The following reslt was suggested by Ralph Eddingthont on Math Overflow.

Theorem. *Let*

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n = z - P(z)$$

be a power series with $a_n \geq 0$, having radius of convergence $r > 0$. If $r_0 \leq r$ is the smallest positive zero of f' then f is univalent in $\{z : |z| < r_0\}$.

Proof. If f' has any zeros in D, let a be the zero of the smallest absolute value, $|a| < r$. Then a is the singular point of the smallest absolute value of the function

$$\frac{1}{f'} = \frac{1}{1 - P'} = 1 + P' + (P')^2 + \dots,$$

which is represented by a Taylor series with positive coefficients. By Pringsheim's theorem, $a > 0$.

If f' has no zeros in D, we set $r_0 = r$.

Now we prove that f is univalent in $\{z : |z| < r_0\}$.

Consider the function P'. This is represented by a power series with positive coefficients, $P'(0) = 0$, and P' is real and strictly increasing on $(0, r)$. As $P'(x) \neq 1$ for $0 \leq x < r_0$, we conclude that for every $\epsilon \in (0, r_0)$ there exists $k \in (0, 1)$ such that $P'(x) \leq k$ for $x \in (0, r_0 - \epsilon)$. As P' has positive coefficients, we conclude that

$$|P'(z)| \leq k \quad \text{for} \quad |z| < r_0 - \epsilon. \tag{1}$$

Now suppose that $f(z_1) = f(z_2)$ for some z_1, z_2 in D_0. We choose ϵ such that $0 < \epsilon < r_0 - \max\{|z_1|, |z_2|\}$ and write

$$0 = f(z_1) - f(z_2) = \int_{z_2}^{z_1} f'(\zeta) d\zeta = z_1 - z_2 - \int_{z_2}^{z_1} P'(\zeta) d\zeta,$$

where the integral is over the straight line segment from z_2 to z_1. Now we estimate the integral using (1):

$$|z_1 - z_2| = \left| \int_{z_2}^{z_1} P'(\zeta) d\zeta \right| \leq k|z_2 - z_1|,$$

which implies that $z_1 = z_2$ since $0 < k < 1$. This proves the theorem.