From Klein’s Platonic Solids
to Kepler’s Archimedean Solids:
Elliptic Curves and *Dessins d’Enfants*
Part I

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Abstract

In 1884, Felix Klein wrote his influential book, “Lectures on the Icosahedron,” where he explained how to express the roots of any quintic polynomial in terms of elliptic modular functions. His idea was to relate rotations of the icosahedron with the automorphism group of 5-torsion points on a suitable elliptic curve. In fact, he created a theory which related rotations of each of the five regular solids (the tetrahedron, cube, octahedron, icosahedron, and dodecahedron) with the automorphism groups of 3-, 4-, and 5-torsion points.

Using modern language, the functions which relate the rotations with elliptic curves are Belyï maps. In 1984, Alexander Grothendieck introduced the concept of a Dessin d’Enfant in order to understand Galois groups via such maps. We will complete a circle of ideas by reviewing Klein’s theory with an emphasis on the octahedron; explaining how to realize the five regular solids (the Platonic solids) as well as the thirteen semi-regular solids (the Archimedean solids) as Dessins d’Enfant; and discussing how the corresponding Belyï maps relate to moduli spaces of elliptic curves.
Research Experiences for Undergraduate Faculty (June 4 – 8, 2012)

http://aimath.org/ARCC/workshops/reuf4.html
Outline of Talk

1 Lectures on the Icosahedron, Part I
   - The Icosahedron
   - Solving the Quintic
   - Examples

2 Lectures on the Icosahedron, Part II
   - Platonic Solids
   - Klein’s Theory of Covariants
   - Solving Cubics, Quartics, and Quintics

3 Dessin d’Enfants
   - Belyi’s Theorem
   - Children’s Drawings
   - Examples
Christian Felix Klein (April, 25 1849 – June 22, 1925)

http://en.wikipedia.org/wiki/Felix_Klein
What is an Icosahedron?

The icosahedron is the collection of 12 vertices

\[ V = \left\{ (z_1 : z_0) \in \mathbb{P}^1(\mathbb{C}) \mid z_1 z_0 (z_1^{10} - 11 z_1^5 z_0^5 - z_0^{10}) = 0 \right\} \]

\[ = \left\{ 0, \left( \zeta_5 + \zeta_5^4 \right) \zeta_5^\nu, \left( \zeta_5^2 + \zeta_5^3 \right) \zeta_5^\nu, \infty \mid \nu \in \mathbb{F}_5 \right\} . \]

It is a regular solid having 30 faces and 20 edges. We embed this object \( V \hookrightarrow S^2(\mathbb{R}) \) into the unit sphere via stereographic projection

\[ \mathbb{P}^1(\mathbb{C}) \sim \rightarrow S^2(\mathbb{R}) = \left\{ (u, \nu, w) \in \mathbb{A}^3(\mathbb{R}) \mid u^2 + \nu^2 + w^2 = 1 \right\} \]

defined as that bijection which sends

\[ (z_1 : z_0) \mapsto \left( \frac{2 \text{Re} (z_1 \overline{z_0})}{|z_1|^2 + |z_0|^2}, \frac{2 \text{Im} (z_1 \overline{z_0})}{|z_1|^2 + |z_0|^2}, \frac{|z_1|^2 - |z_0|^2}{|z_1|^2 + |z_0|^2} \right). \]
Rigid Rotations of the Icosahedron

We have an action \( \circ : PSL_2(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \) defined by

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \circ z = \frac{a z + b}{c z + d}.
\]

By restriction, the alternating group \( A_5 = \langle r, s \mid s^2 = r^3 = (s r)^5 = 1 \rangle \) acts on the icosahedron, that is, \( \circ : A_5 \times V \to V \) via the fractional linear transformations

\[
\begin{align*}
r(z) &= \frac{(\zeta_5 + \zeta_5^4) \zeta_5 - z}{(\zeta_5 + \zeta_5^4) z + \zeta_5} \\
s(z) &= \frac{(\zeta_5 + \zeta_5^4) - z}{(\zeta_5 + \zeta_5^4) z + 1}.
\end{align*}
\]

The following rational function is invariant under action of \( A_5 \):

\[
j(z) = \frac{(z^{20} + 228 z^{15} + 494 z^{10} - 228 z^5 + 1)^3}{z^5 (z^{10} - 11 z^5 - 1)^5}.
\]

Note that the algebraic extension \( \mathbb{Q}(\zeta_5, z)/\mathbb{Q}(\zeta_5, j) \) is Galois.
**Principal Quintic Form**

**Proposition (Felix Klein, 1884)**

- \( j = (\lambda + 3)^3 (\lambda^2 + 11\lambda + 64) = (\mu^2 + 10\mu + 5)^3/\mu \) in terms of

\[
\lambda(z) = \frac{[z^2 + 1]^2 [z^2 + 2(\zeta_5 + \zeta_5^4)z - 1]^2 [z^2 + 2(\zeta_5^2 + \zeta_5^3)z - 1]^2}{z(z^{10} - 11z^5 - 1)}
\]

\[
\mu(z) = \frac{125z^5}{z^{10} - 11z^5 - 1}
\]

- For each \( m, n \in \overline{\mathbb{Q}} \), the five resolvents

\[
x_{\nu} = \frac{m}{\lambda(\zeta_5^\nu z) + 3} + \frac{n}{\lambda(\zeta_5^\nu z) + 3} \frac{\lambda(\zeta_5^\nu z)^2 + 10\lambda(\zeta_5^\nu z) + 45}{\lambda(\zeta_5^\nu z)^2 + 10\lambda(\zeta_5^\nu z) + 45}
\]

are roots of the quintic \( x^5 + A_{m,n,j} x^2 + B_{m,n,j} x + C_{m,n,j} \) where

\[
A_{m,n,j} = -\frac{20}{j} \left[ \left( 2m^3 + 3m^2 n \right) + 432 \frac{6m^2 n^2 + n^3}{1728 - j} \right]
\]

\[
B_{m,n,j} = -\frac{5}{j} \left[ m^4 - 864 \frac{3m^2 n^2 + 2m n^3}{1728 - j} + 559872 \frac{n^4}{(1728 - j)^2} \right]
\]

\[
C_{m,n,j} = -\frac{6}{j} \left[ m^5 - 1440 \frac{m^3 n^2}{1728 - j} + 62208 \frac{15 m n^4 + 4 n^5}{(1728 - j)^2} \right]
\]
Elliptic Curves Associated to Quintic Polynomials

**Theorem (Felix Klein, 1884; Annette Klute, 1997; G, 2003)**

Let \( q(x) = x^5 + Ax^2 + Bx + C \) be over \( \mathbb{Q} \). Set \( K = \mathbb{Q}(\sqrt[5]{5} \cdot \text{Disc}(q)) \), and denote \( L \) as its splitting field.

- \( A = A_{m,n,j} \), \( B = B_{m,n,j} \), and \( C = C_{m,n,j} \) for some \( m, n, j \in K \).
- There exists an elliptic curve \( E \) over \( K \) such that \( L/K \) is the field generated by sum \( x_P + x_{2P} \) of \( x \)-coordinates of the 5-torsion of \( E \).

**Proof:** We find \( \delta^5 j^2 - 1728 \left( \gamma_4^3 - \gamma_6^2 + \delta^5 \right) j + 1728^2 \gamma_4^3 = 0 \) upon eliminating \( m \) and \( n \), where

\[
\begin{align*}
5^4 \cdot \delta &= A^4 - 5B^3 + 25 ABC \\
12^2 5^5 \cdot \gamma_4 &= 128 A^4 B^2 - 144 B^5 - 192 A^5 C - 600 AB^3 C + 1000 A^2 B C^2 + 3125 C^4 \\
12^3 5^{10} \cdot \gamma_6 &= 1728 A^{10} + 10400 A^6 B^3 + 405000 A^2 B^6 - 180000 A^7 B C - 1170000 A^3 B^4 C \\
&
+ 1725000 A^4 B^2 C^2 - 2025000 B^5 C^2 - 1800000 A^5 C^3 + 2812500 A B^3 C^3 - 4687500 A^2 B C^4 - 9765625 C^6
\end{align*}
\]

Then \( L/K \) is the splitting field of \( (\mu^2 + 10 \mu + 5)^3 - j \mu \). For \( P \in E[5] \) on \( E : y^2 = x^3 + 3j/(1728 - j)x + 2j/(1728 - j) \), set \( x = x_P + x_{2P} \). Then

\[
x = -2 \frac{\mu^2 + 10 \mu + 5}{\mu^2 + 4 \mu - 1} \quad \iff \quad \mu = \frac{31104 x^3}{(x - 2)^5 j - 1728 x^3 (x^2 - 10 x + 34)}.
\]
Corollary (Kuang-yen Shih, 1978; G, 2003)

For any $t \in \mathbb{Q}$, consider the quintic polynomials

$$q_{p,t}(x) = \begin{cases} 
  x^5 - 5 (5 t^2 - 1) x - 4 (5 t^2 - 1) & \text{for } p = 2, \\
  x^5 - 20 (5 t^2 - 1)^2 x^2 - 48 (5 t^2 - 1)^3 & \text{for } p = 3;
\end{cases}$$

and define the elliptic curves

$$E_{p,t} : \begin{cases} 
  y^2 = x^3 + 2 x^2 + \frac{1}{2} (1 + \sqrt{5} t) x & \text{for } p = 2, \\
  y^2 + 3 x y + \frac{1}{2} (1 + \sqrt{5} t) y = x^3 & \text{for } p = 3.
\end{cases}$$

- $q_{p,t}(x)$ has Galois group contained in $A_5$.
- $E_{p,t}$ is associated to $q_{p,t}(x)$, and is a $p$-isogenous $\mathbb{Q}$-curve defined over $\mathbb{Q}(\sqrt{5})$. (A $\mathbb{Q}$-curve is an elliptic curve without complex multiplication which is isogenous to each of its Galois conjugates.)

Conversely, given $q(x) = x^5 + A x^2 + B x + C$ a quintic over $\mathbb{Q}$ with Galois group $A_5$ where $A B = 0$, set $t = \sqrt{\text{Disc}(q)}/(125 C^2)$ and $p = 2$ if $A = 0$ or $p = 3$ if $B = 0$. Then $E_{p,t}$ is associated to $q(x)$. 

Number Theory Seminar

From Klein’s Platonic Solids to Kepler’s Archimedean Solids
Can We Generalize?
What is a Platonic Solid?

A regular, convex polyhedron is one of the collections of vertices

\[ V = \left\{ (z_1 : z_0) \in \mathbb{P}^1(\mathbb{C}) \mid \delta(z_1, z_0) = 0 \right\} \]

in terms of the homogeneous polynomials

\[ \delta(z_1, z_0) = \begin{cases} 
    z_1^n + z_0^n & \text{for the regular polygon}, \\
    z_1 (z_1^3 - z_0^3) & \text{for the tetrahedron}, \\
    z_1 z_0 (z_1^4 - z_0^4) & \text{for the octahedron}, \\
    z_1^8 + 14 z_1^4 z_0^4 + z_0^8 & \text{for the cube}, \\
    z_1 z_0 (z_1^{10} - 11 z_1^5 z_0^5 - z_0^{10}) & \text{for the icosahedron}, \\
    z_1^{20} + 228 z_1^{15} z_0^5 + 494 z_1^{10} z_0^{10} - 228 z_1^5 z_0^{15} + z_0^{20} & \text{for the dodecahedron}. 
\end{cases} \]

We embed \( V \hookrightarrow S^2(\mathbb{R}) \) into the unit sphere via stereographic projection.
Platonic Solids

- **Tetrahedron** (four faces)
- **Cube or hexahedron** (six faces)
- **Octahedron** (eight faces)
- **Dodecahedron** (twelve faces)
- **Icosahedron** (twenty faces)

[http://mathworld.wolfram.com/RegularPolygon.html](http://mathworld.wolfram.com/RegularPolygon.html)

Rigid Rotations of the Platonic Solids

Recall the action $\circ : PSL_2(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. We seek Galois extensions $\mathbb{Q}(\zeta_n, z)/\mathbb{Q}(\zeta_n, j)$ for rational $j(z)$.

- $Z_n = \langle r \mid r^n = 1 \rangle$ and $D_n = \langle r, s \mid s^2 = r^n = (s r)^2 = 1 \rangle$ are the rigid rotations of the regular convex polygons, with
  
  $r(z) = \zeta_n z$, \quad $s(z) = \frac{1}{z}$, \quad and \quad $j(z) = 1728 z^n$ or $\frac{6912 z^n}{(z^n + 1)^2}$.

- $A_4 = \langle r, s \mid s^2 = r^3 = (s r)^3 = 1 \rangle \simeq PSL_2(\mathbb{F}_3)$ are the rigid rotations of the tetrahedron, with
  
  $r(z) = \zeta_3 z$, \quad $s(z) = \frac{1 - z}{2 z + 1}$, \quad and \quad $j(z) = -\frac{27 (8 z^3 + 1)^3}{z^3 (z^3 - 1)^3}$.

- $S_4 = \langle r, s \mid s^2 = r^3 = (s r)^4 = 1 \rangle \simeq PGL_2(\mathbb{F}_3) \simeq PSL_2(\mathbb{Z}/4\mathbb{Z})$ are the rigid rotations of the octahedron and the cube, with
  
  $r(z) = \frac{\zeta_4 + z}{\zeta_4 - z}$, \quad $s(z) = \frac{1 - z}{1 + z}$, \quad and \quad $j(z) = \frac{16 (z^8 + 14 z^4 + 1)^3}{z^4 (z^4 - 1)^4}$.

- $A_5 = \langle r, s \mid s^2 = r^3 = (s r)^5 = 1 \rangle \simeq PSL_2(\mathbb{F}_4) \simeq PSL_2(\mathbb{F}_5)$ are the rigid rotations of the icosahedron and the dodecahedron, with
  
  $r(z) = \frac{(\zeta_5 + \zeta_5^4) \zeta_5 - z}{(\zeta_5 + \zeta_5^4) z + \zeta_5}$, \quad $s(z) = \frac{(\zeta_5 + \zeta_5^4) - z}{(\zeta_5 + \zeta_5^4) z + 1}$, \quad and \quad $j(z) = \frac{(z^{20} + 228 z^{15} + 494 z^{10} - 228 z^5 + 1)^3}{z^5 (z^{10} - 11 z^5 - 1)^5}$.
Each \( \delta = \delta(z_1, z_0) \) is invariant under a group. Choose nonzero \( \alpha, \beta, \gamma, \) and \( n \) such that \( c_4^3 - c_6^2 = 1728 \Delta \) for the homogeneous polynomials

\[
c_4(z_1, z_0) = \alpha \cdot \det \begin{bmatrix}
\frac{\partial^2 \delta}{\partial z_1^2}(z_1, z_0) & \frac{\partial^2 \delta}{\partial z_1 \partial z_0}(z_1, z_0) \\
\frac{\partial^2 \delta}{\partial z_0 \partial z_1}(z_1, z_0) & \frac{\partial^2 \delta}{\partial z_0^2}(z_1, z_0)
\end{bmatrix}
\]

\[
c_6(z_1, z_0) = \beta \cdot \det \begin{bmatrix}
\frac{\partial \delta}{\partial z_1}(z_1, z_0) & \frac{\partial \delta}{\partial z_0}(z_1, z_0) \\
\frac{\partial c_4}{\partial z_1}(z_1, z_0) & \frac{\partial c_4}{\partial z_0}(z_1, z_0)
\end{bmatrix}
\]

\[
\Delta(z_1, z_0) = \gamma \cdot \delta(z_1, z_0)^n
\]

These are also invariant under the specified groups. For each \( z \) in the inverse image of the thrice punctured sphere

\[
j^{-1}\left( \mathbb{P}^1(\mathbb{C}) - \{0, 1728, \infty\} \right) = \left\{ z \in \mathbb{P}^1(\mathbb{C}) \mid c_4(z) c_6(z) \Delta(z) \neq 0 \right\}
\]

we form the elliptic curve

\[
E: y^2 = x^3 + \frac{3j(z)}{1728 - j(z)} x + \frac{2j(z)}{1728 - j(z)} \quad \text{where} \quad j(z) = \frac{c_4(z)^3}{\Delta(z)}.
\]
\[ D_n : \]
\[
c_4(z_1, z_0) = -144 z_1^{n-2} z_0^{n-2}
\]
\[
c_6(z_1, z_0) = -864 z_1^{n-3} z_0^{n-3} (z_1^n - z_0^n)
\]
\[
\Delta(z_1, z_0) = -432 z_1^{2n-6} z_0^{2n-6} (z_1^n + z_0^n)^2
\]
\[
j(z_1, z_0) = 6912 z_1^n z_0^n / (z_1^n + z_0^n)^2
\]

\[ A_4 : \]
\[
c_4(z_1, z_0) = 9 z_0 (8 z_1^3 + z_0^4)
\]
\[
c_6(z_1, z_0) = 27 (8 z_1^6 + 20 z_1^3 z_0^3 - z_0^6)
\]
\[
\Delta(z_1, z_0) = -27 z_1 (z_1^3 - z_0^3)^3
\]
\[
j(z_1, z_0) = -27 z_0^3 (8 z_1^3 + z_0^3)^3 / [z_1^3 (z_1^3 - z_0^3)^3]
\]

\[ S_4 : \]
\[
c_4(z_1, z_0) = 4 (z_1^8 + 14 z_1^4 z_0^4 + z_0^8)
\]
\[
c_6(z_1, z_0) = -8 (z_1^{12} - 33 z_1^8 z_0^4 - 33 z_1^4 z_0^8 + z_0^{12})
\]
\[
\Delta(z_1, z_0) = 4 z_1^4 z_0^4 (z_1^4 - z_0^4)^4
\]
\[
j(z_1, z_0) = 16 (z_1^8 + 14 z_1^4 z_0^4 + z_0^8)^3 / [z_1^4 z_0^4 (z_1^4 - z_0^4)^4]
\]

\[ A_5 : \]
\[
c_4(z_1, z_0) = z_1^{20} + 228 z_1^{15} z_0^5 + 494 z_1^{10} z_0^{10} - 228 z_1^5 z_0^{15} + z_0^{20}
\]
\[
c_6(z_1, z_0) = z_1^{30} - 522 z_1^{25} z_0^5 - 10005 z_1^{20} z_0^{10} - 10005 z_1^{10} z_0^{20} + 522 z_1^5 z_0^{25} + z_0^{30}
\]
\[
\Delta(z_1, z_0) = z_1^5 z_0^5 (z_1^{10} - 11 z_1^5 z_0^5 - z_0^{10})^5
\]
\[
j(z_1, z_0) = (z_1^{20} + 228 z_1^{15} z_0^5 + 494 z_1^{10} z_0^{10} - 228 z_1^5 z_0^{15} + z_0^{20})^3 / [z_1^5 z_0^5 (z_1^{10} - 11 z_1^5 z_0^5 - z_0^{10})^5]
\]
Theorem (Felix Klein, 1884)

The rational function
\[ j(z) = \begin{cases} 
  -27 \frac{(8z^3 + 1)^3}{z^3(z^3 - 1)^3} & \text{for } n = 3, \\
  16 \frac{(z^4 + 14z^4 + 1)^3}{z^4(z^4 - 1)^4} & \text{for } n = 4, \\
  \frac{(z^{20} + 228z^{15} + 494z^{10} - 228z^5 + 1)^3}{z^5(z^{10} - 11z^5 - 1)^5} & \text{for } n = 5; 
\end{cases} \]

is invariant under the group \( G = \langle r, s \mid s^2 = r^3 = (s\, r)^n = 1 \rangle \) expressed in terms of the generators
\[ r(z) = \begin{cases} 
  \zeta_3 z & \text{for } n = 3, \\
  \zeta_4 + z & \text{for } n = 4, \\
  \frac{\zeta_4 - z}{(\zeta_5 + \zeta_5^4)z + \zeta_5} & \text{for } n = 5; 
\end{cases} \]
\[ s(z) = \begin{cases} 
  \frac{1 - z}{2z + 1} & \text{for } n = 3, \\
  \frac{1 - z}{1 + z} & \text{for } n = 4, \\
  \frac{\zeta_5 + \zeta_5^4 - z}{(\zeta_5 + \zeta_5^4)z + 1} & \text{for } n = 5. 
\end{cases} \]

In particular, \( \text{Gal}(\mathbb{Q}(\zeta_n, z)/\mathbb{Q}(\zeta_n, j)) \cong G \cong PSL_2(\mathbb{Z}/n\mathbb{Z}). \)
Theorem (Felix Klein, 1884; G, 1999)

Set \( n = 3, 4, 5 \). Let \( q(x) = x^n + A x^{n-3} + \cdots + B x + C \) be over \( K = \mathbb{Q}(\zeta_n) \) with splitting field \( L \), and assume that \( \text{Gal}(L/K) \simeq \text{PSL}_2(\mathbb{Z}/n\mathbb{Z}) \). Then there exists \( j \in K \) such that \( L K = K(E[n]_x) \) is the field generated by the \( x \)-coordinates of the \( n \)-torsion of an elliptic curve \( E \) with invariant \( j \).

If we define the rational functions

\[
\lambda(z) = \begin{cases} 
\frac{8 z^3 + 1}{z (z^3 - 1)} & \text{for } n = 3, \\
(1 + \zeta_4) \left[ z^2 - (1 + \zeta_4) z - \zeta_4 \right] \left[ z^2 - (1 - \zeta_4) z + \zeta_4 \right] \left[ z^2 + (1 + \zeta_4) z - \zeta_4 \right] & \text{for } n = 4, \\
\left[ z^2 + 1 \right]^2 \left[ z^2 + 2 \left( \zeta_5 + \zeta_5^4 \right) z - 1 \right]^2 \left[ z^2 + 2 \left( \zeta_5^2 + \zeta_5^3 \right) z - 1 \right] & \text{for } n = 5;
\end{cases}
\]

then we have the polynomials

\[
q(x) = \prod_{\nu} \left[ x - \frac{1}{\lambda(\zeta_n^\nu) z} \right] = \begin{cases} 
x^3 + \frac{1}{j} & \text{for } n = 3, \\
x^4 + \frac{32}{j} x + \frac{4}{j} & \text{for } n = 4, \\
x^5 - \frac{40}{j} x^2 - \frac{5}{j} x - \frac{1}{j} & \text{for } n = 5.
\end{cases}
\]
“Kepler’s principal goal was to explain the relationship between the existence of five planets (and their motions) and the five regular solids. It is customary to sneer at Kepler for this. . . . It is instructive to compare this with the current attempts to ‘explain’ the zoology of elementary particles in terms of irreducible representations of Lie groups.”

– Shlomo Sternberg, *Celestial Mechanics* (1969), pg. 95
Motivating Question

Let $X$ be a compact Riemann surface. Fix a function $\phi : X \to \mathbb{P}^1(\mathbb{C})$. For each $z$ in the inverse image of the thrice punctured sphere

$$\phi^{-1}\left(\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}\right) \subseteq X$$

form the elliptic curve

$$E : y^2 = x^3 + \frac{3}{\phi(z) - 1} x + \frac{2}{\phi(z) - 1} \quad \text{where} \quad j(E) = \frac{1728}{\phi(z)}.$$ 

What are the properties of this elliptic curve?

- Which types of functions $\phi : X \to \mathbb{P}^1(\mathbb{C})$ are allowed?

- If $\phi(z)$ has “lots of symmetries,” how does this translate into properties of the torsion elements $K(E[n]_x)$?
Belyi’s Theorem

Let $\phi : X \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function on a Riemann surface $X$. We say $z \in X$ is a critical point if $\phi'(z) = 0$, and $w \in \mathbb{P}^1(\mathbb{C})$ is a critical value if $w = \phi(z)$ for some critical point $z \in X$.

Theorem (André Weil, 1956; Gennadii Vladimirovich Belyi, 1979)

Let $X$ be a compact, connected Riemann surface.

- $X$ is a smooth, irreducible, projective variety of dimension 1. In particular, $X$ is an algebraic variety; that is, it can be defined by polynomial equations.

- If $X$ can be defined by a polynomial equation $\sum_{i,j} a_{ij} z^i w^j = 0$ where the coefficients $a_{ij}$ are not transcendental, then there exists a rational function $\phi : X \to \mathbb{P}^1(\mathbb{C})$ which has at most three critical values.

- Conversely, if there exists rational function $\phi : X \to \mathbb{P}^1(\mathbb{C})$ which has at most three critical values, then $X$ can be defined by a polynomial equation $\sum_{i,j} a_{ij} z^i w^j = 0$ where the coefficients $a_{ij}$ are not transcendental.
Proof: Fix a meromorphic $f : X \to \mathbb{P}^1(\mathbb{C})$ such that $|f^{-1}(z)| = n$ for almost all $z \in \mathbb{P}^1(\mathbb{C})$. For any meromorphic $g : X \to \mathbb{P}^1(\mathbb{C})$, consider

$$\prod_{P \in f^{-1}(z)} (w - g(P)) = \sum_{j=0}^{n} c_j(z) w^j.$$ 

Each coefficient $c_j = c_j(z)$ is a well-defined rational function.

$$c_j(z) = \frac{\sum_{i=0}^{m} a_{ij} z^i}{\sum_{i=0}^{m} a_{in} z^i} \quad \implies \quad \sum_{j=0}^{n} c_j(z) w^j = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} z^i w^j \left/ \sum_{i=0}^{m} a_{in} z^i \right..$$

Hence we have a bijective map

$$X \sim \left\{ (z_1 : z_2 : z_0) \in \mathbb{P}^2(\mathbb{C}) \left| \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} z_1^i z_2^j z_0^{e-i-j} = 0 \right. \right\}$$

$$z \longrightarrow (f(z) : g(z) : 1)$$
Proof (cont’d.): Let \( f : X \to \mathbb{P}^1(\mathbb{C}) \) be a rational function. Following Wushi Goldring, we create a series of compositions

\[
\phi : \begin{array}{c}
X \\
\xrightarrow{f} \\
\xrightarrow{g} \\
\xrightarrow{h} \\
\mathbb{P}^1(\mathbb{C}) \\
\mathbb{P}^1(\mathbb{C}) \\
\mathbb{P}^1(\mathbb{C})
\end{array}
\]

Define \( g_0(z) \) as that polynomial over \( \mathbb{Q} \) which contains the affine critical points of \( f(z) \) as its roots; and define \( \Sigma_0 \) as the set of finite critical values of \( g_0 \).

Recursively define

\[
\Sigma_k = \left\{ w \in \mathbb{A}^1(\mathbb{C}) \mid \begin{array}{c}
w = g_k(z) \text{ for some } z \text{ with } g_k'(z) = 0
\end{array} \right\}
\]

\[
g_{k+1}(z) = \prod_{w \in \Sigma_k} (z - w).
\]

Each \( g_k(z) \) has rational coefficients, distinct roots, and \( \deg g_{k+1} < \deg g_k \). As \( \deg g_{k+0+1} = 1 \) for some positive integer \( k_0 \), define the polynomial

\[
g = g_{k_0} \circ \cdots \circ g_2 \circ g_1
\]

which has rational coefficients. Hence \( g \circ f \) is a rational function whose critical points are the critical points of \( f(z) \), and whose critical values lie in \( \mathbb{P}^1(\mathbb{Q}) \).
Proof (cont’d.): Say that the critical values of \( g \circ f \) lie in the set 
\[
\Sigma = \{(m_1 : m_0), (m_2 : m_0), \ldots, (m_n : m_0), (1 : 0)\} \subseteq \mathbb{P}^1(\mathbb{Q})
\]
where each \( m_k \in \mathbb{Z} \). Consider the rational function 
\[
h(z) = \prod_{k=1}^{n} (m_0 z - m_k)^{e_k} \text{ where } e_k = \frac{\prod_{i<j} (m_j - m_i)}{\prod_{i \neq k} (m_k - m_i)}.
\]
One checks that we have the identities 
\[
\frac{1}{m_0} \frac{h'(z)}{h(z)} = \sum_{k=1}^{n} \frac{e_k}{m_0 z - m_k} = \frac{\prod_{i<j} (m_j - m_i)}{\prod_k (m_0 z - m_k)} \quad \text{and} \quad \sum_{k=1}^{n} e_k = 0
\]
In particular, \( \Sigma \) is the set of critical points of \( h \), so that \( \{0, 1, \infty\} \) is the see of its critical values. Hence \( \phi = h \circ g \circ f \) has critical values only at \( \{0, 1, \infty\} \).

The converse to these statements was shown by André Weil in a 1956 paper entitled “The Field of Definition of a Variety”. \( \square \)
Belyǐ Maps

**Theorem (André Weil, 1956; Gennadiĭ Vladimirovich Belyĭ, 1979)**

A compact, connected Riemann surface $X$ can be defined by a polynomial equation $\sum i,j a_{ij} z^i w^j = 0$ where the coefficients $a_{ij}$ are not transcendental if and only if there exists a rational function $\phi : X \to \mathbb{P}^1(\mathbb{C})$ which has at most three critical values $\{0, 1, \infty\}$.

“This discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections, a shift in particular of my centre of interest in mathematics, which suddenly found itself strongly focused. I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact.


**Definition**

A rational function $\phi : X \to \mathbb{P}^1(\mathbb{C})$ which has at most three critical values $\{0, 1, \infty\}$ is called a Belyǐ map.
Fix a Belyï map $\phi : X \rightarrow \mathbb{P}^1(\mathbb{C})$. Denote the preimages

$$
\begin{align*}
B &= \phi^{-1}(\{0\}) \\
W &= \phi^{-1}(\{1\}) \\
E &= \phi^{-1}([0, 1])
\end{align*}
$$

The bipartite graph $\Gamma_\phi = (V, E)$ with vertices $V = B \cup W$ and edges $E$ is called *Dessin d’Enfant*. We embed the graph on $X$ in 3-dimensions.

I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact. This is surely because of the very familiar, non-technical nature of the objects considered, of which any child’s drawing scrawled on a bit of paper (at least if the drawing is made without lifting the pencil) gives a perfectly explicit example. To such a *dessin* we find associated subtle arithmetic invariants, which are completely turned topsy-turvy as soon as we add one more stroke.

Properties of Dessins

**Theorem**

Let \( \phi : X \to \mathbb{P}^1(\mathbb{C}) \) be a Belyi map on a compact, connected Riemann Surface \( X \). Define a bipartite graph \( \Gamma_\phi = (V, E) \) by choosing

\[
V = \phi^{-1}(\{0, 1\}) \quad \text{and} \quad E = \phi^{-1}([0, 1]).
\]

- If \( z \in \phi^{-1}\left(\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}\right) \subseteq X \) then \( z \) is not a critical point for \( \phi \).

- The graph \( \Gamma_\phi \) can be embedded in \( X \) without crossings.

- The number of vertices, edges, and faces is

\[
v = \left| \phi^{-1}(\{0, 1\}) \right| \quad e = \deg \phi \quad f = \left| \phi^{-1}(\{\infty\}) \right|.
\]

- We have \( v - e + f = 2 - 2g \) in terms of the genus

\[
g = 1 + \frac{1}{2} \left[ \deg \phi - \left| \phi^{-1}(\{0, 1, \infty\}) \right| \right].
\]
Last updated on June 23, 2012.

For the past couple of years, I've been thinking about properties of Dessins d'Enfants. During June 4-8, 2012, I lead a research group on the subject at the Research Experiences for Undergraduate Faculty (REUF) at ICERM in Providence, Rhode Island. I've embarked on a project to realize each of the Archimedean Solids as a Dessin d'Enfant. Below you can find my lecture notes from the workshop and an explorer to play around with some Dessins.

REUF4 Lecture Notes
Dessin Explorer (Mathematica Notebook)
Workshop Photos (Flickr Photostream)

Klein constructed the Platonic Solids as the inverse images of the origin $V = \phi^{-1}(\{0\})$ in terms of the rational functions $\phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ given by

$$\phi(z) = \begin{cases} 
\frac{(z^n + 1)^2}{4z^n} & \text{for the regular polygons,} \\
- \frac{64z^3(z^3 - 1)^3}{(8z^3 + 1)^3} & \text{for the tetrahedron,} \\
108z^4(z^4 - 1)^4 & \text{for the octahedron,} \\
\frac{(z^8 + 14z^4 + 1)^3}{108z^4(z^4 - 1)^4} & \text{for the cube,} \\
\frac{1728z^5(z^{10} - 11z^5 - 1)^5}{(z^{20} + 228z^{15} + 494z^{10} - 228z^5 + 1)^3} & \text{for the icosahedron,} \\
\left(\frac{z^{20} + 228z^{15} + 494z^{10} - 228z^5 + 1}{z^{20} + 228z^{15} + 494z^{10} - 228z^5 + 1}\right)^3 & \text{for the dodecahedron.}
\end{cases}$$

These are examples of Belyï maps on $X = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$. 

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**Examples**

**Lectures on the Icosahedron, Part I**

**Lectures on the Icosahedron, Part II**

**Dessin d'Enfants**

**Belyi's Theorem**

**Children's Drawings**

**Examples**

Klein constructed the Platonic Solids as the inverse images of the origin $V = \phi^{-1}(\{0\})$ in terms of the rational functions $\phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ given by

$$\phi(z) = \begin{cases} 
\frac{(z^n + 1)^2}{4z^n} & \text{for the regular polygons,} \\
- \frac{64z^3(z^3 - 1)^3}{(8z^3 + 1)^3} & \text{for the tetrahedron,} \\
108z^4(z^4 - 1)^4 & \text{for the octahedron,} \\
\frac{(z^8 + 14z^4 + 1)^3}{108z^4(z^4 - 1)^4} & \text{for the cube,} \\
\frac{1728z^5(z^{10} - 11z^5 - 1)^5}{(z^{20} + 228z^{15} + 494z^{10} - 228z^5 + 1)^3} & \text{for the icosahedron,} \\
\left(\frac{z^{20} + 228z^{15} + 494z^{10} - 228z^5 + 1}{z^{20} + 228z^{15} + 494z^{10} - 228z^5 + 1}\right)^3 & \text{for the dodecahedron.}
\end{cases}$$

These are examples of Belyï maps on $X = \mathbb{P}^1(\mathbb{C}) \simeq S^2(\mathbb{R})$. 

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**Number Theory Seminar**

**From Klein's Platonic Solids to Kepler's Archimedean Solids**
Rotation Group $D_n$: Regular Convex Polygon

\[
\phi(z) = \frac{(z^n + 1)^2}{4z^n} : \quad v = n + n, \quad e = 2 \cdot n, \quad f = 2
\]
Rotation Group $A_4$: Tetrahedron

$$\phi(z) = -\frac{64 z^3 (z^3 - 1)^3}{(8 z^3 + 1)^3} : \quad v = 4 + 6, \quad e = 2 \cdot 6, \quad f = 4$$
Rotation Group $S_4$: Octahedron

$$\phi(z) = \frac{108 z^4 (z^4 - 1)^4}{(z^8 + 14 z^4 + 1)^3} : \quad v = 6 + 12, \quad e = 2 \cdot 12, \quad f = 8$$
Rotation Group $S_4$: Cube

$$\phi(z) = \frac{(z^8 + 14z^4 + 1)^3}{108z^4(z^4 - 1)^4} : \quad v = 8 + 12, \quad e = 2 \cdot 12, \quad f = 6$$
Rotation Group $A_5$: Icosahedron

$$
\phi(z) = \frac{1728 z^5 (z^{10} - 11 z^5 - 1)^5}{(z^{20} + 228 z^{15} + 494 z^{10} - 228 z^5 + 1)^3} : v = 12 + 30, e = 2 \cdot 30, f = 20
$$
Rotation Group $A_5$: Dodecahedron

\[ \phi(z) = \frac{(z^{20} + 228 z^{15} + 494 z^{10} - 228 z^5 + 1)^3}{1728 z^5 (z^{10} - 11 z^5 - 1)^5} \]

: $v = 20 + 30$, $e = 2 \cdot 30$, $f = 12$
Thank You!

Questions?