quantum affine algebras of type $A_1^\lambda$ (the Kats-Moody quantum algebras were introduced independently in [8, 1]).

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**LITERATURE CITED**


A HYPOTHESIS OF LITTLEWOOD AND THE DISTRIBUTION OF VALUES OF ENTIRE FUNCTIONS

A. É. Eremenko and M. L. Sodin

For a function f, meromorphic in $\mathbb{C}$, we denote by $\rho_f$ the spherical derivative $\rho_f(z) = |f'(z)|/1+|f(z)|$. Let $D(r) = \{z: |z| < r\}$, and let $m_2$ be Lebesgue measure in $\mathbb{C}$. Following Littlewood [1], we consider the quantities $\psi(n) = \sup \sum_{D(n)} \rho_f dm_2$ as $n \to \infty$, where the upper bound is taken over all polynomials $f$ of degree $n$. We denote analogous quantities for rational functions by $\psi(n)$. It follows from the Schwarz-Bunyakovskii inequality that

$$\psi(n) \leq \left( \sum_{D(n)} m_2 \sup_{D(n)} \rho_f \right)^{1/2} \leq \pi \sqrt{n}.$$

The best known lower bounds were obtained by Hayman [2]: $\psi(n) \geq A_1 \log n$, $\psi(n) \geq A_2 \sqrt{n}$. Here and in the sequel the $A_k$ are absolute constants. In [1] it was conjectured that

$$\psi(n) \leq A_2 n^{1/2}$$

for some $\alpha > 0$.

**THEOREM 1.** $\psi(n) = o(\sqrt{n})$, $n \to \infty$.

From the hypothesis (1) Littlewood derived a remarkable result, which may be stated thus: For an arbitrary entire function $f$ of finite nonzero order an infinitely small portion $S$ of the plane can be found such that for almost all $w$ the roots of the equation $f(z) = w$ lie in $S$, with a negligible exception. The analysis of elliptic functions in [2] shows that this assertion is invalid if entire functions are replaced by meromorphic functions.

Example. $f(z) = \exp z$. We can put $S = \{x + iy: |y| > x^2\}$. For an arbitrary $w$ all the roots of the equation $f(z) = w$, with the exception of a finite number, belong to $S$. The set $S$ has zero density, $m_2(S \cap D(r)) = o(r^2)$, $r \to \infty$.

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THEOREM 2. Let $f$ be an entire function of finite order, and let $\lambda(r)$ be its proximate order. Then there exists a set $S \subset C$ of zero density such that for an arbitrary $w \in C$ the relation $n(r, w) = n_S(r, w) + o(r^{\lambda(r)}), r \to \infty$ is satisfied. Here $n_S(r, w)$ is the number of roots of the equation $f(z) = w$ in $S \cap D(r)$.

The proofs of Theorems 1 and 2 are based on an elementary lemma from potential theory, a particular case of which is contained in [3, 4].

**LEMMA.** Let $u \geq 0$ be a subharmonic function and let $\mu$ be its Riesz measure. Then $\{z: u(z) = 0\} = E \cup L$, where $\mu(E) = 0$, $m_2(L) = 0$.

As $E$ we can take a points having the density of the set $S$.

**Proof of Theorem 1.** We assume that we can find an infinite set of numbers $N_1$ and polynomials $f_n, \deg f_n = n \in N_1$, such that

$$
\sum_{D(1)} \rho_n \ dm_z \geq A_n \sqrt{n}, \quad n \in N_1.
$$

We consider the family of subharmonic functions $v_n(z) = -\frac{1}{n} \log V + \frac{1}{n} |f_n(z)|^2$ with Riesz measures $\mu_n$. A direct calculation shows that the Laplacian

$$
\Delta v_n(z) = \frac{2}{n} \rho_n^2(z).
$$

In particular, $\mu_n(C) = 1$. Selecting a subsequence, we can assume that $\mu_n \to \mu$ weakly in each disk $D(r), r > 0, n \in N_2 \subset N_1$.

Two cases are possible

1°. $\liminf v_n + \infty$. Selecting a subsequence, $N_3 \subset N_2$, we assume that $v_n \to u$ in the mean disk, $n \in N_3$. Applying the lemma to the function $u \geq 0$, we obtain three sets $M, L, E$ such that $u > 0$ on $M$, $\mu(M) = 0$, $m_2(L) = 0$, $D(1) = M \cup L \cup E$. We fix an $\varepsilon > 0$, sufficiently small. We select $\delta, 0 < \delta < \varepsilon$, so that the set $M = \{z \in D(1): u(z) > 2\delta\}$ will possess the property $m_2(M \setminus M') < \varepsilon$. Following this, we select a closed set $E' \subset E$ so that the inequality $m_2(E \setminus E') < \varepsilon$ is satisfied. It is obvious that $\mu(E') = 0$; therefore, for sufficiently large $n \in N_3$, we have $\mu_n(E') < \varepsilon$.

Let us put $L = D(1) \setminus (E' \cup M')$. Then

$$
m_2(L') < 2\varepsilon.
$$

From the convergence of $v_n \to u$, it follows that sets $L_n$ can be found such that $m_2(L_n) < \varepsilon$ and $v_n(z) \geq \delta$ for $z \in M \setminus M', n \in N_3$.

For an arbitrary measurable set $T \subset D(1)$ the Schwarz–Bunyakovskii inequality yields

$$
\int_T \rho_n \ dm_z \leq \left( m_2 \int_{D(1)} \rho_n^2 \ dm_z \right)^{1/2}.
$$

If in inequality (7) we put $T = L' \cup L_n$, we obtain, by virtue of the relations (5) and (6),

$$
\int_L \rho_n \ dm_z \leq \left( 3\varepsilon \int_{D(1)} \rho_n^2 \ dm_z \right)^{1/2} \leq V^2 \varepsilon n, \quad n \in N_3.
$$

Choosing $T = E'$ in inequality (7) and applying relations (3) and (4), we obtain

$$
\int_{E'} \rho_n^2 \ dm_z \leq (\pi^2 \varepsilon \mu_n(E'))^{1/2} \leq \pi \sqrt{\varepsilon n}, \quad n \in N_3.
$$

We note now, by virtue of inequalities (6), that the image of the set $M' \setminus L_n$ under the action of the function $f_n$ has a spherical area not exceeding $2\pi \exp (-2\delta)$ (taking multiplicity into account). Applying inequality (7) with $T = M' \setminus L_n$, we obtain

$$
\int_{M' \setminus L_n} \rho_n \ dm_z \leq (\pi^2 \varepsilon \mu_n(M'))^{1/2} \leq \varepsilon (1), \quad n \in N_3.
$$

Adding this relationship to inequalities (8) and (9), we obtain a contradiction with inequality (2).
2°. \( v_n \to +\infty \). Then, for sufficiently large \( n \in \mathbb{N} \), we have \( v_n(z) \geq 1 \) for \( z \not\in D(1) \setminus L_n \), where \( m_2(L_n) \to 0 \). We then reason as we did in 1°.

This completes the proof of the theorem.

Conjecture. Let \( 0 \leq u \leq 1 \) be a subharmonic function in \( D(1) \). For arbitrary \( \varepsilon > 0 \) we have \( \{ z : u(z) < \varepsilon = L_n \} \), where \( \mu(L_n) \leq \alpha \varepsilon^3 \), \( m_2(L_n) \leq \alpha \varepsilon^3 \), with some absolute constant \( \beta > 0 \).

The proof of Theorem 1 shows that this conjecture would imply the inequality (1) with \( \alpha < \beta/2 \).

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LITERATURE CITED


DYNAMICS OF THE CALOGERO-MOSER SYSTEM AND THE REDUCTION OF HYPERELLIPTIC INTEGRALS TO ELLIPTIC INTEGRALS

A. R. Its and V. Z. Ènol'skii

We consider the algebraic curve \( C = (\alpha, \lambda) \),

\[ \lambda^3 - 3\lambda\psi(\alpha) - \psi'(\alpha) = 0, \quad \psi^2 = 4\psi^3 - s_3\psi - s_2, \]

the three-sheeted covering torus \( M = (\theta, \psi) \) [2], \( \pi_M : C \to M \). The curve (1) represents one of the curves \( \mathcal{K}_n \), introduced by Krichever [1]: \( \det L - \lambda E = 0, \ E_{ij} = \delta_{ij}, \ L_{ij} = (\delta_{ij} - 1) \Phi_{ij} + \delta_{ij}/2, \ \Phi_{ij} = \Phi(x_i - x_j; \alpha), \ i, j = 1, \ldots, n, \ \Phi(x_i; \alpha) = \sigma(x - \alpha) \exp(\xi(x_1) x_i - \sigma(x_1) \sigma(\alpha)), \) whose coefficients \( l_1, \ldots, l_n \) are the motion integrals of the Calogero-Moser system

\[ H = \sum_{j=1}^n \frac{1}{2} y_j^2 - \sum_{i<j} \psi_{ij} y_i y_j = \psi(x_i - x_j), \quad n = \frac{g(g + 1)}{2}, \quad g \in \mathbb{N}. \]

if the quantities \( I_j(x, y), \ j = 1, \ldots, n \), are defined on the locus \( \mathcal{L}_n \) [3],

\[ \mathcal{L}_n = \{(y) : y_j = 0, \ j = 1, \ldots, n\} \times L_n, \quad l_n = \{(x) : \sum_{j=1}^n \psi_{ij} = 0, \ i = 1, \ldots, n\} \]

[i.e., on the set of fixed points of (2)], then for \( n = 3 \) the curve \( \mathcal{K}_n \) has the form (1).

**Lemma.** The curve \( C \) is birationally equivalent to the curve \( \hat{C} = (z, w) \),

\[ w^2 = (z^2 + 3s_3)(z + 3s_3)(z + 3s_3), \]

\[ (3) \]

**Proof.** The curve \( C \) has genus \( g = 2 \) (the number of branchings of \( \pi_M \) is equal to two) and therefore it is hyperelliptic. In the neighborhoods of the points at infinity \( P_j \in C, \ j = 1, 2, 3 \) (situated over \( \alpha = 0 \)), the expansion of \( \lambda(\alpha) \) has the form \( \lambda = l(\alpha + \alpha V_{10}^1 + O(\alpha^2)), \lambda = -2\alpha + \alpha V_{11} + O(\alpha^2) \), respectively. Therefore, the meromorphic function of second order \( z = (\lambda - \psi(\alpha))/3 \) establishes on \( C \) a canonical hyperelliptic structure (the point \( P_3 \) is a Weierstrass point). The asserted birational equivalence of the curves (1) and (3) follows from the equality

\[ \psi = (z^{2/27} + s_3)(z^{2/3} - s_3)^{-1}, \]

which is proved by inserting \( z \) into (1). The equality (4) gives the covering \( \pi_M : C \to M \).