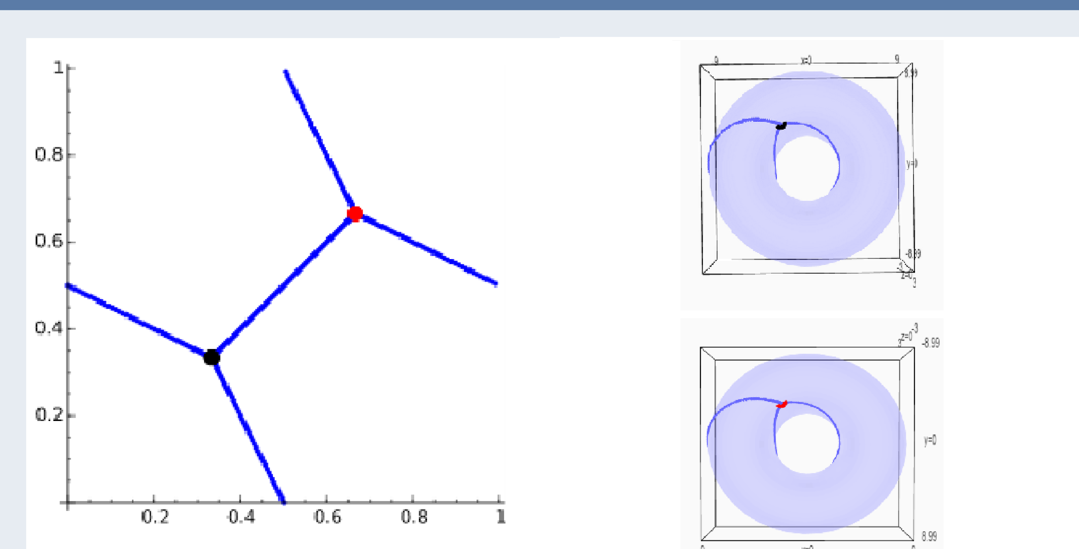
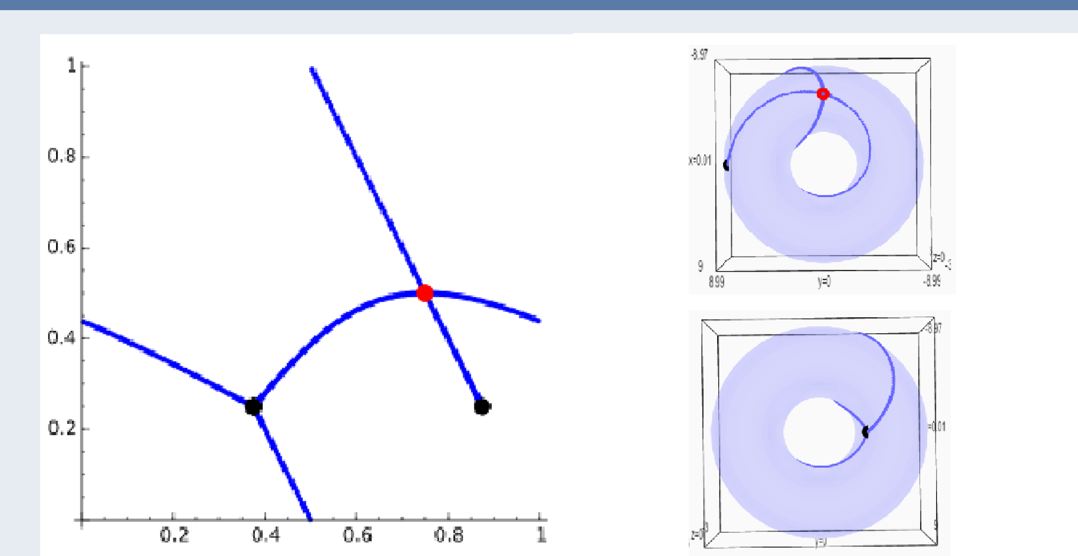
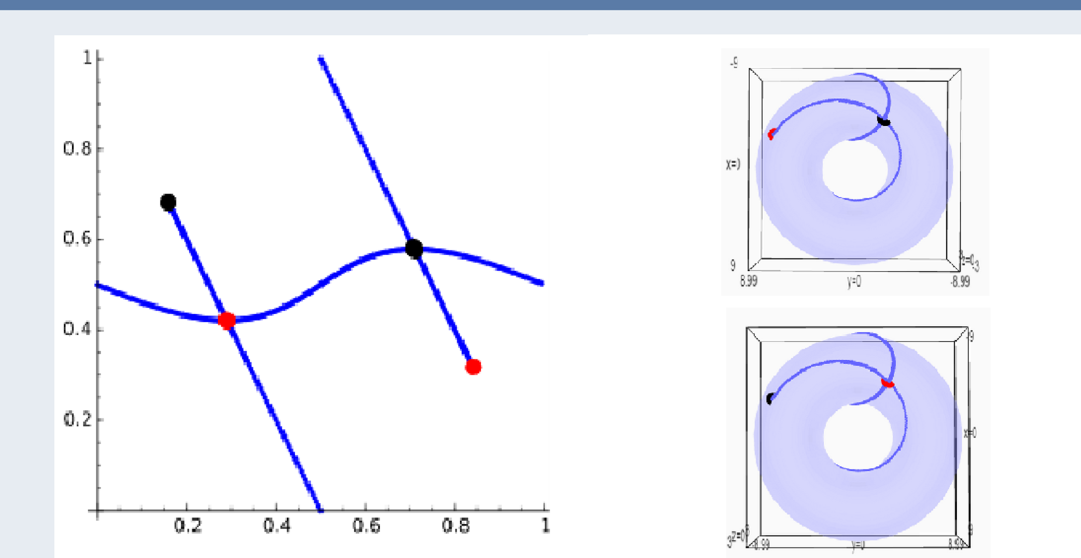
Results



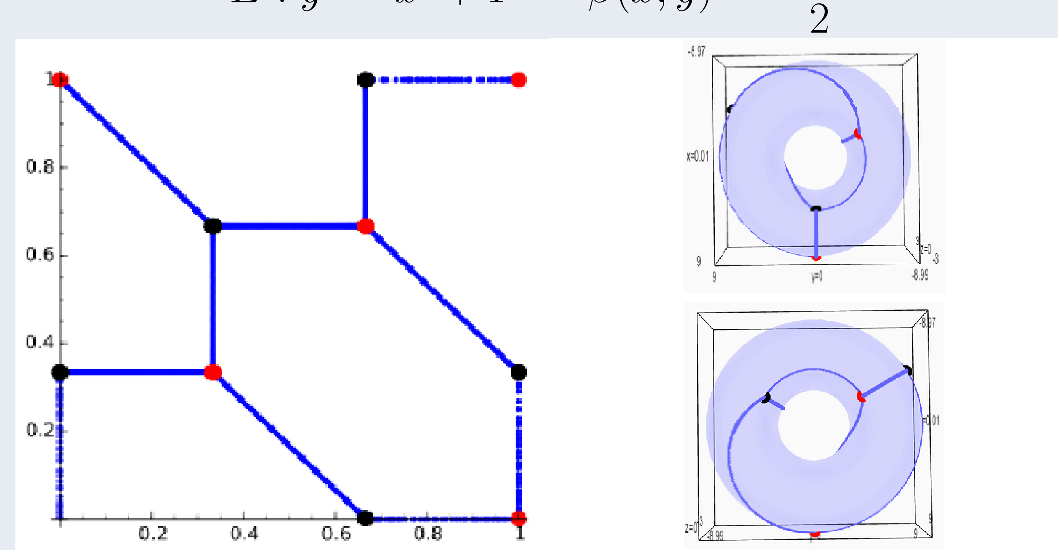
$E: y^2 = x^3 + 1 \qquad \beta(x,y) = \dfrac{y+1}{2}$



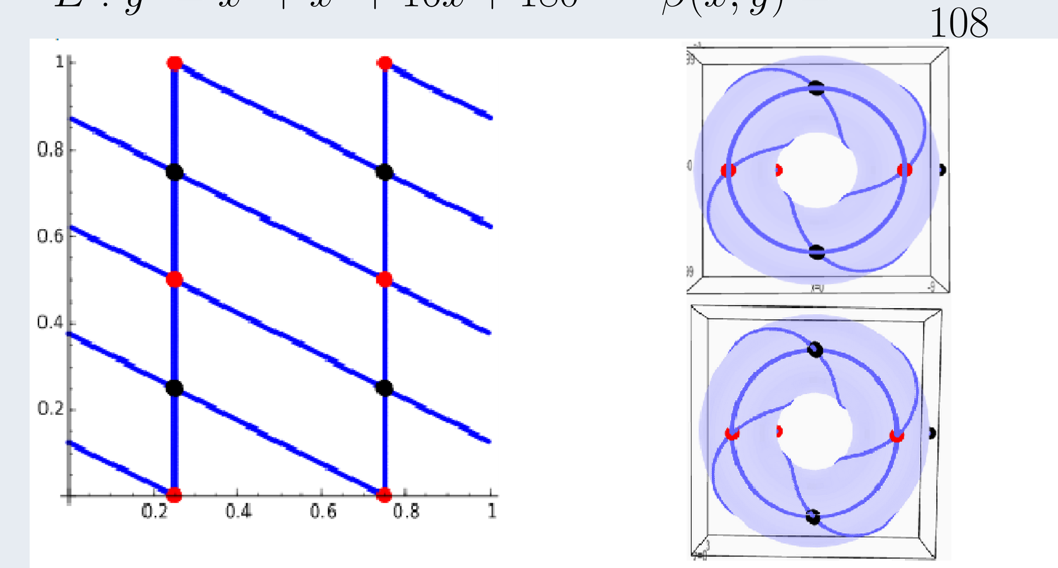
$E: y^2 = x^3 + x^2 + 16x + 180 \qquad \beta(x,y) = \dfrac{x^2 + 4y + 56}{108}$



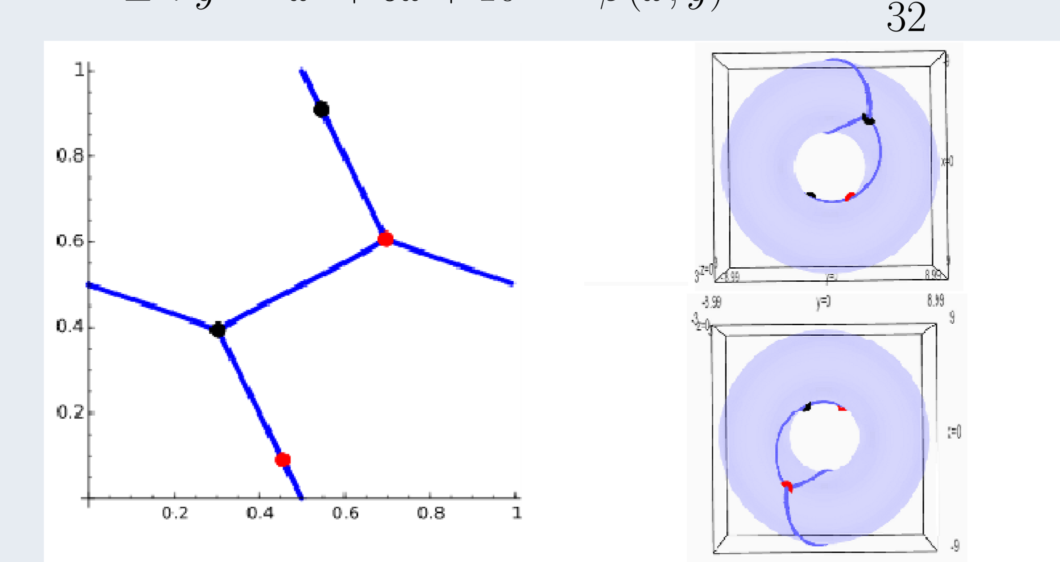
$E: y^2 = x^3 + 5x + 10 \qquad \beta(x,y) = \dfrac{(x-5)y + 16}{32}$



$E: y^2 = x^3 - 432 \qquad \beta(x,y) = \dfrac{216x^3}{(y+36)^3}$



$E: y^2 = x^3 - x \qquad \beta(x,y) = \dfrac{(x^2 + 2x - 1)^4}{16\,y^2(x^2+1)^2}$



$E: y^2 = x^3 - 120x + 740 \qquad \beta(x,y) = \dfrac{(x+5)y + 162}{324}$