1. Suppose that $f$ is a continuous complex-valued function on the real line, and $|f(x)| \leq 1 + |x|^4$. Let $g = \hat{f}$, Fourier transform in the sense of temperate distributions. State the following properties of $f$ in terms of the distribution $g$:

a) $f$ is real,
b) $f$ has period $T > 0$,
c) $f$ is even,
d) $f$ is odd,
e) $|(1 + x^N)f(x)|$ is bounded for each $N$.

2. Suppose that the Fourier transform $\hat{f}$ of a function $f$ of one variable is given. Express the Fourier transforms of the following functions $g$ in terms of $\hat{f}$:

a) $g(x) = xf(-x)$,
b) $g(x) = f''(x) + xf(x)$,
c) $g(x) = f^2(3x - 2)$,
d) $g(x) = \hat{f}(x)$,
e) $g(x) = f(x)/(1 + x^2)$.

3. a) Bessel’s function $y = J_0(x)$ of order 0 satisfies the Bessel equation

$$xy'' + y' + xy = 0.$$ 

Derive a differential equation for the function $w(x) = J_0(\sqrt{x})$.

b) Consider the partial differential equation

$$u_{tt} = (xu_x)_x, \quad t > 0, \quad 0 < x < 1,$$

with the boundary conditions $u(1, t) = 0$ and $|u(0, t)| < \infty$. Separate the variables, find the eigenvalues of the related Sturm-Liouville problem, and write the general solution of the problem satisfying the boundary conditions as a series of the form

$$\sum_n f_n(t)\phi_n(x).$$

The answer should be given in terms of elementary functions, Bessel functions and zeros of Bessel functions.
4. Find the Fourier transform in $\mathbb{R}^3$ of the function

$$f(x) = \begin{cases} 
1, & |x| < 1, \\
0, & |x| \geq 1.
\end{cases}$$

5. Find explicitly the convolution $f \ast g$, where

$$f(x) = \begin{cases} 
x, & x \geq 0, \\
0, & x < 0.
\end{cases} \quad \text{and} \quad g(x) = \begin{cases} 
e^x, & x < 0 \\
0, & x > 0.
\end{cases}$$

6. For a rectangular membrane $R$ given by $0 \leq x \leq 1$ and $0 \leq y \leq 2$, whose oscillations are described by the equation

$$u_{tt} = 2(u_{xx} + u_{yy})$$

and the $u(x, y) = 0$ on the whole boundary of $R$, find the 5 smallest oscillation frequencies.