Complex numbers

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1. Complex numbers are expressions of the form $z = a + bi$ where a and b are real numbers. These real numbers are called the real part and the imaginary part of the complex number z. Notation:

$$a = \operatorname{Re} z, \quad b = \operatorname{Im} z.$$

Complex numbers can be added, subtracted and multiplied, so that the usual rules (commutativity, distributivity and associativity) hold, and i^2 is replaced by -1 everywhere.

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a + bi)(c + di) = ac + (ad + bc)i + bdi^2 = (ac - bd) + (ad + bc)i.$$

There formulas define addition and multiplication.

Real numbers can be identified with complex numbers of the form $a + 0i$. For such numbers the definitions above agree with the usual multiplication and addition of real numbers. In particular, real number 0 is identified with $0 + 0i$ and 1 with $1 + 0i$.

Moreover we can dispense with those zeros and simply write

$$a = a + 0i, \quad bi = 0 + bi.$$

It is important that division is possible on every non-zero number:

$$\frac{1}{a + bi} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i. \tag{1}$$

Here we used that for real a, b, $a^2 + b^2 \neq 0$ unless $a = 0$ and $b = 0$.

Therefore, all arithmetic operation can be performed on complex numbers by the same rules as for real (or rational) numbers.

In particular, one can solve systems of linear equations, compute determinants, etc. with complex numbers.

Such collection of objects with two operations is called a field. Other examples of fields that you know are: rational numbers, real numbers, and perhaps the field of two elements, which consists of just two "numbers" 0 and 1, and the operations are defined as follows

$$\begin{array}{ccc} + & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{ccc} \times & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}.$$

Integers do not form a field because division of integers does not always give an integer. Positive rational numbers also do not (subtraction is not always possible).

For a complex number $z = a + bi$ we denote $\overline{z} = a - bi$. This is called the conjugate number. This operation commutes with all algebraic operations:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \quad \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}.$$

It follows that whenever you have a true equality, and you conjugate everything in it, you obtain a true equality. Real numbers are characterized as those complex numbers which satisfy $\overline{z} = z$. Also notice that

$$\overline{\overline{z}} = z$$

for all complex z.

As an important application, consider a polynomial P *with real coefficients*. If z is a root, that is $P(z) = 0$ then $P(\overline{z}) = 0$ so $\overline{z}$ is also a root. Thus non-real roots of a real polynomial come in complex conjugate pairs (and members of the pair have the same multiplicity).

Real and imaginary parts of a complex number z can be expressed in terms of $z, \overline{z}$:

$$\operatorname{Re} z = (z + \overline{z})/2, \quad \operatorname{Im} z = (z - \overline{z})/(2i).$$

If $z = a + bi$ then $z\overline{z} = a^2 + b^2 \geq 0$, and the positive square root of this is called the *absolute value* and is denoted by $|z|$. This matches the usual absolute value if z happens to be real.

Absolute value has the following properties:

$$\max\{|\mathrm{Re}\, z|, |\mathrm{Im}\, z|\} \leq |z| \leq |\mathrm{Re}\, z| + |\mathrm{Im}\, z|, \tag{2}$$

$$|z_1 z_2| = |z_1||z_2|, \tag{3}$$

$$|z_1 + z_2| \leq |z_1| + |z_2|. \tag{4}$$

The first two properties easily follow from the definitions; to prove the third one we compute:

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}\,(z_1 z_2) \\ &\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1|^2 + |z_2|^2)^2. \end{aligned}$$

Taking the square root we obtain (4).

Complex numbers can be visualized as:

a) vectors in the plane, or

b) points in the plane.

To each complex number $z = a + bi$ we can put into correspondence the vector $(a, b)^T$, or a point with coordinates (a, b). (Here T is the transposition. The vectors are always column-vectors).

Addition of complex numbers and vectors is performed by the same rule, so we have a geometric interpretation of addition: it can be visualized as the "parallelogram rule" of addition of vectors.

With this interpretation, $|z|$ is the length of the vector (or the distance of a point from the origin if we use interpretation b)). Inequality (4) becomes the *triangle inequality* which says that the length of a side is at most the sum of the lengths of two other sides.

Absolute value permits to define the *distance* between two complex numbers z_1 and z_2 as $|z_1 - z_2|$. Once the distance is defined the limits of sequences can be also defined.

Inequality (2) shows that a sequence of complex numbers converges if and only if the sequences of real parts and imaginary parts converge separately.

We can introduce polar coordinates in the plane: for a point (x, y) they are $r = \sqrt{a^2 + b^2}$, the distance from the origin, and the "polar angle" ϕ, counted counterclockwise from the positive direction of the x-axis to the direction of the vector $(x, y)^T$. Then, as you know from analytic geometry, or calculus,

$$x = r \cos \phi, \quad y = r \sin \phi.$$

Thus every complex number $z = x + yi$ can be written as

$$z = r(\cos\phi + i\sin\phi), \quad \text{where} \quad r = |z| \geq 0. \tag{5}$$

This angle ϕ is called an *argument* of z, denoted by $\arg z$. Strictly speaking, every number $z \neq 0$ has infinitely many arguments: they differ by multiples of 2π. That's why I wrote "an argument", rather than "the argument". Notation $\arg z$ means the set of all these numbers, so it is not a usual function; it takes infinitely many values for each z.

Representation (5) is called the "trigonometric form" of a complex number. This form is convenient for multiplication. Indeed, multiplying two numbers of this form, we obtain

$$r_1 r_2 \left((\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2) + i(\cos\phi_1\sin\phi_2 + \sin\phi_1\cos\phi_2)\right)$$

$$= r_1 r_2 \left(\cos(\phi_1 + \phi_2) + i\sin(\phi_2 + \phi_2)\right).$$

So, when we multiply two numbers, their absolute values are multiplied (which we already know (3)) and *arguments are added*.

2. The exponential function can be defined as

$$e^{x+iy} = e^x(\cos y + i\sin y). \tag{6}$$

Then

$$e^0 = 1, \tag{7}$$

$$e^{z_1+z_2} = e^{z_1}e^{z_2}. \tag{8}$$

For example,

$$i = e^{i\pi/2}, \quad -1 = e^{i\pi}, \quad 1 = e^{2i\pi}.$$

Now every non-zero complex number can be written as

$$z = |z|e^{i\phi}, \tag{9}$$

where is a polar angle, $\phi \in \arg z$. I recall that $\arg z$ stands for infinitely many numbers, adding any multiple of $2\pi i$ to ϕ does not change the expression (9).

We can also write

$$|e^z| = e^{\operatorname{Re} z}, \quad \arg e^z = \operatorname{Im} z + 2\pi n,$$

where n is any integer.

With this approach, properties of trigonometric functions are assumed to be known, and Euler's formula is simply a definition.

Alternatively, one can define

$$e^z = \sum_{n=1}^{\infty} \frac{z^n}{n!},$$

then prove that this series is convergent for all z, and derive the main properties (7) and (8). Then one can *define* sin and cos by the formula

$$\cos t + i \sin t = e^{it},$$

and *derive* all trigonometric formulas from (7) and (8).

Exercises. All answers must be written in the form $a + bi$, where a and b are real numbers.

1. Compute

$$1/i, \quad \frac{1-i}{1+i}, \quad (1 + i\sqrt{3})^{10}.$$

2. Solve the system of linear equations

$$\begin{array}{rcrcl} (1+i)x & + & 2y & = & 1 \\ x & + & 2iy & = & i. \end{array}$$

3. Find all solutions of the quadratic equation

$$z^2 - (1+i)z + 2 = 0.$$

4. Show that for every complex $w \neq 0$ the equation

$$z^n = a$$

has exactly n distinct solutions and write them explicitly using trigonometric functions.

5. To each complex number $x + iy$ one can put into correspondence a real matrix

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

We obtain a one-to-one correspondence between complex numbers and matrices of this kind. Check that this correspondence agrees with addition and multiplication (takes sums to sums and product of complex numbers to products of matrices).

What is the interpretation of $|z|^2$ in terms of this matrix? What is the geometric meaning of these matrices? Find

$$\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}^{20}.$$