Disconjugacy and the Secant Conjecture

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Let V be a real vector space of dimension n whose elements are real functions on an interval $[a, b]$. We say that V is disconjugate if one of the following equivalent conditions is satisfied:

a) Every $f \in V \backslash \{0\}$ has at most $n - 1$ zeros, or

b) For every n distinct points $z_1, \ldots, z_n$ on $[a, b]$ and every basis $f_1, \ldots, f_n$ of V we have $\det(f_i(x_j)) \neq 0$.

One can replace "every basis" by "some basis" in b) and obtain an equivalent condition.

A space of real functions on an open interval is called disconjugate if it is disconjugate on every closed subinterval.

We are only interested here in spaces V consisting of polynomials.

Suppose that a positive integer d is given, and V consists of polynomials of degree a most d. Then every basis $f_1, \ldots, f_n$ of V defines a real rational curve $\mathbf{RP}^1 \to \mathbf{RP}^{n-1}$ of degree d. Indeed, we can replace very $f_j(x)$ by a homogeneous polynomial $f_j^*(x_0, x_1)$ of two variables of degree d, such that $f_j(x) = f_j^*(1, x)$, and then $f_1^*, \ldots, f_n^*$ define a map $f : \mathbf{RP}^1 \to \mathbf{RP}^{n-1}$ (if polynomials have a common root, divide it out).

Then the geometric interpretation of disconjugacy is:

c) The curve f constructed from a basis in V is *convex*, that is intersects every hyperplane at most $n - 1$ times.

Taking any basis $f_1, \ldots, f_n$ in V we can consider its Wronski determinant $W = W(f_1, \ldots, f_n)$. Changing the basis results in multiplication of W by a non-zero constant, so the roots of W only depend on V.

Conjecture 1. *Suppose that all roots of W are real. Then V is disconjugate on every interval that does not contain these roots.*

This is known for $n = 2$ with arbitrary d (see below), and for $n = 3, d \leq 5$ by direct verification with a computer.

This conjecture arises in real enumerative geometry (Schubert calculus), and we explain the connection. The problem of enumerative geometry we are interested in is the following:

Let $m \geq 2$ and $p \geq 2$ be given integers. Suppose that mp linear subspaces of dimension p in general position in $\mathbf{C}^{m+p}$ are given. How many linear subspaces of dimension m intersect all of them?

The answer was obtained by Schubert in 1886 and it is

$$d(m,p) = \frac{1!2!\dots(p-1)!(mp)!}{m!(m+1)!\dots(m+p-1)!}.$$

Now suppose that all those given subspaces are real. Does it follow that all p-subspaces that intersect all of them are real? The answer is negative, and we are interested in finding a geometric condition on the given p-spaces that ensure that all $d(m,p)$ m-subspaces that intersect all of them are real.

One such condition was proposed by B. and M. Shapiro. Let $F(x) = (1, x, \dots, x^d), d = m + p - 1$ be a rational normal curve, a. k. a. moment curve. Suppose that the given p-spaces are osculating F at some real points $F(x)$. This means that subspaces X_j are spanned by the (row)-vectors $F(x_j), F'(x_j), \dots, F^{(p-1)}(x_j)$ for some real $x_j, 1 \leq j \leq mp$. Then all m-subspaces that intersect all X_j are real.

This was conjectured by B. and M. Shapiro and proved by E. Mukhin, V. Tarasov and A. Varchenko (MTV) [3].

We are interested in the following generalization of this result.

Secant Conjecture. *Suppose that each of the mp subspaces $X_j, 1 \leq j \leq mp$ is spanned by p distinct real points $F(x_{j,k}), 0 \leq k \leq p-1$, and that these sets of points are separated, that is $x_{j,k} \in I_j$, where $I_j \subset \mathbf{RP}^1$ are disjoint intervals. Then all m-subspaces which intersect all X_j are real.*

This is known when $p = 2$, [1] and has been tested on a computer for $p = 3$ and small m, [2]. The special case when the groups $\{x_{j,k}\}_{k=0}^{p-1}$ form arithmetic progressions, $x_{j,k} = x_{j,0} + kh$ has been also established [4].

Next we show how the Secant Conjecture follows from Conjecture 1 and the results of MTV.

Let I_j be the intervals with disjoint closures which contain the $x_{j,k}$. We may assume without loss of generality that $\infty \notin I_j$. We place on each I_j

a point y_j, and consider the $d(m,p)$ real rational curves $\mathbf{R} \to \mathbf{RP}^p$ with inflection points at y_j. These curves exist by the MTV theorem, and they depend continuously on the y_j.

Let $f = (f_0 \ldots, f_{p-1})$ be one of these curves. Fix $k \in \{1, \ldots, mp\}$. For all $j \neq k$, fix all $y_j \in I_j$. When y_k moves on I_k from the left end to the right end, the determinant $\det(f_i(x_{k,m}))_{i,m=0}^{p-1}$ must change sign. So this determinant is 0 for some position of y_k on I_k.

Then it follows by a well-known topological argument that one can choose all $y_j \in I_j$ in such a way that $\det(f_i(x_{j,m})) = 0$ for all j.

Thus we have constructed $d(m,p)$ real solutions of the secant problem. As the total number of solutions is also $d(m,p)$, for generic data, we obtain the result.

Proof of the Conjecture for $n = 2$. We have two real polynomials f_1 and f_2, such that $f_1' f_2 - f_1 f_2'$ has only real zeros. This means that the rational function $F = f_1/f_2$ is real and all its critical points are real. Let $I \subset \mathbf{R}$ be a closed interval without critical points. Then f is a local homeomorphism on I, so $F(I + i\epsilon)$ belongs to one of the halfplanes $\overline{\mathbf{C}} \backslash \overline{\mathbf{R}}$, for all sufficiently small $\epsilon > 0$. Suppose without loss of generality that it belongs to the upper half-plane H. Let D be the component of $F^{-1}(H)$ that contains $F(I + i\epsilon)$. Then D is a region in H with piecewise analytic boundary, and $I \subset \partial D$. The map $F : D \to H$ is a covering because it is proper and has no critical points. As H is simply connexted, D must be simply connected and $F : D \to H$ must be a conformal homeomorphism. Then $F^{-1} : H \to D$ is a conformal homeomrophism. As ∂D is locally connected, this homeomprhism extends to $F^{-1} : \overline{H} \to \overline{D}$. This last map must be injective because this is a left inverse of a function. Thus $F^{-1} : \overline{H} \to \overline{D}$ is a homeomorphism. Then $F : \overline{D} \to \overline{H}$ must be also a homeomorphism, in particular F is injective on I.

This implies that the linear span of f_1, f_2 is disconjugate.

References

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[4] E. Mukhin, V. Tarasov and A. Varchenko, On reality property of Wronski maps, Confluentes Math., 1 (2009) 225–247.