Analytic invariant curves of rational functions

A. Eremenko

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In the very last paragraph of his famous memoire [2] on iteration of rational functions Fatou writes:

"In nous resterait à étudier les courbes analytiques invariantes par une transformation rationnelle et dont l'étude est intimement liée á celle des fonctions étudiées dans ce Chapitre. Nous espérons y revenir bientôt.

As far as I know, Fatou never returned to this question in his published work.

Which Jordan analytic curves γ in the Riemann sphere can be invariant under rational functions? Of course, γ can be a circle, and it is easy to describe all rational functions which leave a given circle invariant. I know only two classes of examples where γ is not a circle.

1. Let D be a rotation domain (a Siegel disc or an Herman ring) of a rational function f and ϕ the linearizing map from D onto a disc or onto a round ring centered at the origin. Then the ϕ-preimage of any circle centered at the origin is an analytic invariant curve.

If f is a polynomial then the only possible invariant analytic curves are either circles of preimages of circles under the linearizing function of a Siegel disc [1].

2. Let f be a Lattés function satisfying

$$\wp(2z) = f \circ \wp(z),$$

where $\wp$ is the Weierstrass function with periods 1 and i. Let L be the line $\mathrm{Re}\, z = 1/3$. It is easy to see that $\wp(L)$ is an analytic Jordan curve. As $2L \equiv -L$ we conclude that $\wp(L)$ is invariant under f.

Are there any other examples, not coming from the rotation domains or Lattés functions?

The question is related to factorization theory of rational and meromorphic functions, namely to classifying all triples of rational functions that satisfy the equation $f \circ h = h \circ g$.

About this equation, the question is the following: Suppose that f has an invariant circle C. Is it true that $h(C)$ is contained in a circle, unless f and g are Lattés examples?

Peter Müller recently constructed a new class of examples, showing that the answer to the last question is negative.

It remains a question whether there exist invariant curves mapped by a rational function homeomorphically, different from circles, and different from those curves in the Siegel discs or Hermann rings.

References

[1] Yu. Azarina, Invariant analytic curves for entire functions, English version: Siberian Math. J., 30 (1989) 349–353 (1990).

[2] A. Eremenko, Invariant curves and semiconjugacies of rational functions, Fund. Math., 219, 3 (2012) 263–270.

[3] P. Fatou, Sur les équations fonctionnelles, Bull. Soc. Math. Fr., t 48 (1920) 209–314.

[4] Peter Müller, Circles and rational functions, http://mathoverflow.net/questions/103949