1. Solve the heat equation for a rod

\[ u_t = u_{xx}, \quad 0 \leq x \leq \pi, \quad t \geq 0 \]

with insulated ends, \( u_x(0, t) = u_x(\pi, t) = 0, \quad t \geq 0 \) and the initial temperature \( u(x, 0) = \sin x \). How much time (approximately) is needed for the initial temperature to level, so that its variation along the segment \([0, \pi]\) does not exceed 1%?
2. Suppose that \( f \) is a smooth \( 2\pi \)-periodic function, and

\[
f(t) = \sum_{-\infty}^{\infty} c_n e^{int}
\]

its Fourier expansion. How can one find out from the sequence \((c_n)\) that:

a) \( f \) is real
b) \( f \) is even
c) \( f \) has \( \pi \) as a period.

3. Consider the Sturm-Liouville problem

\[
y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0.
\]

a) Is this problem self-adjoint? Explain your answer.

b) How many eigenvalues \( \lambda_j \) satisfy \( 0 < \lambda_j < 4\pi \)?
4. Given that the Fourier sine series for $f(x) = x(\pi - x)$, $0 \leq x \leq \pi$ is

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin nx,$$

find the solution of the inhomogeneous heat equation

$$u_t = 2u_{xx} + e^{-t}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = f(x).$$
5. Expand the function \( x/|x| \) into a Fourier series on \((-\pi, \pi)\) and use the Parseval identity to find the sum of the series

\[
\sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2}.
\]