

Let F be a polynomial in two variables. Assume that the differential equation

$$F(y^{(k)}, y) = 0$$

has a solution $y(z)$ which is meromorphic in the whole plane. What can be said about y ?

For $k = 1$, a complete answer is well known. Namely, all meromorphic solutions belong to the class W , which consists by definition of: all elliptic functions, all functions of the form $R(\exp(az))$, R rational and $a \neq 0$ is a complex number, and all rational functions. And vice versa, each function of the class W satisfies some differential equation of the above form with $k = 1$.

For $k = 2$ Emile Picard proved in 1880 that all meromorphic solutions also belong to the class W . This result was forgotten, in 1978 E. Hille stated it as a conjecture, and in 1981 S. Bank and R. Kaufman gave a proof, more complicated than the one Picard published 101 years before.

It follows from another theorem of Picard that non-constant meromorphic solutions are possible only if the genus of the curve $F(x, y) = 0$ is at most 1.

In the case when the genus is equal to 1 I proved for every k , that all meromorphic solutions must be elliptic functions.

Thus only the case of genus 0 remains. I also proved the following. If k is odd and a solution y is meromorphic in the plane and has at least one pole, it must belong to W .

The simplest equation not covered by these results is

$$y^{(IV)} = 24y^5.$$

It evidently has solutions of the form $1/(z - b)$. Does it have any other meromorphic solutions?

Added on May 22, 2007. L. W. Liao, T. W. Ng and the present author recently proved that for every k , all meromorphic solutions of $F(y^{(k)}, y) = 0$ having at least one pole belong to W . The preprint will be posted on this site soon.

This proof also implies that the only meromorphic functions that satisfy $y^{(k)} = y^m$ with $m > 1$ are of the form $c(z - b)^{-n}$.

The question of description of entire solutions remains open. Notice that non-trivial entire solutions of $y''' = y$ do not belong to W . So it is not completely clear what is the right conjecture about entire solutions.