Midterm exam solutions

1. The boundary of the NE quarter of the unit circle consists of three curves, which we orient positively (so that the region is on the left).

   The first curve is parametrized as \( z(t) = t, 0 \leq t \leq 1 \). Joukowski image is parametrized as \( w(t) = J(z(t)) = (1 + t^{-1})/2 \). When \( t \) runs from 0 to 1 this describes the ray \((+\infty, 1]\) of the real line.

   The second curve is parametrized as \( z(t) = e^{it}, 0 \leq t \leq \pi/2 \). the image is parametrized as \( w(t) = (e^{it} + e^{-it})/2 = \cos t \). When \( t \) runs from 0 to \( \pi/2 \), this describes the interval \([1, 0]\).

   The third curve is parametrized as \( z(t) = \text{where } t \text{ is from 1 to 0} \). Then \( w(t) = (it + (it)^{-1})/2 = i(t - t^{-1})/2 \). As \( t \) runs from 1 to 0, this changes from 0 to \(-\infty\). So the image is the negative part of the imaginary axis. Thus the image of the boundary is the boundary of the 4-th quadrant.

   It was proved in class that \( J \) is one-to-one in the upper half-plane so \( J \) maps that NE quarter to the 4-th quadrant.

2. Write \( f(z) = u(x, y) + iv(x, y) \).

   a) Then

   \[
   \overline{f(z)} = u(x, -y) - iv(x, -y) =: U(x, y) + iV(x, y).
   \]

   Now verify the C-R condition for \( U \) and \( V \), using the CR conditions for \( u \) and \( v \):

   \[
   U_x = u_x = v_y = V_y, \quad U_y = -u_y = v_x = -V_x.
   \]

   b) From C-R conditions for \( f \) we have \( u_x = v_y \) and from C-R conditions for \( \overline{f} \) we have \( u_x = -v_y \), so both are zero. So the function is constant.

3. Unit circle is parametrized by \( z(t) = e^{it}, 0 \leq t \leq 2\pi \). So \( dz = ie^{it}dt \) and \( \overline{z} = e^{-it} \). By definition of integral, it is equal to

   \[
   \int_0^{2\pi} e^{-imt}ie^{it}dt = i \int_0^{2\pi} e^{i(1-m)t}dt = \begin{cases} 2\pi i, & m = 1 \\ 0, & m \neq -1 \end{cases}.
   \]

4.

\[ \Im \sin(x+iy) = \Im(e^{i(x+iy)} - e^{-i(x+iy)})/2i = (e^y \cos x - e^{-y} \cos x)/2 = \cos x \sinh x. \]

This is 0 when \( x = \pi/2 + \pi k, k = 0, \pm 1, \pm 2, \ldots \), or \( y = 0 \).
5. Do the partial fraction decomposition:

\[
\frac{z}{z^2 - 1} = \frac{1}{2} \left( \frac{1}{z - 1} + \frac{1}{z + 1} \right).
\]

As both poles ±1 are inside the contour, each contributes \(2\pi i\), so the answer is \((2\pi i + 2\pi i)/2 = 2\pi i\).

6. a) Yes. Let \(a\) be a point in \(D\), such that \(f(z) \neq 0\). Let \(G\) a little disc around \(a\). Then a branch of \(\log(f(z))\) is defined in \(G\), and its imaginary part is constant. So \(f\) must be constant in \(G\). Then by uniqueness theorem \(f\) is constant in \(D\).

b) No. For example, \(x^2\) is not harmonic.

c) Yes. The region is simply connected so Cauchy theorem applies to every closed curve.

d) No. Example: \(f(z) = 1/z\), and the curve is the unit circle.

e) Yes. This is Morera’s theorem.