ON THE DISTRIBUTION OF VALUES OF MEROMORPHIC FUNCTIONS OF FINITE ORDER

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Given a sequence of complex numbers $z_k \to \infty$, we define a counting measure $\kappa$ by setting $\kappa(E) = \text{card}\{k : z_k \in E\}$ for every $E \subset \mathbb{C}$.

Let $f$ be a meromorphic function of finite order $\rho > 0$. We fix pairwise distinct $a_1, \ldots, a_q \in \mathbb{C}$, $q \geq 3$, and we denote by $\mu_k$ the counting measures of the sequences of $a_k$-points of the function $f$, taking account of multiplicity. We study the asymptotic behavior of the measures $\mu_k$ and the relations among them by using the methods of the theory of limit sets of subharmonic functions [1]-[5]. We employ the standard notation of Nevanlinna theory.

1. Let $V(r) = r^{\rho(r)}$, where $\rho(r)$ is a proximate order of the function $f$. There is a representation $f = f_1/f_2$, where the $f_i$ are entire functions with the property

$$T(r, f_i) = O(V(r)), \quad r \to \infty, \quad i = 1, 2$$

(the $f_i$ may vanish simultaneously). Let $\mu_0$ be the counting measure for the sequence of common zeros of the functions $f_1$ and $f_2$. We consider the subharmonic functions

$$u_k = \log |f_1 - a_k f_2|$$

(if $a_k = \infty$, then $u_k = \log |f_2|$). The Riesz measure of the function $u_k$ equals $\mu_k + \mu_0$. We introduce another subharmonic function:

$$u = \log \frac{|f_1|^2 + |f_2|^2}{4},$$

the Riesz measure of which has the form $\mu + \mu_0$, where $\mu$ is an absolutely continuous measure with density $\pi^{-1}|f'|^2/(1 + |f|^2)^2$. Then

$$T(r, f) = \int_0^r \frac{dt}{t} \mu(\{z : |z| \leq t\}),$$

$$N(r, a_k, f) = \int_1^r \frac{dt}{t} \mu_k(\{z : |z| \leq t\}) + O(\log r), \quad r \to \infty.$$ 

For each $r \geq 1$, we define the operator $L_r$ acting on functions $w$ and on measures $\kappa$ by the formulas

$$(L_r w)(z) = w(rz)/V(r), \quad (L_r \kappa)(E) = \kappa(rE)/V(r),$$

where $E \subset \mathbb{C}$ is an arbitrary Borel set. Furthermore, we employ the topology of the space $D'$ of generalized functions—the dual to the space of infinitely differentiable, compactly supported functions. It follows from (1) that the families $(L_r u)$, $(L_r u_k)$, $(L_r \mu)$, and $(L_r \mu_k)$, $r \geq 1$, are precompact in $D'$. Let $r_j \to \infty$ be an arbitrary sequence, and let $L_{j} = L_{r_j}$. Choosing a subsequence, if necessary, we may assume that $L_j u_k \to v_k$, $1 \leq k \leq q$; $L_j \mu_k \to v_k$, $0 \leq k \leq q$; and $L_j u \to v$ and $L_j \mu \to \nu$,
$j \to \infty$; here $v$ and $v_k$ are subharmonic functions with Riesz measures $\nu + \nu_0$ and $\nu_k + \nu_0$ respectively. The limit measures $\nu_k$ characterize the asymptotic distribution of $\alpha$-points of the function $f$, and the measure $\nu$ describes the distribution of $\alpha$-points for almost all $\alpha \in \mathcal{C}$ \cite{4}.

The set of functions $w : \mathcal{C} \to \mathbb{R}$ is provided with a natural partial order: $w_1 \leq w_2$ if $w_1(z) \leq w_2(z)$, $z \in \mathcal{C}$. The space of charges in $\mathcal{C}$ (a charge is the difference of two locally finite Borel measures) is also provided with a partial order: $\kappa_1 \leq \kappa_2$ if $\kappa_2 - \kappa_1$ is a measure. We denote the least upper bound and greatest lower bound of finite families of charges or functions (relative to the above ordering) by $\vee$ and $\wedge$ respectively.

It follows from (2) and (3) that $u(z) = u_j(z) \vee u_k(z) + O(1)$, $z \to \infty$, $1 \leq j < k \leq q$, whence

\begin{equation}
(4) \quad v = v_j \vee v_k, \quad 1 \leq j < k \leq q.
\end{equation}

In particular, we have $v = \bigvee_{1 \leq k \leq q} v_k$. Relation (4) was used in \cite{5} to derive an inequality analogous to the second fundamental theorem of Nevanlinna theory. Here we derive a sharper relation.

We recall that the fine topology in $\mathcal{C}$ is the smallest topology in which all subharmonic functions are continuous \cite{7}. We consider the fine open sets

\begin{equation}
(5) \quad E_k = \{z : v_k(z) < v(z)\}.
\end{equation}

It follows from (4) that $E_j \cap E_k = \emptyset$ when $j \neq k$. We denote by $\nu_k^*$ the restriction of the charge $\nu_k$ to the set $\mathcal{C} \setminus E_k$.

**Theorem.** It follows from (4) that

\begin{equation}
(6) \quad \sum_{k=1}^{q} (\nu - \nu_k^*) = 2\nu - \bigwedge_{1 \leq k < j \leq q} (\nu_k^* + \nu_j^*).
\end{equation}

If $f$ is entire, we take $a_q = \infty$. Then $\nu_q = 0$, and (6) gives

\begin{equation}
(q - 2)\nu = \sum_{k=1}^{q-1} \nu_k^* - \bigwedge_{1 \leq k \leq q-1} \nu_k^*.
\end{equation}

In particular, when $q = 3$ we obtain the relation $\nu = \nu_1^* \vee \nu_2^*$.

By weakening (6), we obtain that for meromorphic functions

\begin{equation}
(7) \quad \sum_{k=1}^{q} (\nu - \nu_k) \leq 2\nu.
\end{equation}

Hence we may deduce the second fundamental theorem of Nevanlinna theory in the form

\begin{equation}
\sum_{k=1}^{q} m(r, a_k, f) \leq 2T(r, f) + o(V(r)), \quad r \to \infty.
\end{equation}

Details and refinements are contained in \cite{5}.

Relation (6) has two advantages in comparison with the second fundamental theorem. Firstly, (6) and (7) have a local character, which makes it possible to investigate Borel rays, filling discs, etc. \cite{4}. Secondly, (6) is an equality, while the second fundamental theorem is an inequality. Our approach does not use the derivative. This is both an advantage (the possibility of generalizations \cite{5}) and a disadvantage: (6) does not contain information on multiple points of the function $f$. 

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2. Auxiliary assertions from potential theory.

**Lemma 1** ([7], p. 186). Let \( w_1 \) and \( w_2 \) be subharmonic functions with Riesz measures \( \kappa_1 \) and \( \kappa_2 \). If \( w_1(z) = w_2(z) \), \( z \in E \), then the restrictions of the measures \( \kappa_1 \) and \( \kappa_2 \) to the fine interior of the set \( E \) coincide.

A function \( w \) that is representable as the difference of two subharmonic functions is called \( \delta \)-subharmonic. We denote by \( \mu[w] \) the Riesz charge of the \( \delta \)-subharmonic function \( w \). The operators \( \wedge \) and \( \vee \), applied to finite families, do not go out of the class of \( \delta \)-subharmonic functions.

**Lemma 2.** Let \( w_1, \ldots, w_n \) be \( \delta \)-subharmonic functions. Then

\[
\mu \left[ \bigwedge_{k=1}^{n} w_k \right] \geq \bigwedge_{1 \leq k < j \leq n} \mu[w_k \wedge w_j].
\]

This assertion was proved in [5] for continuous functions \( w_1, \ldots, w_n \). B. Fuglede kindly explained to the authors that an analogous proof carries over to the general case if one uses results from fine potential theory [7]–[9].

We will say that a relation between charges holds on the set \( X \) if it is true for the restrictions of these charges to \( X \).

**Lemma 3** [10]. Let \( w_1 \) and \( w_2 \) be \( \delta \)-subharmonic functions with \( w_1 \leq w_2 \). If \( E = \{ z : w_1(z) = w_2(z) \} \), then \( \mu[w_1] \leq \mu[w_2] \) on \( E \).

3. Proof of the theorem. We may assume that \( \nu_0 = 0 \), for otherwise we subtract the potential of the measure \( \nu_0 \) from all the functions \( \nu \) and \( \nu_k \). The new functions satisfy (4) as before, and their Riesz measures will be \( \nu \) and \( \nu_k \) respectively. The sets \( E_k \) and the measures \( \nu_k^* \) do not change.

Fixing \( k \), we will verify (6) on \( E_k \). By definition,

\[
\nu_k^* = 0 \quad \text{on} \quad E_k.
\]

Furthermore, if \( j \neq k \), then \( \nu_j^*(z) = \nu(z) \) for \( z \in E_k \) in view of (4). Inasmuch as the set \( E_k \) is fine open, we conclude by Lemma 1 that

\[
\nu_j^* = \nu_j = \nu \quad \text{on} \quad E_k \quad \text{if} \quad j \neq k.
\]

It follows from (8) and (9) that (6) holds on \( E_k \).

We now prove (6) on \( E = \mathbb{C} \setminus \bigcup E_k = \{ z : v(z) = \bigwedge_{1 \leq k \leq q} v_k(z) \} \). We set

\[
w_j = v + v_j;
\]

\[
w = \bigwedge_{1 \leq j \leq q} w_j = v + \bigwedge_{1 \leq j \leq q} v_j = \sum_{j=1}^{q} v_j - (q - 2)v
\]

(the last equality holds by (4)). From (11) we obtain

\[
\mu[w] = \sum_{j=1}^{q} \nu_j - (q - 2)v.
\]

In particular,

\[
\mu[w] = 2v - \sum_{j=1}^{q} (\nu - \nu_j^*) \quad \text{on} \quad E.
\]
We estimate the restriction of the charge \( \mu[w] \) from above and below. In view of (4), for all \( k \neq j \) we have \( w \leq v_k + v_j \), where there is equality on \( E \). We conclude by Lemma 3 that
\[
\mu[w] \leq \nu_k + \nu_j = \nu_k^* + \nu_j^* \quad \text{on } E,
\]
or
\[
(13) \quad \mu[w] \leq \bigwedge_{1 \leq k < j \leq q} (\nu_k^* + \nu_j^*) \quad \text{on } E.
\]
On the other hand, for all \( j \) and \( k \) we set
\[
(14) \quad w_{jk} = w_j \land w_k \geq v_j + v_k
\]
and note that (14) reduces to equality on \( E \). By Lemma 3,
\[
\mu[w_{jk}] \geq v_j + v_k = v_j^* + v_k^* \quad \text{on } E.
\]
Finally, applying Lemma 2, we obtain
\[
(15) \quad \mu[w] \geq \bigwedge_{1 \leq k < j \leq q} \mu[w_{jk}] \geq \bigwedge_{1 \leq k < j \leq q} (\nu_j^* + \nu_k^*) \quad \text{on } E.
\]
The relations (13) and (15) together with (10) give (6) on \( E \). The theorem is proved.

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