In the definition of a (dis-)connected set $E$ one can replace the requirement that $A \cap B = \emptyset$ by the weaker requirement that $A \cap B \cap E = \emptyset$.

(This is actually the definition in the book. But the two definitions are equivalent, as the following simple argument shows.)

**Theorem.** Let $E$ be a set in $\mathbb{R}^n$. Suppose that $A$ and $B$ are open,

$$A \cap E \neq \emptyset, \quad B \cap E \neq \emptyset, \quad E \subset A \cup B,$$

and

$$A \cap B \cap E = \emptyset.$$  \hspace{1cm} (2)

Then one can find two other open sets $A_1$ and $B_1$ such that all properties (1) are satisfied by $A_1$ and $B_1$, and (2) is replaced by a stronger property

$$A \cap B = \emptyset.$$ \hspace{1cm} (3)

**Proof.** Let $x \in A \cap E$. Then there exists $r(x) > 0$ such that

$$B(x, r(x)) \cap B \cap E = \emptyset.$$ \hspace{1cm} (4)

Here $B(x, r)$ is the open ball of radius $r$. Indeed, if this is not so, then one can find a sequence $y_n \in B \cap E$ which tends to $x$. Then, because $A$ is open, all $y_n$ with sufficiently large $n$ belong to $A$, and this contradicts (2). This proves the existence of $r(x)$ such that (4) holds.

Similarly, for every $y \in E \cap B$, there exists $r(y) > 0$ such that

$$B(y, r(y)) \cap A \cap E = \emptyset.$$ \hspace{1cm} (5)

Now put

$$A_1 = \bigcup_{x \in A \cap E} B(x, r(x)/3), \quad B_1 = \bigcup_{y \in B \cap E} B(y, r(y)/3).$$

These sets are open, as unions of open balls, they both intersect $E$ and their union contains $E$. It remains to show that their intersection is empty.

Suppose that this is not the case. Then some ball $B(x, r(x)/3)$ intersects some ball $B(y, r(y)/3)$. Suppose, without loss of generality, that $r(x) \geq r(y)$. Then $\|x - y\| \leq r(x)/3 + r(y)/3 < r(x)$, so $B(x, r(x))$ contains $y$, but this contradicts the definition of $r(x)$, see (4), because $y \in B \cap E$. If $r(y) \geq r(x)$, one argues similarly using (5) instead of (4).

This ends the proof.

**Remark.** The argument works in any metric space, not only in $\mathbb{R}^n$. 

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