

# Linear dependence of exponentials

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(Letter written in 1990-s)

## Abstract

Any finite set of distinct exponentials is linearly independent. People ask me whether infinitely many exponentials  $\exp(\lambda_k z)$  are linearly independent. The answer is “well known” but the only reference I have is in Russian. People ask so frequently that I decided to post the answer on the web. A complete study of these questions can be found in the books of the Russian mathematician A. Leontiev.

Dear Steven and Sherman,

I recently saw the problem on “linear independence of exponentials” posted on Steven’s web page, with my comments, and decided to send more specific comments. Here they are:

1. Let  $(\lambda_n)$  be a given sequence of exponents. Consider an entire function  $L(\lambda)$  which has simple zeros exactly at  $\lambda_n$ . For example, we can take  $L$  to be the canonical product with these zeros). By the Residue Theorem

$$\sum_{k:|\lambda_k|<r} \frac{e^{\lambda_k z}}{L'(\lambda_k)} = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{e^{\lambda z}}{L(\lambda)} d\lambda. \quad (1)$$

2. Now it is easy to construct a sequence  $(\lambda_n)$  such that:

a) The series

$$\sum_{k=1}^{\infty} \frac{e^{\lambda_k z}}{L'(\lambda_k)}$$

is absolutely convergent, and

b) the integral in the right hand side of (1) tends to zero as  $r \rightarrow \infty$ , uniformly when  $z$  is restricted to any compact set.

The simplest example can be obtained by taking  $\lambda_n$  at the lattice points  $\{m + in : m, n, \in \mathbf{Z}\}$ . Then  $L$  is a Weierstrass sigma-function, and

$$\log |L'(\lambda_k)| \sim c|\lambda_k|^2$$

with some positive constant  $c$ . Furthermore,

$$\log |L(\lambda)| \sim c|\lambda|^2,$$

when  $\lambda \rightarrow \infty$  avoiding exponentially small neighborhoods of the points  $\lambda_k$ . Thus properties a) and b) above are satisfied.

3. The main positive results are the following.

A. Suppose that  $k = O(\lambda_k)$  and a series

$$\sum_{k=1}^{\infty} a_k \exp(\lambda_k z) \equiv 0 \tag{2}$$

converges uniformly and absolutely in the whole plane. Then  $a_k = 0$ . This follows from Theorem 3.1.4 on p. 201 of Leontiev's book: Series of Exponentials, Moscow, 1976.

B. Suppose that all  $\lambda_k$  are real, and the series (2) converges in the whole complex plane to 0. Then  $a_k = 0$ . This can be found in many books, for example, in S. Mandelbrojt, *Séries de Dirichlet*. Paris, Gauthier-Villars, 1969, Th. I.3.1.

4. Here is a brief proof of Leontyev's theorem. Let  $\lambda_k$  and  $L$  be the same as before, and put

$$L_k(\lambda) = \frac{L(\lambda)}{\lambda - \lambda_k},$$

so that

$$L_k(\lambda_n) = \begin{cases} 0, & n \neq k, \\ L'(\lambda_k), & n = k \end{cases} \tag{3}$$

Let  $f_k$  be the Laplace (Borel) transform of  $L_k$  and  $K$  the conjugate diagram of  $L$ . (See, for example, B. Levin, *Lectures on Entire Functions*, or any other book on entire functions). Take any closed curve  $\gamma$  going once around  $K$ , multiply (2) by  $f_k$  and integrate along this curve. A uniformly and absolutely convergent series can be integrated term-by-term, so

$$0 = \sum_{k=1}^{\infty} a_k \int_{\gamma} f_k(w) e^{\lambda_k w} dw = a_k L_k(\lambda_k),$$

so  $a_k = 0$  because  $L_k(\lambda_k) = L'(\lambda_k) \neq 0$ .

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