Midterm exam. Solutions and comments

1. a) $|z| < \pi/2$ (because $\pi/2$ is the singular point closest to the origin).
   b) $|z| < 3^{-1/2}$ (by Hadamard’s formula)
   c) $|z| < e$. If one uses Hadamard’s formula, one needs to know Stirling’s formula

   \[ n! = n^n e^{-n} \sqrt{2\pi n + o(n)}. \]

   The elementary approach is to use the ratio test.

   \[ |a_n/a_{n+1}| = |z|^{-1} (1 + 1/n)^n \to e/|z|, \]

   so the series converges for $|z| < e$ and diverges for $|z| > e$.

   d) This is not a power series, but a simple change of variable $w = e^z$ reduce it to a power series. Using this or arguing directly we conclude that the region of convergence is the right half-plane.

   e) The RHS has a pole at the origin which is the closest singularity to the point 1, so the series is convergent for $|z - 1| < 1$.

2. a) essential singularity at $z = 1$. In d) and e), essential singularity at 0.

   Most of you had difficulties with b) and c).

   In b), $1 - \cos z$ has a double zero at the origin. The reciprocal has double poles at $2\pi n$, but the double pole at 0 is canceled by $z^3$. So the answer is: removable singularity at 0 and double poles at $2\pi n$ for all integers $n \neq 0$.

   In c), $\cot \pi z$ has simple poles at the integers. and $2/(1 - z^2)$ has simple poles at $\pm 1$. So these points $\pm 1$ need additional investigation: The poles might cancel when we subtract two functions. For example $\cot z - 1/z$ has a removable singularity at the origin. In our case they do not cancel, because

   \[ \frac{2}{1 - z^2} = \frac{1}{1 - z} + \frac{1}{1 + z} \]

   has residues 1 at both poles, while $\cot \pi z$ has residues $1/\pi$. To see this, we write

   \[ \cot \pi z = \frac{\cos \pi z}{\sin \pi z} = \frac{1}{\pi z} + \ldots \]

   near zero. So the residue at 0 is $1/\pi$. Then it is the same at all other integers by periodicity. Thus the answer is: simple poles at all integers.

3. Make a substitution $w = e^{iz}$. Then

   \[ \frac{w - w^{-1}}{i(w + w^{-1})} = 2i. \]
This is a quadratic equation whose solutions are

\[ w = \pm i\sqrt{3}. \]

So

\[ z = -iw = i\log\sqrt{3} \pm \frac{\pi}{2} + 2\pi k. \]

4. Parametrize the circle by \( z = 1 + e^{it} \). Then the integral is equal to

\[
\int_0^{2\pi} (1 + e^{it})(1 + e^{-it})ie^{it} \, dt = i \int_0^{2\pi} (2e^{it} + e^{2it} + 1) \, dt = 2\pi i,
\]

because the integrals of the exponentials from 0 to \( 2\pi \) are equal to zero. (Using sines, cosines and square roots was a bad idea: calculations with complex exponents are almost always simpler than trig calculations).

5. In a) some interpreted the “circle” literally, and others interpreted it as a “disc”. The answer depends on the interpretation, and I gave full credit for a complete explanation in both cases.

“Does there exist a fractional-linear transformation which sends the unit circle to the right half-plane, \( 1/2 \) to 1 and \( 2 \) to \(-1\)?”

No. Because fractional-linear transformations send circles to circles, and a half-plane is not a circle. (References to 1-to-1 or continuity are not enough: there are 1-to-1 maps of a circle to a half-plane, and there are continuous maps of a circle to a right half-plane. It is true that there is no 1-to-1 and continuous map like this, but this is a deep result).

“Does there exist a fractional-linear transformation which sends the unit disc to the right half-plane, \( 1/2 \) to 1 and \( 2 \) to \(-1\)?”

Yes. \( f(z) = 3(1 - z)/(1 + z) \).

b) No. Because \( 1/2 \) and \( 2 \) are symmetric with respect to the unit circle while 1 and 3 are not symmetric with respect to the imaginary axis. Or because \( 1/2 \) and \( 2 \) are in different components of the complement of the unit disc, while 1 and 3 are in the same component of the complement of the imaginary axis.