1. Simplify

\[ 2i \log \frac{1 - i}{1 + i}. \]

(The answer should be in the form \(a + bi\), where \(a\) and \(b\) are real.)

Solution.

\[ \frac{1 - i}{1 + i} = \frac{(1 - i)^2}{2} = -i = e^{-\pi i/2}. \]

\[ \log(-i) = -\pi i/2, \]

so the answer is \(\pi\).
2. Find all solutions of the equation \( \sin z = i \) and make a picture of them. (All solutions should be written as \( a + ib \), where \( a, b \) are real.)

Solution. Set \( w = e^{iz} \). Then

\[
\sin z = (w - w^{-1})/2i = i, \quad w^2 + 2w - 1 = 0,
\]

and we have two roots \( w_1 = -1 + \sqrt{2} \) and \( w_2 = -1 - \sqrt{2} \). Now we have to solve

\[
e^{iz} = w_k, \quad k = 1, 2.
\]

For the first root,

\[
iz = \log |1 - \sqrt{2}| + i \text{Arg} (-1 + \sqrt{2}) + 2\pi in, \quad n = 0, \pm 1, \pm 2, \ldots.
\]

Notice that \(-1 + \sqrt{2} > 0\) and thus \( \text{Arg} (-1 + \sqrt{2}) = 0 \). So the first series of solutions is

\[
z_{1,n} = -i \log (\sqrt{2} - 1) + 2\pi n.
\]

To make an accurate picture, notice that \( 0 < \sqrt{2} - 1 < 1 \), therefore the \( \log \) of it is negative. For the second series we have \(-1 - \sqrt{2} < 0\), therefore its argument is \( \pi + 2\pi n \), and absolute value is \( > 1 \). and the second series is

\[
z_{2,n} = -i \log (1 + \sqrt{2}) + \pi + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots.
\]

To make an accurate picture it remains to notice that \(|w_1w_2| = 1\) so \(-i \log (\sqrt{2} - 1)\) is on the positive imaginary axis, while \(-i \log (1 + \sqrt{2})\) on the negative, opposite to it.
3. Show that the function $u(x, y) = e^x (y \cos y + x \sin y)$ is harmonic in the whole plane. Find an analytic function $f$ for which $u$ is the real part.

Solution. To check that it is harmonic just compute the Laplacian

$$u_{xx} + u_{yy}.$$

It is the real part of the function

$$-iz e^x = e^x (y \cos y + x \sin y) + i(y \sin y - x \cos y + c),$$

where $c$ is any real constant.
4. Let $f$ be an analytic function. Can $|f|^2$ be harmonic? Describe all $f$ for which this is the case.

Solution. Harmonic means

$$\Delta = (|f|^2)_{xx} + (|f|^2)_{yy} = 0,$$

Let $f = u + iv$. Then

$$\Delta = 2(u_x^2 + u_y^2 + v_x^2 + v_y^2) + 2(u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy})).$$

The second expression in parentheses is zero because both $u$ and $v$ are harmonic. Thus the first expression in parentheses must be zero. But it is a sum of real squares, so each $u_x, u_y, v_x, v_y$ is zero, and the function must be constant.
5. a) For each integer \( n \) (positive or negative, or 0), compute

\[
\int_{\gamma} (z)^n \, dz,
\]

where \( \gamma \) is the circle of radius \( R \) centered at the origin, and described counterclockwise.

b) For which \( n \) is the function \( f(z) = (z)^n \) analytic in the plane? Justify your answer.

Solution.

a) Parametrize the curve: \( z(t) = R \, e^{it}, \ 0 \leq t \leq 2\pi, \ dz = iR e^{it} \, dt \), so the integral equals

\[
iR^{n+1} \int_0^{2\pi} e^{it(1-n)} \, dt.
\]

When \( n \neq 1 \) this is equal to 0, when \( n = 1 \) to \( 2\pi iR^2 \).

b) When \( n = 0 \) this function is constant, so it is analytic. For all other \( n \) it is not.

To prove the last statement, let \( f(z) = \overline{z} \) and \( g(z) = z^n \). Then denote \( h(z) = g(f(z)) \). Proving by contradiction, suppose that \( h \) is analytic in a neighborhood of some point \( w \neq 0 \). Let \( g^{-1} \) be some inverse branch of \( g \) near \( w \). Then

\[
f(z) = g^{-1}(h(z)),
\]

so \( f \) would be analytic as a composition of analytic functions. But we know from Cauchy-Riemann equations that \( f(z) = \overline{z} \) is nowhere analytic.

Another proof. \( (\overline{z})^n = |z|^{2n}/z^n \), so if \( (\overline{z})^n \) were analytic, then \( |z|^{2n} \) would be analytic, as a product of two analytic functions, but it is real-valued, so must be constant. Thus our function is analytic only when \( n = 0 \).

Remark. That the integrals in a) happen to be 0 is just an accident. On other curves they are not zero, and they are path-dependent.
6. a) Let $D$ be a half-plane or a disc, $f$ an analytic function in $D$, and $|f'(z)| \leq 1$ for every $z \in D$. Prove that

$$|f(z_1) - f(z_2)| \leq |z_1 - z_2|, \quad (1)$$

for all $z_1, z_2$ in $D$.

b) Give an example of a region $D$ and an analytic function $f$ in $D$ with the property $|f'(z)| \leq 1$, $z \in D$ such that the inequality (1) does not hold for some $z_1$ and $z_2$ in $D$.

Hint: it is one of the examples given several times in this class for various reasons.

Solution. a) Let $\gamma$ be the straight line segment from $z_1$ to $z_2$. Then

$$|f(z_1) - f(z_2)| \leq \left| \int_{\gamma} f'(\zeta)d\zeta \right| \leq |z_1 - z_2|,$$

where we used the Newton–Leibniz formula, and then the main estimate of the integral and the condition that $|f'(\zeta)| \leq 1$.

b). Take $D = \{z : |z| > 1\} \setminus (-\infty, 0]$, (exterior of the disc cut on the negative ray), and $f(z) = \log z$. Then

$$|f'(z)| = 1/|z| < 1,$$

on the other hand, $|f(-2 + i\epsilon) - f(-2 - i\epsilon)|$ is approximately $2\pi$ when $\epsilon > 0$ is very small, while the distance between these two points is only $2\epsilon$.  