UNIFORM APPROXIMATION OF $\operatorname{sgn}(x)$ BY RATIONAL FUNCTIONS WITH PRESCRIBED POLES

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Dedicated to the memory of B. Ya. Levin.

Abstract. For $a \in (0,1)$ let $L_m^k(a)$ be the error of the best approximation of the function $\operatorname{sgn}(x)$ on the two symmetric intervals $[-1,-a] \cup [a,1]$ by rational functions with the pole of degree $2k-1$ at the origin and of $2m-1$ at the infinity. Then the following limit exists

$$\lim_{m\to\infty} L_m^k(a) \left(\frac{1+a}{1-a}\right)^{m-\frac{1}{2}} (2m-1)^{k+\frac{1}{2}} = \frac{2}{\pi}\left(\frac{1-a^2}{2a}\right)^{k+\frac{1}{2}} \Gamma\left(k+\frac{1}{2}\right). \tag{0.1}$$

1. Introduction

This is our second step (for the first one see [5]) on the way to understand better difficulties that up to now not allow to find the Bernstein constant. Recall that Sergey Natanovich found [3, 4] that for the error $E_n(p)$ of the best uniform approximation of $|x|^p$, p is not an even integer, on $[-1,1]$ by polynomials of degree n the following limit exists:

$$\lim_{m\to\infty} n^p E_n(p) = \mu(p) > 0.$$

This result for $p=1$ was obtained by Bernstein in 1914, and he asked the question, whether one can express $\mu(1)$ in terms of some known transcendental functions. This question is still open.

Actually, we solve here a quite interesting problem on asymptotics of the best approximation of $\operatorname{sgn}(x)$ on the union of two intervals $[-1,-a]\cup[a,1]$ by rational functions. In 1877, E. I. Zolotarev [6, 2] found an explicit expression, in terms of elliptic functions, of the rational function of given degree which is uniformly closest to $\operatorname{sgn}(x)$ on this set. This result was subject to many generalizations, and it has applications in electric engineering. In the Zolotarev's case the position of poles of the rational function is free, a natural question is to find the best approximation with the poles and multiplicities in them fixed. Of course, the most natural among them is just to find the best polynomial approximation (the problem that we solved in [5]). Next step is to allow the rational function to have one more pole in the second gap, that is, as the most symmetric case, to allow two poles — at infinity and in the origin.

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Thus the problem is:

Problem 1.1. For $k, m \in \mathbb{N}$, find the best approximation of the function $\operatorname{sgn}(x)$, $|x| \in [a, 1]$, by functions of the form

$$f(x) = \frac{a_{-(2k-1)}}{x^{2k-1}} + \ldots + a_{2m-1}x^{2m-1}$$

and the approximation error $L_m^k(a)$.

One can be interested in many different asymptotic for $L_m^k(a)$ then m or k or they both in a certain prescribed way go to infinity. In this work we concentrate on the case when k is fixed and $m \to \infty$. Note, however, that due to the evident symmetry $L_m^k(a) = L_k^m(a)$ and a bit less evident (6.2) we have simultaneously asymptotic for $k \to \infty$, m is fixed and $k \to \infty$, $m \to \infty$ so that $k = m$.

As it appears all tricks that we used to find precise asymptotic in [5] work in this general case (so we have a method in hands):

1. For each particular k and m we reveal the structure of the extremal function by representing it with the help of an explicitly given conformal mapping.
2. The system of conformal mappings (k is fixed, m is a parameter) converges (in the Caratheodory sense) after an appropriate renormalization. The limit map does not depend on a, thus we obtain asymptotic for $L_m^k(a)$ in terms of a–depending parameters, that we use for renormalization, (an explicit formula) and a k–depending constant say Y_k, which is a certain characteristic of this final conformal map (kind of capacity).

Of course, it is very tempting to guess Y_k directly from the given explicitly conformal map. It might be that we have here special functions that are given in such form that we unable to recognize them. In any case, we would consider this way of finding Y_k as a very interesting open problem. However we are able to find Y_k using the third step of our strategy. Problem 1.1 in an evident way is equivalent to

Problem 1.2. For $p = 2k - 1$ and $n = 2(k + m - 1)$, find the best weighted *polynomial* approximation and the minimal deviation

$$E_n^*(p, a) = \inf_{\{P : \deg P \le n\}} \sup_{|x| \in [a,1]} \left| \frac{|x|^p - P(x)}{x^p} \right|. \tag{1.1}$$

Thus we have $E_n^*(p, a) = L_m^k(a)$. Note that Bernstein himself solve the unweighted

Problem 1.3. For a fixed non even p, find asymptotic for the minimal deviation

$$E_n(p, a) = \inf_{\{P : \deg P \le n\}} \sup_{|x| \in [a,1]} \left| |x|^p - P(x) \right|, \tag{1.2}$$

when n goes to infinity through the even integers.

3. Due to the evident

$$\lim_{a \to 1} \lim_{n \to \infty} \frac{E_n^*(p, a)}{E_n(p, a)} = 1,$$

we can recalculate the constant in Problem 1.3 to the constant related to Problem 1.2 and thus to get explicitly $e^{Y_k} = \frac{\Gamma\left(k+\frac{1}{2}\right)}{2^{k+\frac{1}{2}}\pi}$.

This interplay between Problems 1.2 and 1.3 indicates that most likely one can find our asymptotic formula (0.1) by using original Bernstein's method, though up to the last step our consideration are very direct and simple. However we can go in the opposite direction. In particular in this work we show that the extremal polynomials of Problem 1.3, at least for $p = 1$, also have special representations in terms of conformal mappings. The boundary of the corresponding domains are not so explicit as in our case, they are described in terms of certain functional equations involving unknown function, its Hilbert transform and independent variable (7.2). Precise constants that characterize these equations (counterparts of the constants Y_k), related to them conformal mappings and their asymptotics leave enough space for the hope that for $a = 0$ one also would be able to characterize very similar equations in terms of classical constants.

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2. Special Functions

In this section we introduce certain special conformal mappings that we need in what follows. They marked by a natural parameter k, but in this section k can be just real, $k > 1/2$.

The given k, consider the domain

$$\Pi_k = \mathbb{C}_+ \setminus \{w : \mathrm{Re} w = -\log t,\ |\mathrm{Im} w - k\pi| \le \arccos t,\ t \in (0,1]\} \tag{2.1}$$

Define the conformal map

$$H_k : \mathbb{C}_+ \to \Pi_k$$

normalized by $H_k(0) = \infty_1$, $H_k(\infty) = \infty_2$ (on the boundary we have two infinite points that we denote respectively ∞_1, ∞_2), and moreover

$$H_k(\zeta) = \zeta + \dots, \quad \zeta \to \infty,$$

(that is the leading coefficient is fixed). By D_k we denote the positive number such that $H_k(-D_k) = 0$.

Note that for H_k we have the following integral representation

$$H_k(\zeta) = \zeta + D_k + \int_0^\infty \left(\frac{1}{t-\zeta} - \frac{1}{t+D_k}\right) \rho_k(t)dt, \tag{2.2}$$

where $\rho_k(t) = \frac{1}{\pi}\mathrm{Im} H_k(t)$. Evidently $\rho_k(t) \to k + \frac{1}{2}$, $t \to +\infty$.

Lemma 2.1. *The function H_k possesses the asymptotic*

$$\lim_{\zeta \to -\infty} \left\{ H_k(\zeta) - \zeta + \left(k + \frac{1}{2}\right) \log(-\zeta) \right\} = Y_k, \tag{2.3}$$

where

$$Y_k := D_k + \left(k + \frac{1}{2}\right) \log D_k - \int_0^\infty \frac{\rho_k(t) - \left(k + \frac{1}{2}\right)}{t + D_k} dt. \tag{2.4}$$

Proof. Since

$$\int_0^\infty \left(\frac{1}{t-\zeta} - \frac{1}{t+D_k}\right) \left(\rho_k(t) - \left(k + \frac{1}{2}\right)\right) dt \to -\int_0^\infty \frac{\rho_k(t) - \left(k + \frac{1}{2}\right)}{t + D_k} dt, \tag{2.5}$$

and

$$\left(k + \frac{1}{2}\right) \int_0^\infty \left(\frac{1}{t-\zeta} - \frac{1}{t+D_k}\right) dt = -\left(k + \frac{1}{2}\right) (\log(-\zeta) - \log D_k) \tag{2.6}$$

we get (2.3). □

Finally note that Y_k, as it was defined here, has sense for all real $k > \frac{1}{2}$. As it is shown in Sect. 5 for an integer k we have

$$Y_k = \log\Gamma\left(k+\frac{1}{2}\right) - \left(k+\frac{1}{2}\right)\log 2 - \log\pi.$$

We do not know if these values coincide for non integers k.

3. Extremal problem

Problems 1.1 and 1.2 are related in a trivial way. Recall, for $p = 2k-1$ and $n = 2(k+m-1)$, we have

$$E_n^*(p,a) = L_m^k(a) = \inf_{\{P:\deg P\le 2(m+k-1)\}} \sup_{|x|\in[a,1]} \left|\frac{|x|^{2k-1} - P(x)}{x^{2k-1}}\right|, \tag{3.1}$$

where $a \in (0,1)$, $k, m \in \mathbb{N}$. Evidently, $L_m^k(a)$ can be rewritten in the terms of the best approximation of the function $\operatorname{sgn}(x)$ by functions of the form

$$f(x) = \frac{a_{-(2k-1)}}{x^{2k-1}} + ... + a_{2m-1}x^{2m-1}.$$

Also, it is trivial that the extremal polynomial is even in the first case and the extremal function $f = f(x;k,m;a)$ is odd.

For a parameter $B > 0$ and $k, m \in \mathbb{N}$, $\Omega_m^k(B)$ denote the subdomain of the half strip

$$\{w = u + iv : v > 0,\ 0 < u < (k+m)\pi\}$$

that we obtain by deleting the subregion

$$\{w = u + iv : |u - \pi k| \le \arccos\left(\frac{\cosh B}{\cosh v}\right),\ v \ge B\}.$$

Let $\phi(z) = \phi(z;k,m;B)$ be the conformal map of the first quadrant onto $\Omega_m^k(B)$ such that $\phi(0) = \infty_1$, $\phi(1) = (k+m)\pi$, $\phi(\infty) = \infty_2$. Let $a = \phi^{-1}(0)$. Then a is a continuous strictly increasing function of B, moreover $\lim_{B\to 0} a(B) = 0$ and $\lim_{B\to\infty} a(B) = 1$. Thus we may consider the inverse function $B(a) = B_m^k(a)$, $a \in (0,1)$.

Theorem 3.1. *The error of the best approximation is*

$$L_m^k(a) = \frac{1}{\cosh B_m^k(a)} \tag{3.2}$$

and the extremal function is of the form

$$f(x;k,m;a) = 1 - (-1)^k L_m^k(a)\cos\phi(x;k,m;B(a)), \quad x > 0.$$

Proof. Basically the proof is the same as in [5]. A comparably important difference is as follows. We have to note and prove that on the imaginary axis the extremal function has precisely one zero (there are no critical points and the behavior at $i0$ and at $i\infty$ is evident). At this point $\phi = k\pi + iB$ and we have (3.2). □

4. ASYMPTOTIC

Theorem 4.1. *The following limit exists*

$$\lim_{m\to\infty}\left\{B_m^k(a)-\left(m-\frac{1}{2}\right)\log\frac{1+a}{1-a}-\left(k+\frac{1}{2}\right)\log(2m-1)\right\}$$
$$=\left(k+\frac{1}{2}\right)\log\frac{a}{1-a^2}-Y_k. \tag{4.1}$$

Proof. As in [5] we use the symmetry principle and make a convenient changes of variable to have a conformal map $\Phi_m(Z)=\Phi(Z;k,m;B)$ of the upper plane in the region

$$i(\Omega_m^k(B)\cup\overline{\Omega_m^k(B)})\cup(0,i\pi(m+k)).$$

This conformal map has the following boundary correspondence

$$\Phi_m:(-C_m,-A_m,0,A_m,C_m)\to(-\infty_2,-\infty_1,0,\infty_1,\infty_2),$$

here $A_m=aC_m$ and C_m will be chosen a bit later.

For Φ_m we have the following integral representation

$$\Phi_m(Z)=\left(m-\frac{1}{2}\right)\log\frac{1+\frac{Z}{C_m}}{1-\frac{Z}{C_m}}+\int_{A_m}^{\infty}\left[\frac{1}{X-Z}-\frac{1}{X+Z}\right]v_m(X)\,dX,$$

where

$$v_m(X)=\begin{cases}\frac{1}{\pi}\mathrm{Im}\Phi_m(X), & A_m\le X\le C_m\\ k+\frac{1}{2}, & X>C_m\end{cases} \tag{4.2}$$

Put now

$$H_m^k(\zeta)=\Phi_m(Z)-B_m,\quad Z=A_m+\zeta,$$

then

$$H_m^k(\zeta)=\left(m-\frac{1}{2}\right)\log\frac{1+a+\frac{\zeta}{C_m}}{1-a-\frac{\zeta}{C_m}}+\int_0^{\infty}\left[\frac{1}{t-\zeta}-\frac{1}{t+A_m+\zeta}\right]\hat{v}_m(t)\,dt-B_m,$$

where $\hat{v}_m(t)=v_m(t+A_m)$. Let us rewrite H_m^k in the form that is close to the integral representation of H_k:

$$H_m^k(\zeta)=\left(m-\frac{1}{2}\right)\log\frac{1+\frac{\zeta}{C_m(1+a)}}{1-\frac{\zeta}{C_m(1-a)}}+D_k+\int_0^{\infty}\left[\frac{1}{t-\zeta}-\frac{1}{t+D_k}\right]\hat{v}_m(t)\,dt$$
$$+\left(m-\frac{1}{2}\right)\log\frac{1+a}{1-a}-D_k+\int_0^{\infty}\left[\frac{1}{t+D_k}-\frac{1}{t+A_m+\zeta}\right]\hat{v}_m(t)\,dt-B_m \tag{4.3}$$

Now, we put

$$C_m=\frac{2m-1}{1-a^2}.$$

In this case the first line in (4.3) converges to $H_k(\zeta)$. Since

$$\lim_{m\to\infty}\int_0^{\infty}\left[\frac{1}{t+D_k}-\frac{1}{t+A_m+\zeta}\right]\left(\hat{v}_m(t)-\left(k+\frac{1}{2}\right)\right)dt$$
$$=\int_0^{\infty}\frac{\rho_k(t)-\left(k+\frac{1}{2}\right)}{t+D_k}dt \tag{4.4}$$

and

$$\int_0^\infty \left[\frac{1}{t+D_k} - \frac{1}{t+A_m+\zeta}\right] dt = \log\frac{A_m}{D_k} + \log\left(1+\frac{\zeta}{A_m}\right) \tag{4.5}$$

we have from the second line in (4.3) that

$$\begin{aligned} &\lim_{m\to\infty}\left\{B_m - \left(m-\frac{1}{2}\right)\log\frac{1+a}{1-a} - \left(k+\frac{1}{2}\right)\log A_m\right\} \\ &= -D_k - \left(k+\frac{1}{2}\right)\log D_k + \int_0^\infty \frac{\rho_k(t)-\left(k+\frac{1}{2}\right)}{t+D_k} dt = -Y_k. \end{aligned} \tag{4.6}$$

Thus we get (4.1). In order to prove (0.1) we have to find the constant $2e^{Y_k}$.

□

5. The constant

From the point of view of the best weighted polynomial approximation of the function $|x|^p$ (see Sect. 3) our current result has the form

$$\lim_{m\to\infty}\left(\frac{1+a}{1-a}\right)^{\frac{n}{2}+1} n^{\frac{p}{2}+1}E_n^*(p,a) = \left(\frac{(1+a)^2}{a}\right)^{\frac{p}{2}+1} c(p). \tag{5.1}$$

On the other hand for the uniform approximation of $|x|^p$ (details see in Appendix 1)

$$\lim_{m\to\infty}\left(\frac{1+a}{1-a}\right)^{\frac{n}{2}+1} n^{\frac{p}{2}+1}E_n(p,a) = 2^{\frac{p}{2}+1}a^{\frac{p}{2}-1}\frac{(1+a)^2}{2\left|\Gamma\left(-\frac{p}{2}\right)\right|}. \tag{5.2}$$

Since

$$\lim_{a\to 1}\lim_{n\to\infty}\frac{E_n^*(p,a)}{E_n(p,a)} = 1,$$

we obtain

$$c(p)2^{\frac{p}{2}+1}\left|\Gamma\left(-\frac{p}{2}\right)\right| = 2.$$

Using $\left|\Gamma\left(-\frac{p}{2}\right)\right|\Gamma\left(\frac{p}{2}+1\right) = \pi$, we have

$$c(p) = \frac{2}{\pi}2^{-\frac{p}{2}-1}\Gamma\left(\frac{p}{2}+1\right).$$

This finishes the proof of (0.1).

6. Case $m=k$, $m\to\infty$

It is quite evident that the final configuration of the conformal mapping in this case should be just a symmetrization of the map that we had in the case $k=0$, $m\to\infty$. However it's even much simpler to make this reduction by a suitable change of variable. First we put $a=\alpha^2$, then $x\in[a,1]$ means $y=\frac{x}{\alpha}\in[\alpha,\alpha^{-1}]$ and we have one more symmetry $y\mapsto 1/y$. Therefore the extremal function is symmetric and possesses the representation

$$\tilde{f}(y;m,m) := f(x;m,m;a) = P_{2m-1}\left(\frac{y+y^{-1}}{\alpha+\alpha^{-1}}\right), \tag{6.1}$$

where $P_{2m-1}(t)$ is the best polynomial approximation of sgn (t) on $\left[-1, -\frac{2\alpha}{1+\alpha^2}\right] \cup \left[\frac{2\alpha}{1+\alpha^2}, 1\right]$. Thus we just have

$$L_m^m(a) = L_m^0\left(\frac{2\sqrt{a}}{1+a}\right), \tag{6.2}$$

and

$$\lim_{m\to\infty} L_m^m(a)\left(\frac{1+\sqrt{a}}{1-\sqrt{a}}\right)^{2m-1}(2m-1)^{\frac{1}{2}} = \frac{1-a}{\sqrt{\pi\sqrt{a}(1+a)}}. \tag{6.3}$$

7. UNWEIGHTED EXTREMAL POLYNOMIAL VIA CONFORMAL MAPPING

Let $P_m(z, a)$ be the best uniform (unweighted) approximation of $|x|$ by polynomials of degree not more than $2m$ on two intervals $[-1, -a] \cup [a, 1]$ and $L = L_m(a)$ be the approximation error.

In this section we prove

Theorem 7.1. *There is a curve $\gamma = \gamma_m(a)$ inside the half–strip*

$$\{w = u + iv : u \in (0, (m+1)\pi),\ v > 0\} \tag{7.1}$$

such that the extremal polynomial possesses the representation

$$P_m(z, a) = z + L\cos\phi_m(z, a)$$

where $\phi_m(z, a)$ is the conformal map of the first quadrant onto the region in the half strip (7.1) bounded on the left by $\gamma_m(a)$, which normalized by $\phi_m(a, a) = 0$, $\phi_m(1, a) = (m+1)\pi$ and $\phi_m(\infty, a) = \infty$. Moreover, the curve γ is the image of the imaginary half–axis under this conformal map that satisfies the following functional equation

$$\gamma_m(a) = \{u + iv = \phi_m(iy) :\ L\sin u(y)\sinh v(y) = y,\ y > 0\}. \tag{7.2}$$

Proof. First we clarify the shape of the extremal polynomial. In particular, we prove that $P_m(0, a) > L$. On the way we show the fact that probably interesting on its own: $P_m'(x, a)$ looks pretty similar to the polynomial of the best approximation of sgn (x) on two symmetric intervals [5], with the only difference that the deviations in area should be equal, instead of the maximum modulus. Note that $P_m'(x, a)$ is not however the best L^1 approximation of sgn (x).

Due to the symmetry of $P_m(x, a)$, we can use the Chebyshev theorem with respect to the best approximation of $\sqrt{x}$ on $[a^2, 1]$ by polynomials of degree m. It gives us that $P_m(z, a)$ has $m+2$ points $\{x_j\}$ on interval $[a, 1]$ where $P_m(x_j, a) = x_j \pm L$ (the right half of the Chebyshev set in this case). Moreover, $x_0 = a$ and $x_{m+1} = 1$. At all other points, in addition, we have $P_m'(x_j, a) = 1$, $1 \le j \le m$. Between each two of them we have a point y_j, where $P_m''(y_j) = 0$. Therefore we obtain $2(m-1)$ zeros of the second derivative in $(-1, -a) \cup (a, 1)$ and this is precisely its degree. Thus there is *no other critical points* of $P_m'(z, a)$, in particular, in $(-a, a)$ and on imaginary axis.

From the first consequence, we conclude that on $(-a, a)$ the $P_m'(z, a)$ increases. That is on (a, x_1) the graph of $P_m(z, a)$ is under the line $x \pm L$, depending on the value $P_m(a, a)$, that, recall, should be $a + L$ or $a - L$. Therefore, it is under the line $x + L$ and $P_m(a, a) - a = L$, $P_m(x_1, a) - x_1 = -L$. Continuing in this way we get values of $P_m(x_j, a)$ at all other points x_j by alternance principle. Note that as byproduct we get

$$\int_{x_{i-1}}^{x_i} |P'_m(x,a) - 1|\, dx = 2L$$

for all $1 \le i \le m+1$.

From the second consequence we have that $\mathrm{Im} P'_m(iy) \ge 0$ on the imaginary axis, that is $P_m(iy, a)$, being real, decreases with y, starting from $P_m(0,a) > L$ to $-\infty$. From this remark and the argument principle we deduce that the equation

$$P_m(z,a) - z = tL \tag{7.3}$$

has no solution in the open first quarter for all $t \in (-1,1)$.

Indeed, since $P_m(z,a) - z$ alternate between $\pm L$ in the interval [a,1], (7.3) has $m+1$ solutions, which we denote by $x_j(t)$. Consider now the contour that runs on the positive real axis till $x_j(t) - \epsilon$, then it goes around x_j on the half–circle of the radius ϵ clockwise. After the last of x_j's we continue to go along the contour till the big positive R. Next piece of the contour is a quarter–circle till imaginary axis. Finally, from iR we go back to the origin. On each half–circle of the radius ϵ the argument of the function changes by $-\pi$. On the quarter circle it changes by about $2m \times \frac{\pi}{2} = m\pi$. On the imaginary axis we have $\mathrm{Re}(P_m(iy,a) - iy) = P_m(iy,a)$ and $\mathrm{Im}(P_m(iy,a) - iy) = -y$. Since $P_m(iy,a)$ decreases and much faster than $-y$ (degree of P_m is at least two), the change of the argument on the last piece of the contour is about π. Thus the whole change is $-(m+1)\pi + m\pi + \pi = 0$. Since the function has no poles, it has no zeros in the region.

Thus $\arccos \frac{P_m(z,a)-z}{L}$ is well define in the quarter–plane. We finish the proof by inspection of the boundary correspondence. □

Note two facts: the curve (7.2) has the asymptote $u \to \pi$, $v \to +\infty$ $(y \to +\infty)$ and we have uniqueness of the solution of the functional equation (7.2) due to uniqueness of the extremal polynomial.

8. Appendix 1

From [1], problem 42:

$$E_l\left[\frac{1}{(b+x)^s}\right] \sim \frac{l^{s-1}}{|\Gamma(s)|}\frac{(b-\sqrt{b^2-1})^l}{(b^2-1)^{\frac{s+1}{2}}} \quad (b>1,\ s \ne 0), \tag{8.1}$$

where $E_l[f(x)]$ is the error of the approximation of the function $f(x)$ on the interval $[-1,1]$ by polynomials of degree not more than l.

We change the variable

$$y = \frac{b+x}{b+1}$$

and put $a^2 = \frac{b-1}{b+1}$. Then we have

$$\inf_{P:\deg P \le l} \max_{y \in [a^2,1]} |y^{-s} - P(y)| = (1+b)^s E_l\left[\frac{1}{(b+x)^s}\right].$$

That is

$$E_{2l}(-2s, a) = (1+b)^s E_l\left[\frac{1}{(b+x)^s}\right]. \tag{8.2}$$

Note that
$$b=\frac{1+a^2}{1-a^2},\quad b^2-1=\frac{4a^2}{(1-a^2)^2},$$
and therefore
$$\sqrt{b^2-1}=\frac{2a}{1-a^2},\quad b-\sqrt{b^2-1}=\frac{1-a}{1+a}.$$
Thus from (8.1) and (8.2) we get
$$\begin{aligned}E_{2l}(-2s,a)\sim&\left(\frac{2}{1-a^2}\right)^s\frac{l^{s-1}}{|\Gamma(s)|}\left(\frac{1-a}{1+a}\right)^l\left(\frac{1-a^2}{2a}\right)^{s+1}\\=&a^{-s}\frac{l^{s-1}}{|\Gamma(s)|}\left(\frac{1-a}{1+a}\right)^l\left(\frac{1-a^2}{2a}\right)\\=&a^{-s-1}\frac{l^{s-1}}{|\Gamma(s)|}\left(\frac{1-a}{1+a}\right)^{l+1}\frac{(1+a)^2}{2}.\end{aligned}$$

9. APPENDIX 2

Here we present a "solvable model" for the problem under consideration: we substitute a comparably complicated configuration that we remove from the strip, see Sect. 2, by just two slits. We used this model on the first step of rough understanding a form of the asymptotic and it might be useful for a reader, in particular, it contains the hint that in non–model case the asymptotic of $L_m^{qm}(a)$ for $m\to\infty$ can also be found for an arbitrary $q\in\mathbb{N}$ fixed, see (9.12).

For $B>0$, consider the conformal map $w=\phi(z)$ of the upper half plane $\mathbb{C}_+$ onto the strip
$$\Pi=\{w:0<\mathrm{Im}w<(k+m)\pi\}\tag{9.1}$$
with the cut
$$\gamma_B=\{w:\mathrm{Im}w=k\pi,|\mathrm{Re}w|\geq B\},\tag{9.2}$$
under the normalizations
$$\phi(0)=0,\quad\phi(\pm1)=\pm\infty_2,\tag{9.3}$$
where ∞_2 denote the point on the boundary of the domain when we go to infinity on the level $k\pi<\mathrm{Im}w<(k+m)\pi$. By ∞_1 we denote the point on the boundary that corresponds to the level $0<\mathrm{Im}w<k\pi$. Put $a=\phi^{(-1)}(+\infty_1)$ (therefore $-a=\phi^{(-1)}(-\infty_1)$).

Let us find the precise formula for this map as well as the relation between a and B. We have
$$\begin{aligned}\phi(z)=&k\int_a^\infty\left(\frac{1}{x-z}-\frac{1}{x+z}\right)dx+m\int_1^\infty\left(\frac{1}{x-z}-\frac{1}{x+z}\right)dx\\&+k\log\frac{x-z}{x+z}\Big|_a^\infty+m\log\frac{x-z}{x+z}\Big|_1^\infty\\=&k\log\frac{a+z}{a-z}+m\log\frac{1+z}{1-z}.\end{aligned}\tag{9.4}$$

Further, for $a<x<1$ we have
$$\mathrm{Re}\phi(x)=k\log\frac{x+a}{x-a}+m\log\frac{1+x}{1-x}\tag{9.5}$$

and B corresponds to the critical value of this function on the given interval. For the critical point c we have

$$(\mathrm{Re}\phi)'(c) = -\frac{2ka}{c^2-a^2} + \frac{2m}{1-c^2} = 0. \tag{9.6}$$

Therefore

$$c = \sqrt{\frac{ma^2+ka}{m+ka}} \tag{9.7}$$

and

$$B = k\log\frac{c+a}{c-a} + m\log\frac{1+c}{1-c}. \tag{9.8}$$

Let us mention that the relation between a and B is monotonic, and a runs from 0 to 1 as B runs from 0 to ∞.

As the next step we calculate the asymptotic behavior of B for the fixed a as $m \to \infty$. First we write the asymptotic for c

$$c = \sqrt{\frac{ma^2+ka}{m+ka}} = a + \frac{k}{2m}(1-a^2) + ... \tag{9.9}$$

Therefore

$$\begin{aligned} B =& k\log\left(2a + \frac{k}{2m}(1-a^2) + ...\right) - k\log\left(\frac{k}{2m}(1-a^2)+...\right) \\ &+ m\log\frac{1+a+\frac{k}{2m}(1-a^2)+...}{1-a-\frac{k}{2m}(1-a^2)+...} \\ =& k\log\frac{2a}{1-a^2} + k\log\frac{2m}{k} + ... \\ &+ m\log\frac{1+a}{1-a} + m\log\frac{1+\frac{k}{2m}(1-a)+...}{1-\frac{k}{2m}(1+a)+...} \\ =& m\log\frac{1+a}{1-a} + k\log 2m + k\log\frac{2a}{1-a^2} + k - k\log k + ... \end{aligned} \tag{9.10}$$

Actually it was important for us to note that in the second (logarithmic) term in asymptotic we have the factor k.

To finish this section let us discuss asymptotic for the case

$$k = qm, \quad m \to \infty$$

for a fixed q. Note that now c is just a constant

$$c = \sqrt{\frac{a^2+qa}{1+qa}} \tag{9.11}$$

and we have

$$B = m\left(q\log\frac{c+a}{c-a} + \log\frac{1+c}{1-c}\right). \tag{9.12}$$

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