Find maximum of one function of one variable

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1. Consider the rectangular lattice \( \Lambda = \{ an + ibm : n, m \in \mathbb{Z} \} \) where \( a \) and \( b \) are real numbers,

\[ a^2 + b^2 = 1, \quad a \in (0, 1). \] (1)

Let \( U \) be the unit disc, and \( f_a : U \to \mathbb{C} \setminus \Lambda \) the universal covering map with \( f_a(0) = (a + ib)/2, \ f'_a(0) > 0 \).

Maximize \( f'_a(0) \) for \( a \in (0, 1) \).

2. Of course, this comes from the attempt to find the exact constant in the Landau theorem. Landau’s theorem says that if \( f \) is analytic in the unit disc, and \( f'(0) = 1 \) then the image of \( f \) contains some disc of radius \( L - \epsilon \), for every \( \epsilon > 0 \). The problem is to find the optimal constant \( L \).

Let \( D \) be the image domain, and let \( R \) be the inner radius of \( D \) (which is the sup of the radii of discs contained in \( D \)). Let \( \lambda \) be the (linear) density of the hyperbolic metric in \( D \). Then for every analytic function \( f : U \to D \) we have \( |f'(0)| \leq 1/\lambda(f(0)) \), and this is optimal. So one needs the infimum of \( \lambda(w) \) over all \( w \in D \) and all \( D \) with fixed inner radius.

This seems hopeless. One can restrict the problem by considering only those regions \( D \) whose complements are lattices. A. Baernstein proved that among such regions the local minimum of \( \lambda \) occurs in the center of the complement of the hexagonal lattice.

Our problem is to find the similar infimum for rectangular lattices.

3. The function \( f_a \) in section 1 has almost explicit representation. Consider the quadrilateral \( Q \) in the unit disc bounded by arcs of circles orthogonal to the unit circle, symmetric with respect to the coordinate axes, and having one vertex \( e^{i\theta} \), where \( 0 < \theta < \pi/2 \). Then there is a unique \( a \in (0, 1) \) such that \( Q \) is conformally equivalent to the rectangle \( [0, a, a+ib, ib] \), \( b = \sqrt{1-a^2} \).
by a conformal map which sends vertices to vertices and \( \exp(i\theta) \) to \( a + ib \). It is easy to see that \( f_a \) is the inverse to this map. One can show that our inverse map can be expressed in terms of solutions of the Lamé differential equation

\[
w'' = (\wp - c)w,
\]

but determining the accessory parameter \( c \) is not a trivial matter.