Fast Fourier Transform. Some answers and solutioins

1.3 Look at the remainders modulo 3: $2^0 = 1,\ 2^1 = 2\ 2^2 = 1$, and so on, the pattern repeats, and 2^k and 2^{k+1} never give equal remainders. So by exercise 1.2, the sums of digits are never equal.

1.5 Modulo 4: $1, 3$; modulo 5: $1, 2, 3, 4$.

1.6 Those and only those remainders have multiplicative inverse which are mutually prime with the modulus. (Two integers are called mutually prime if they have no common factor, except 1 and -1).

2.3 $1 + i = \sqrt{2}\exp(i\pi/4),\ 1/2 + i\sqrt{3}/2 = \exp(\pi i/3)$.

3.1 Begin by factoring $z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1)$. It remains to solve

$$z^4 + z^3 + z^2 + z + 1 = 0.$$

Divide by z^2:

$$z^2 + z + 1 + z^{-1} + z^{-2} = 0.$$

Put $w = z + z^{-1}$ then $w^2 = z^2 + z^{-2} + 2$; substituting to our equation we obtain

$$w^2 + w - 1 = 0,$$

So $w_{1,2} = -(1/2) \pm \sqrt{5}/2$. It remains to solve two quadratic equations for z:

$$z + z^{-1} = -1/2 + \sqrt{5}/2 \quad \text{and} \quad z + z^{-1} = -1/2 - \sqrt{5}/2.$$

Solutions of the second equation are

$$-\frac{1+\sqrt{5}}{4} \pm \frac{i}{2}\sqrt{\frac{5-\sqrt{5}}{2}},$$

and solutions of the first:

$$-\frac{1-\sqrt{5}}{4} \pm \frac{i}{2}\sqrt{\frac{5+\sqrt{5}}{2}}.$$

Remark. On the other hand, the 5-th roots are $\exp(2\pi ik/5),\ k = 0, 1, 2, 3, 4$. Can you figure out which is which? Then you obtain precise expressions for sin and cos of several angles like $72°, 54°, 26°$.

3.2 They are roots of the polynomial $z^N - 1$. By the Fundamental theorem of Algebra, the polynomial factors into linear factors:

$$f(z) = z^N - 1 = (z - z_1)(z - z_2) \dots (z - z_N).$$

One still has to show that all these factors are distinct. If some factor $(z - z_k)$ appears at least twice, then z_k will be a root of the defivative $f'(z)$. Indeed, if

$$f(z) = (z - z_k)^2 g(z) \quad \text{then} \quad f'(z) = 2(z - z_k)g(z) + (z - z_k)g'(z).$$

But $f(z) = z^N - 1$ and $f'(z) = Nz^{N-1}$ have no common roots.

3.3. For degree 12: $\exp(\pi i k/6)$, where $k = 1, 5, 7, 11$ are primitive,the rest are not.

In general, $\exp(2\pi i k/N)$ is primitive if and only if N and k are mutually prime.

4.2 $(2, 4, -2, 4)$.

5.2 Let $F(N)$ be the time spent on calculating N points. We are testing the hypothesis that $F(N) = cN \log_2 N$. Dividing $F(N)$ by $N \log_2 N$, we obtain the series

$$5/12,\ 25/64,\ 15/40,\ 35/96,$$

which is slowly decreasing from .42 to .37. I would say that $c \approx .395$ from these data. Probably the slow increase of performance was due to computor's training in the process of calculation.

5.3 $(a + e, b + f, c + g, d + h)$. Computation: denote

$$\mathbf{f} = (a, b, c, d, e, f, g, h),\ \mathbf{F} = (A, B, C, D, E, F, G, H),\ \mathbf{G} = (A, C, E, G),$$

and let $\mathbf{g}$ be the vector we are trying to find. By definition, $\mathbf{G}(k) = \mathbf{F}(2k),\ 0 \le k \le 3$. By definition of FT:

$$\mathbf{F}(2k) = \sum_{n=0}^{7} w_8^{2nk} \mathbf{f}(n) = \sum_{n=0}^{7} w_4^{nk} \mathbf{f}(n).$$

As $w_4^{nk}, 0 \le n \le 7$ repeat with period 4, the last expression equals

$$\sum_{n=0}^{3} w_4^{nk} (\mathbf{f}(n) + \mathbf{f}(n + 4)).$$

This means that $\mathbf{G}$ is the FT of the sequence $\mathbf{g} = (a + e, b + f, c + g, d + h)$.