On metrics of constant positive curvature with four conic singularities on the sphere

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Abstract

We show that for given four points on the sphere and prescribed angles at these points, which are not multiples of 2π, the number of metrics of curvature 1 having conic singularities with these angles at these points is finite.

Key words: Heun's equation, WKB asymptotics, entire functions, surfaces, positive curvature.

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1. Introduction.

We consider metrics of positive curvature with n conic singularities on the sphere. Without loss of generality we assume that curvature equals 1. Such a metric can be described by the length element $\rho(z)|dz|$, where z is a local conformal coordinate and ρ satisfies

$$\Delta \log \rho + \rho^2 = 2\pi \sum_{j=0}^{n-1} (\alpha_j - 1)\delta_{a_j},$$

where a_j are the singularities with angles $2\pi\alpha_j$.

For the recent results of such metrics we refer to [4], [7], [8], [11], [12]. It is believed that when none of the α_j is an integer the number of such metrics with prescribed a_j and α_j is finite. This number has been found in some very special cases, [17], [4], [6], [7], [8], [10], in particular, there is only one

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such metric with angles not integer multiples of 2π when $n \leq 3$, [17], [4]. If $\alpha_j < 1$ for all j, then the metric is unique [10], but for large angles there is usually more than one metric [6, 7].

In this paper we address the case $n = 4$. The case when at least one of the four angles is a multiple of 2π has been completely settled in [7], so we assume here that none of the angles is a multiple of 2π.

We briefly recall the reduction of the problem to a problem about Heun's equation, see [6].

If S is the sphere equipped with such a metric, one can consider a developing map $f : S \to \overline{\mathbf{C}}$, where $\overline{\mathbf{C}}$ is the Riemann sphere equipped with the standard metric of curvature 1. Strictly speaking, f is defined on the universal cover of S, but we prefer to consider it as a multi-valued function with branching at the singularities. One can write $f = w_1/w_2$ where w_1 and w_2 are two linearly independent solutions of the Heun equation, a Fuchsian equation with four singularities. These four singularities are the singularities of the metric, and the angles at the singularities are 2π times the exponent differences. Heun's equation can be written as

$$w'' + \left(\sum_{j=0}^{2} \frac{1-\alpha_j}{z-a_j} \right) w' + \frac{Az - \lambda}{(z-a_0)(z-a_1)(z-a_2)} w = 0, \tag{1}$$

where the singularities are a_0, a_1, a_2, ∞, the angles are $2\pi\alpha_j$, $0 \leq j \leq 3$, and

$$A = (2 + \alpha_3 - \alpha_0 - \alpha_1 - \alpha_2)(2 - \alpha_3 - \alpha_0 - \alpha_1 - \alpha_2)/4.$$

Three singularities can be placed at arbitrary points, so one can choose, for example $(a_0, a_1, a_2) = (0, 1, t)$. So for given angles, the set of Heun's equation essentially depends on 2 parameters: t which describes the quadruple of singularities up to conformal equivalence, and λ which is called the *accessory parameter*.

The metric and the differential equation (1) can be pulled back to the plane via the regular ramified covering having simple ramification points over the singularities. Assuming that the singularities are at e_1, e_2, e_3, ∞, where $e_1 + e_2 + e_3 = 0$ this ramified covering is

$$\wp : \mathbf{C} \to S,$$

the Weierstrass function with primitive periods ω_1, ω_2. We denote $\omega_0 = 0$, $\omega_3 = \omega_1 + \omega_2$, then $e_j = \wp(\omega_j/2)$. The resulting differential equation in the plane is called the Heun equation in the elliptic form:

$$w'' = \left(\sum_{j=0}^{3} k_j \wp(z - \omega_j/2) + \lambda \right) w. \qquad (2)$$

The two parameters are now the modulus of the torus τ and λ. We will use both forms of the Heun equation. The ratio of two solutions $f = w_1/w_2$ is a developing map of a metric in question if and only if the projective monodromy group of the Heun equation is conjugate to a subgroup of $PSU(2)$. In this case we say that the monodromy is *unitarizable*. The exponents at the singularity $\omega_j/2$ are $\rho_j^{\pm} = 1/2 \pm \sqrt{1/4 + k_j}$, so the angle at this singularity is $4\pi\sqrt{1/4 + k_j}$, which is $4\pi\alpha_j$, twice the angle of the original metric on the sphere.

The problem is for given ω_1, ω_2 and $k_j > -1/4,\ 0 \leq j \leq 3$, to find the values of λ for which the monodromy is unitarizable. The main result of this paper is

Theorem 1. *If none of the numbers $\sqrt{1/4 + k_j}$ is an integer, then the set of λ for which the projective monodromy of (2) is unitarizable is finite.*

Corollary. *For every four points $a_0, \ldots, a_3$ on the Riemann sphere and every $\alpha_j \in \mathbf{R}_+ \backslash \mathbf{Z}$, there exist at most finitely many metrics of curvature 1 and conic singularities at a_j with angles $2\pi\alpha_j$.*

Unfortunately, our proof is non-constructive, and does not give any explicit upper estimate.

The proof consists of two parts: first we prove that the set of λ corresponding to the unitarizable monodromy is bounded; this part is based on the asymptotic analysis of equation (2) as $\lambda \to \infty$.

Compactness of the set of metrics with prescribed angles in the given conformal class was recently proved by Mondello and Panov [12] for metrics on arbitrary compact Riemann surfaces with any number of singularities, and for generic angles. For the case of 4 singularities on the sphere their condition on the angles is the following:

None of the sums $\sum_{j=0}^{3} \pm\alpha_j$ is an even non-zero integer.

The second part of the proof shows that the set of accessory parameters defining unitarizable monodromy is discrete. A general theorem of Luo ([9]) implies that equations with unitarizable monodromy correspond to a real analytic surface in the complex two dimensional space of all Heun's equations with prescribed angles. Assigning the position of singularities means that we take the intersection of this real analytic surface with a complex line. In general, such an intersection does not have to be discrete.

Finding the accessory parameters corresponding to unitarizable monodromy requires solving an equation of the form

$$g(\lambda) = 0, \tag{3}$$

where λ and $g(\lambda)$ are complex, but g is only real analytic (not complex analytic). In fact, g is a (complex) harmonic map [2], and there is no general method of proving that the set of solutions of (3) is discrete, or to estimate the number of solutions from above. See [8], [2] where a very special case is solved.

To investigate equation (3) in our case, we use a general theorem of Stephenson [16] which reduces the local question about discreteness of the set of solutions of (3) to a question about asymptotic behavior at infinity of entire functions (traces of the generators of monodromy), and this question is solved using the asymptotic behavior established in the first part of the proof and a theorem of Baker [1] on compositions of entire functions. It is not clear whether this proof can be generalized to the case $n > 4$.

We finish this introduction with a brief description of the situation when the angles can be integer multiples of 2π and a statement of a conjecture.

Two metrics with developing maps f_1, f_2 are called *equivalent* if $f_1 = \phi \circ f_2$ for some $\phi \in PSL(2, \mathbf{C})$. So developing maps of equivalent metrics are ratios of distinct pairs of solutions of the same Fuchsian equation, and there is a bijective correspondence between the Fuchsian equations with unitarizable monodromy and equivalence classes of metrics. We conjecture that *for every compact Riemann surface, any set of singularities and any prescribed angles at the singularities, there exist at most finitely many Fuchsian equations with unitarizable monodromy.*

An equivalence class can contain more than one metric only if the projective monodromy representation is reducible (that is elements of the projective monodromy group have a common fixed point). For unitarizable monodromy, this happens if and only if the projective monodromy is co-axial (isomorphic to a subgroup of the circle). If the projective monodromy is trivial, the equivalence classes are real one- or three-parametric families. If the projective monodromy is a nontrivial subgroup of the unit circle, then the equivalence classes are one-parametric families or consist of one element [3].

When S is the sphere, $n \geq 3$ and none of the angles is a multiple of 2π, the monodromy representation is irreducible [5].

We also notice that the angles of the metric on a torus defined by equation (2) are twice the angles of the metric defined by (1). When all angles on the sphere are multiples of π, the local monodromy of (2) is trivial, and the monodromy representation of (2) which is a homomorphism of the commutative fundamental group of the torus is co-axial, or $\mathbf{Z}^2 \times \mathbf{Z}^2$. In the co-axial case each equation (2) with unitarizable monodromy defines a one-parametric family of equivalent metrics on the torus but only one member of this family corresponds to a metric on the sphere. This shows that the Corollary indeed follows from Theorem 1.

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2. Asymptotics of properly normalized solutions

Consider a differential equation of the form

$$w'' = \left(\frac{k}{z^2} + h^2 + \phi(z) \right) w, \tag{4}$$

where $k > -1/4$, $h^2 = \lambda$ is a large complex parameter, ϕ is holomorphic on the segment $[0, z]$ for some complex z, and $\phi(0) = 0$.

The singularity at 0 is regular, the exponents are $\rho_{1,2} = 1/2 \pm \sqrt{1/4 + k}$, and we assume that the difference $\rho_1 - \rho_2 > 0$ is not an even integer. We call a solution w_j *properly normalized* if

$$w_j(z, h) = z^{\rho_j}(1 + g_j(z, h)), \quad j \in \{1, 2\}, \tag{5}$$

where $z \mapsto g_j(z, h)$ is holomorphic at 0, and $g(0, h) = 0$. Here and in the following we use principal branches of the power functions. So there are exactly two properly normalized solutions.

When $\rho_1 - \rho_2$ is an odd integer, we make an additional assumption that ϕ is an even function;

then there are still two properly normalized solutions w_j as in (5), and they contain no logarithms in their expansion. Functions $z \mapsto w_j(z, h)$ are multi-valued, so when speaking of their values, one has to specify $\arg z$ and a path from 0 to z. We will always use the straight line segment as the path.

It is well-known that for every properly normalized solution and every z in the domain of ϕ, the function $h \mapsto w(z,h)$ is entire and even. To see this, one substitutes w of the form (5) where g is a formal power series in z and h^2 into (4) and shows that there are exactly two solutions of this form. Then one uses the majorant method to prove that this series converges in $D \times \mathbf{C}$, where D is the disk where ϕ is holomorphic. For the details we refer to [14], [15].

For example, when $k = 0$ and $\phi = 0$, the equation is $w'' = h^2 w$. It has solutions $\cosh zh$, $(\sinh zh)/h$ and e^{zh}. The first two are properly normalized, while the third one is not and cannot be properly normalized by multiplying on a factor which depends only on h. The first two solutions are even functions of h, while the third one is not.

Theorem 2. *Let w_j be any properly normalized solution of (4), $j \in \{1,2\}$. Then for every $\delta > 0$ and every z such that $|\arg z| < \pi/2$,*

$$w_j(z,h) = c_j \cosh(zh + O(1/h)), \quad h \to \infty, \quad zh \notin S_\delta, \tag{6}$$

$$w_j'(z,h) = c_j h \sinh(zh + O(1/h)), \quad h \to \infty, \quad zh \notin S_\delta, \tag{7}$$

where $S_\delta = \{h \in \mathbf{C} : |\mathrm{Re}\,(zh)| < \delta |\mathrm{Im}\,(zh)|\}$. Moreover, if $w_j(-z,h) := w_j(e^{i\pi}z,h)$ (an analytic continuation along the half-circle $\{ze^{it} : 0 \leq t \leq \pi\}$), then

$$w_j(-z,h) = c_j e^{\pi i \rho_j} \cosh(zh + O(1/h)), \quad h \to \infty, \quad zh \notin S_\delta, \tag{8}$$

$$w_j'(-z,h) = -c_j e^{\pi i \rho_j} h \sinh(zh + O(1/h)), \quad h \to \infty, \quad zh \notin S_\delta. \tag{9}$$

Here and in what follows we denote by c_j and C_j various constants which depend only on the exponents at the singularities, for example on k in (4).

Proof of Theorem 2. Since $h \mapsto w(z,h)$ is even, it is sufficient to prove (6) for $|\arg(zh)| < \pi/2 - \delta$. Then (7) follows from (6) by differentiation. Furthermore, $e^{-\pi i \rho_j} w_j(-z,h)$ is a properly normalized solution of (4) with $\phi(-z)$ instead of $\phi(z)$, and applying (6) and (7) to this solution, we obtain (8) and (9). So it remains to prove (6) for $|\arg(zh)| \leq \pi/2 - \delta$.

Put $w(z) = y(zh)$, $\zeta = zh$ in (4), then

$$y'' = \left(\frac{k}{\zeta^2} + 1 + h^{-2}\phi(\zeta/h) \right) y. \tag{10}$$

If Y is a properly normalized solution of (10), $Y(\zeta) \sim \zeta^\rho$ as $\zeta \to 0+$, then $w(z) = h^{-\rho}Y(zh)$ is a properly normalized solution of (4).

Let $\theta = \arg(zh)$, $\zeta_0 = e^{i\theta}$. Let Y_j, $j = 1, 2$, be properly normalized solutions of (10). Then, as $h \to \infty$,

$$Y_1(\zeta_0, h) \to y_+(\zeta_0), \quad \text{and} \quad Y_1'(\zeta_0, h) \to y_+'(\zeta_0), \tag{11}$$

$$Y_2(\zeta_0, h) \to y_-(\zeta_0), \quad \text{and} \quad Y_2'(\zeta_0, h) \to y_-'(\zeta_0), \tag{12}$$

where $y_\pm$ are properly normalized solutions of the model equation

$$y'' = \left(\frac{k}{\zeta^2} + 1\right) y, \tag{13}$$

which is simply related to a (modified) Bessel equation with parameter $\nu^2 = k + 1/4$. We assume without loss of generality that $\nu > 0$, then

$$y_\pm(\zeta) = 2^{\pm\nu}\Gamma(1 \pm \nu)\zeta^{1/2} I_{\pm\nu}(\zeta) = \zeta^{1/2\pm\nu} \sum_{n=0}^{\infty} \frac{\zeta^{2n}}{2^n n! \Gamma(n \pm \nu + 1)}.$$

To prove (11), (12) we write Y_j in the form (5) where g_j are power series in ζ and h^2. These series are convergent in $\mathbf{C} \times \mathbf{C}$, and (13) immediately follows.

We will need three solutions of (13), $y_1 = y_+$, $y_2 = y_-$ and

$$y_3(\zeta) = \zeta^{1/2} K_\nu(\zeta) = C_1 \zeta^{1/2}(I_{-\nu}(\zeta) - I_\nu(\zeta)),$$

where we use the standard notation for Bessel functions as in [13], [18]. Solution y_3 is not properly normalized; it is exponentially decreasing in the right half-plane. Asymptotics of y_1, y_2, y_3 for large ζ can be found in [13], [18] and elsewhere.

We define A_j and B_j by

$$Y_j(\zeta_0, h) = A_j(h) y_1(\zeta_0) + B_j(h) y_3(\zeta_0), \quad j \in \{1, 2\}.$$

Then

$$Y_j'(\zeta_0, h) = A_j(h) y_1'(\zeta_0) + B_j(h) y_3'(\zeta_0).$$

It follows from (11), (12) that $A_1(h) \to 1$ as $h \to \infty$ while $A_2(h)$ and $B_j(h)$ have some finite limits which depend on ν and j.

We solve the Cauchy problem for equation (10) on the interval $[\zeta_0, zh]$ with initial conditions at ζ_0 that match $Y_j(\zeta_0, h)$, $Y_j'(\zeta_0, h)$. We can write

$$Y_j = A_j(h)(y_1 + a_1) + B_j(h)(y_3 + a_3), \quad j \in \{1, 2\}, \tag{14}$$

where $y_j + a_j$ are solutions of (10) and $a_j(\zeta_0) = a_j'(\zeta_0) = 0$. Substituting $y_j + a_j$ to (10) we obtain

$$a_j'' - a_j \left(\frac{k}{\zeta^2} + 1 \right) = (y_j + a_j)\psi, \tag{15}$$

where

$$\psi(\zeta) = h^{-2}\phi(\zeta/h). \tag{16}$$

Considering (15) as a non-homogeneous equation and applying the method of variation of constants, we obtain integral equations for a_j:

$$a_j(\zeta) = \int_{\zeta_0}^{\zeta} K(\zeta, t)\psi(t) \left(y_j(t) + a_j(t)\right) dt, \quad j \in \{1, 3\}. \tag{17}$$

where

$$K(\zeta, t) = C_2 \left(y_1(\zeta)y_3(t) - y_3(\zeta)y_1(t)\right), \tag{18}$$

and the integration is over the segment $[\zeta_0, zh]$. Here C_2 is the reciprocal of the constant Wronskian of y_1, y_3. Now we consider two cases.

Case $j = 1$ in (17). We introduce $b = a_1/y_1$ and re-write the integral equation as

$$b(\zeta) = \int_{\zeta_0}^{\zeta} K_1(\zeta, t)\psi(t) \left(1 + b(t)\right) dt.$$

The kernel

$$K_1(\zeta, t) = C_2 \left(y_1(t)y_3(t) - \frac{y_3(\zeta)}{y_1(\zeta)} y_1^2(t) \right) \tag{19}$$

is bounded on our integration path, as can be seen from the asymptotics of I_ν and K_ν in [13] or elsewhere.

The integral equation for b is solved by iteration. We set $b_0 = 0$, and

$$b_{n+1} = \int_{\zeta_0}^{\zeta} K_1(\zeta, t)\psi(t) \left(1 + b_n(t)\right) dt, \quad n \geq 0.$$

Then the solution will be

$$b = \lim_{n\to\infty} b_n = b_0 + (b_1 - b_0) + (b_2 - b_1) + \ldots.$$

We will prove by induction that

$$|b_{n+1}(\zeta) - b_n(\zeta)| \leq C_3^{n+1}\Psi^{n+1}(\zeta)/(n + 1)!, \tag{20}$$

where

$$\Psi(\zeta) = \int_{\zeta_0}^{\zeta} |\psi(t)||dt| \leq \frac{1}{|h|}\int_0^z |\phi(t)||dt| \tag{21}$$

in view of (16).

For $n = 0$ we have

$$|b_1(\zeta) - b_0(\zeta)| \leq C_3 \int_{\zeta_0}^{\zeta} |\psi(t)||dt| \leq C_3\Psi(\zeta).$$

Then

$$|b_{n+1}(\zeta) - b_n(\zeta)| \leq C_3 \int_{\zeta_0}^{\zeta} |\phi(t)| C_3^n \frac{\Psi^n(t)}{n!}(t)|dt| = C_3^{n+1}\frac{\Psi^{n+1}(\zeta)}{(n+1)!},$$

which proves (20).

So

$$|b(\zeta)| \leq \sum_1^{\infty} C_3^n \Psi^n(\zeta)/n! = e^{C_3\Phi(\zeta)} - 1.$$

Combining this with (21) we obtain that $b(\zeta) = O(h^{-1})$, and $a_1(zh) = O(e^{zh})/h,\ h \to \infty$.

Case $j = 3$ in (17). We do the same as in the previous case, setting again $b = a_3/y_1$. Then the integral equation becomes

$$b(\zeta) = \int_{\zeta_0}^{\zeta} K_1(\zeta, t)\phi(t)\left(\frac{y_3(t)}{y_1(t)} + b(t)\right) dt,$$

with the same kernel K_1 as in (19), and it is treated exactly in the same way as before, using the fact that y_3/y_1 is bounded on $[\zeta_0, zh]$.

We obtain that $a_3(zh) = O(e^{zh}/h)$.

Returning to our solution Y_j we have from (14)

$$\begin{aligned} w_j(z,h) &= Y_j(zh) = A_j(h)(y_1(zh) + a_1(zh)) + B_j(h)(y_3(zh) + a_3(zh)) \\ &\sim c_j \cosh(zh)(1 + O(1/h)), \end{aligned}$$

as $h \to \infty$, $|\arg(zh)| < \pi/2 - \delta$. This completes the proof of (6) and of Theorem 2.

3. Asymptotics of the trace of monodromy

Now we consider the differential equation

$$w'' = \left(\frac{k_1}{(z-a_1)^2} + \frac{k_2}{(z-a_2)^2} + h^2 + \phi(z) \right) w, \tag{22}$$

where $k_i > -1/4$, ϕ is holomorphic on $[a_1, a_2]$, and h^2 is the large parameter.

Let w_{ij} be properly normalized solutions at a_i with exponents ρ_{ij}, where $\rho_{i1} > \rho_{i2}$. Then we have $\rho_{i1} + \rho_{i2} = 1$, so

$$\rho_{i1} = (1+\rho_i)/2, \quad \rho_{i2} = (1-\rho_i)/2, \tag{23}$$

where ρ_i are the exponent differences at a_i, so that $\rho_i = \rho_{i1} - \rho_{i2}$.

Notice that the Wronskian determinants of pairs of solutions are constant,

$$W(w_{i1}, w_{i2}) = -\rho_i, \quad i = 1, 2. \tag{24}$$

Consider the connection matrix $F(h) = (f_{ij}(h))$ such that

$$\mathbf{w}_1 = FM_2\mathbf{w}_2, \quad \mathbf{w}_i = \begin{pmatrix} w_{i1} \\ w_{i2} \end{pmatrix}, \quad M_2 = \begin{pmatrix} e^{-\pi i \rho_{21}} & 0 \\ 0 & e^{-\pi i \rho_{22}} \end{pmatrix}. \tag{25}$$

To compute the elements of F we differentiate with respect to z:

$$\begin{aligned} w_{1j} &= f_{j1} w_{21} e^{-\pi i \rho_{21}} + f_{j2} w_{22} e^{-\pi i \rho_{22}}, \\ w'_{1j} &= f_{j1} w'_{21} e^{-\pi i \rho_{21}} + f_{j2} w'_{22} e^{-\pi i \rho_{22}}. \end{aligned}$$

Solving this by Cramer's rule and using Theorem 2, at the point $z = (a_1+a_2)/2$,

$$f_{jk} = c_{jk} h \sinh\left((a_2 - a_1)h + O(1/h)\right), \tag{26}$$

$$\text{for all} \quad j, k \in \{1, 2\}, \quad \arg(a_2 - a_1)h \notin S_\delta. \tag{27}$$

So we determined the asymptotic behavior of the elements of the connection matrix.

Now we consider the differential equation of the form

$$w'' = \left(\sum_{j=-\infty}^{\infty} \frac{k_j}{(z - j\omega/2)^2} + h^2 + \phi(z) \right) w, \tag{28}$$

where ϕ is an even function with period ω, and all k_j are equal to k_1 or k_2, according to the parity of j.

We are going to compute the asymptotics of the "monodromy", that is the connection matrix between $\mathbf{w}_0(z)$ and $\mathbf{w}_2 = \mathbf{w}_0(z-\omega)$ where $\mathbf{w}_j$ is the vector of properly normalized solutions at $z = j\omega/2$.

This matrix is evidently given by

$$\mathbf{w}_0 = F^{-1}M_1FM_2\mathbf{w}_2(z), \tag{29}$$

Here F and M_2 are the same as in (25), and

$$M_1 = \begin{pmatrix} e^{-\pi i\rho_{11}} & 0 \\ 0 & e^{-\pi i\rho_{12}} \end{pmatrix}.$$

Theorem 3. *The trace of the monodromy matrix of solutions of (28) has asymptotics*

$$T(h) := \mathrm{Tr}\,(F^{-1}M_1FM_2^{-1}) = ch^2\sinh\left(\omega h + O(1/h)\right), \quad h\to\infty,$$

where

$$h \notin S_\delta = \{h : |\mathrm{Re}\,\omega h| < \delta|\mathrm{Im}\,\omega h|\},$$

and $\delta > 0$ is arbitrary.

Proof. As in the previous section, we denote by C_j various non-zero constants which depend only on the exponents ρ_{ij}, or, which is the same, only on k_i in (22).

As the Wronski determinants $W(w_{i1}, w_{i2})$ are independent of h (see (24)), we conclude that $\det F$ is independent of h. Using the expressions in (29), (25), we compute our trace, and obtain

$$\begin{aligned} C_1T &= f_{11}f_{22}\left(e^{-\pi i(\rho_{11}+\rho_{21})} + e^{-\pi i(\rho_{22}+\rho_{12})}\right) \\ &- f_{12}f_{21}\left(e^{-\pi i(\rho_{11}+\rho_{22})} + e^{-\pi i(\rho_{12}+\rho_{21})}\right). \end{aligned}$$

Using (23), this simplifies to

$$C_1T = -C_2f_{11}f_{22} + C_3f_{12}f_{21},$$

where $C_2 = \cos\pi(\rho_1+\rho_2)/2$ and $C_3 = \cos\pi(\rho_1-\rho_2)/2$. As neither ρ_1 nor ρ_2 is an even integer, we conclude that $C_2 \neq C_3$. On the other hand $\det F = f_{11}f_{22} - f_{12}f_{21} =: C_4$, so

$$C_1T = -C_2C_4 + (C_3 - C_2)f_{12}f_{21},$$

and the statement of Theorem 3 follows from (26) with $a_2 - a_1 = \omega/2$.

4. Boundedness of the set of accessory parameters.

Now we prove that the set of values of parameters λ in (2) for which the monodromy is unitarizable is bounded. Potential in (2) has two non-collinear periods ω_1 and ω_2, $\operatorname{Im} \omega_2/\omega_1 \neq 0$. So it can be written in the form (28) in two ways: with $\omega = \omega_1$ and with $\omega = \omega_2$. Monodromy of (2) can be unitarizable only if the traces of monodromy transformations $T_j(h)$ corresponding to these two periods satisfy $|T_j(h)| \leq 2,\ j = 1, 2$. Theorem 3 implies that this cannot happen for large h. This proves the first part of Theorem 1, that the set of $\lambda = h^2$ corresponding to unitarizable monodromy is bounded.

5. The real case.

To prove discreteness of the set of accessory parameters corresponding to unitarizable monodromy, we first address the real case: we assume that $(a_0, a_1, a_2) = (0, 1, t)$, in (1) and that t and λ are real. Here we essentially follow Smirnov [14], [15], who investigated the case of $SL(2, \mathbf{R})$ monodromy and all angles $\alpha_j < 1$.

We assume without loss of generality that $t < 0$. Let w_{01}, w_{02} be properly normalized solutions of (1) at 0; they are real on $(0, 1)$. We also consider two solutions w_{11}, w_{12} which are proportional to the properly normalized solutions at 1 but both real on $(0, 1)$:

$$w_{1j}(z) = |z - 1|^{\rho_{1j}}(1 + g(z)), \quad z \in (0, 1),$$

where g is analytic near 1, $g(1) = 0$, and ρ_{1j} are the exponents at 1, $\rho_{11} = \alpha_1,\ \rho_{12} = 0$.

Then we have the connection matrix

$$F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

such that

$$\mathbf{w}_0 = F\mathbf{w}_1, \quad \text{where} \quad \mathbf{w}_i = \begin{pmatrix} w_{i1} \\ w_{i2} \end{pmatrix}, \quad i \in \{0, 1\}.$$

The entries of F are entire functions of λ, real on the real line [14], [15]. To obtain the projective monodromy, we consider $f_i = w_{i1}/w_{i2},\ i \in \{0, 1\}$, which are related by a linear-fractional transformation $f_0 = L(f_1)$ which is represented by the matrix F. Projective monodromy of f_0 at 0 is an elliptic transformation with fixed points 0 and ∞. Projective monodromy of f_0 at $z = 1$ is an elliptic transformation with fixed points $u_1 = f_{11}/f_{21}$ and $u_2 = f_{12}/f_{22}$. These points are real. Projective monodromies at 0 and 1 are simultaneously unitarizable if and only if the product of these fixed points in negative. Indeed, an elliptic transformation is a rotation of the Riemann sphere if and only if its fixed points u_1, u_2 are diametrally opposite, that is

$$u_1\overline{u_2} = -1. \tag{30}$$

In our case both u_1, u_2 are real so the bar can be dropped. Choosing the fixed points of projective monodromy at 0 to be $0, \infty$, we still can multiply f_0 by a constant μ. This will result in multiplying both fixed points of the projective monodromy at 1 by μ, so (30) can be achieved for these fixed points if and only if $u_1u_2 < 0$.

Similar considerations apply to the interval $(t, 0)$. If we denote the connection matrix on $(t, 0)$ by

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$

then g_{ij} are entire functions on λ, real on the real line, and the fixed points of the projective monodromy of f_0 at t are $v_1 = g_{11}/g_{21}$ and $v_2 = g_{12}/g_{22}$.

So the condition of unitarizability is

$$\frac{f_{11}f_{12}}{f_{21}f_{22}} = \frac{g_{11}g_{12}}{g_{21}g_{22}} < 0. \tag{31}$$

This includes the condition that two meromorphic functions of λ take equal values, so the set of λ satisfying this condition is discrete, unless the equality in (31) is satisfied identically.

To show that the equation in (31) cannot be satisfied identically in λ, one can use the asymptotics obtained in section 2, but an easier way is to see this is directly from the Heun equation in the form (1), which in our case can be written as

$$w'' + p(z)w' + q(z)w = 0,$$

where

$$q(z) = \frac{Az - \lambda}{z(z-1)(z-t)}, \quad t < 0.$$

When λ is large positive, $q(z)$ is large negative on $(0,1)$, so solutions oscillate on $(0,1)$, and functions f_{ij} have infinitely many positive zeros, while on the interval $(t,1)$ we have $q(z) > 0$, solutions do not oscillate, and functions g_{ij} have no large positive zeros. Thus (31) cannot hold identically, and the set of real λ for which the monodromy is unitarizable is discrete.

4. Completion of the proof of Theorem 1.

To prove the second part of Theorem 1, discreteness of the set of λ corresponding to unitarizable monodromy, we consider two entire functions $h \mapsto T_j(h)$ introduced in section 3. They are the traces of the generators of the monodromy corresponding to the periods ω_1, ω_2.

If for some $\lambda = h^2$ the monodromy is unitarizable, then $T_j(h) \in [-2, 2]$, so if the set of such h is not discrete, there is a non-degenerate curve γ such that both T_j are real on γ.

Now we use the following

Theorem of Stephenson ([14, Thm. 13]. *Let $g_j,\ j = 1,2$ be two entire functions which are both real on a non-degenerate curve γ. Then*

$$g_j = G_j \circ \phi, \tag{32}$$

where ϕ, G_j are entire, G_j are real on the real line, and ϕ is real on γ.

Recalling that our functions T_j are even functions of h, and $h^2 = \lambda$, we introduce entire functions

$$g_j(\lambda) = T_j(\sqrt{\lambda}) = c_j e^{\sqrt{(\omega_j^2 + o(1))\lambda}}, \quad \lambda \to \infty, |\arg \lambda| \leq \pi - \epsilon, \tag{33}$$

where the asymptotics is obtained from Theorem 3. These two functions have two *different* directions of maximal growth and their zeros have arguments accumulating in the directions opposite to the directions of maximal growth. To be more precise, we say that an entire function g has a single direction of maximal growth θ if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $r > r_0$ we have

$$\max\{|g(re^{it})| : \theta + \epsilon \leq t \leq \theta + 2\pi - \epsilon\} \leq (1 - \delta)\{\max |g(z)| : |z| = r\}.$$

Each of our functions $g_j,\ j = 1,2$, has a single direction of maximal growth $\theta_j = \arg(\omega_j^{-2})$, and these directions are distinct because ω_1/ω_2 is not real. It follows that G_j in (32) cannot be polynomials, and ϕ cannot be a polynomial of degree greater than 1. Now we use

Theorem of Baker [1]. *If an entire function g of finite order has a representation (32) with some entire transcendental functions G and all zeros of g except finitely many lie in a sector of opening less than π, then ϕ must be a polynomial of degree* 1.

From this we conclude that the curve γ in Stephenson's theorem must be an interval of a straight line ℓ, and both g_j are symmetric with respect to this line, that is $g_j \circ s(z) = \overline{g_j}(z)$, where s is the reflection with respect to ℓ. Comparing this with asymptotics (33) we conclude that the directions of maximal growth of the g_j must be collinear, and since they are distinct, they must be opposite, which gives $\omega_1 = i\omega_2$. So our torus must be rectangular. In terms of equation (1), this means that the singularities are real. This implies that our functions g_j are real on the real line.

Now we prove that ℓ is the real line. First ℓ cannot cross the real line, because a function with two lines of symmetry will have at least two directions of maximal growth, while our functions have only one. Second, it cannot be parallel to the real line, because in this case our functions g_j would be periodic which is incompatible with their asymptotics (33).

This reduces the general case to the real case considered in the previous section and completes the proof of Theorem 1.

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