On Deviations of Meromorphic Functions of Finite Lower Order

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For a function $f$ meromorphic in the finite plane $\mathbb{C}$ let

$$\beta(a, f) = \lim_{r \to \infty} \frac{\ln M(r, (f - a)^{-1})/T(r, f)}{a \neq \infty},$$

and

$$\beta(\infty, f) = \lim_{r \to \infty} \ln M(r, f)/T(r, f).$$

Here and below we use the standard notation from the theory of meromorphic functions [1].

Several papers in recent years have dealt with the study of convergence for series $\sum_a \delta^\alpha(a, f)$, $\alpha < 1$, for meromorphic functions of finite lower order. The strongest result in this direction is due to Weitsman [2], who proved that if $f$ is a meromorphic function of finite lower order, then

$$\sum_a \delta^1/3(a, f) < \infty. \quad (0.1)$$

Hayman ([1] §4.3) proved that the series $\sum_a \delta^1/3-\varepsilon(a, f)$ can diverge for any $\varepsilon > 0$. A detailed history of the problem is given in [2]. The quantities $\beta(a, f)$ (they are called the deviation values) were systematically studied by Petrenko [3], who proved, in particular, that

$$\sum_a \beta^{1/2}(a, f) \ln^{-1/2-\varepsilon} \frac{1}{\beta(a, f)} < \infty$$

for any $\varepsilon > 0$ for functions of finite lower order [4].

In the present paper we prove the

**THEOREM.** Let \( f \) be a meromorphic function of finite lower order. Then

\[
\sum_a \beta^{1/2}(a, f) < \infty. \tag{0.2}
\]

This relation was conjectured in [3]. The theorem was announced by Barsegyan in Akad. Nauk Armyan. SSR Dokl. 67 (1978), no. 5. According to a letter from him, his proof contained a gap.

It is easy to deduce (0.1) from (0.2). Indeed, let

\[
\theta(r, a) = \text{meas}\{\theta \in [0, 2\pi] : \ln |f(re^{i\theta}) - a|^{-1} \geq \ln r\}, \quad a \in \mathbb{C}.
\]

For any finite collection \( a_1, \ldots, a_q \in \mathbb{C} \) the inequality \( \sum_j \theta(r, a_j) \leq 2\pi \) holds for sufficiently large \( r \). On the other hand, it is easy to see that

\[
\delta(a_j, f) \leq \theta(r, a_j) \beta(a_j, f) + o(1), \quad r \to \infty.
\]

By Hölder's inequality,

\[
\sum \delta^{1/3}(a, f) \leq (2\pi)^{1/3} \left( \sum \beta^{1/2}(a, f) \right)^{2/3}.
\]

Therefore, (0.1) follows from (0.2).

Our theorem is sharp in the following sense. First, examples are known of meromorphic function of infinite lower order such that \( \beta(a, f) > 0 \) for uncountable sets of numbers \( a \in \mathbb{C} \) ([3], p. 82). Second, analysis of known examples ([3], p. 47) shows that for any sequence \( (\eta_n) \) of numbers with \( \eta_n > 0 \) and \( \sum_1^\infty \eta_n = 1 \) there is a meromorphic function of normal type of order 1 such that \( \beta(a_n, f) \geq \eta_n^2/4 \), where \( (a_n) \subset \mathbb{C} \) is a previously specified sequence.

Needed auxiliary results are given in §1, and the theorem is proved in §2.

1. **LEMMA 1.** Let \( f \) be a meromorphic function, and let \( a \in \mathbb{C} \). Then

\[
\log^+ \left| \frac{f'(z)}{f(z) - a} \right| = o(T(12|z|, f)), \quad z \to \infty, \; |z| \not\in I,
\]

where \( I \subset (0, \infty) \) is such that \( \text{meas}(I \cap (0, r)) = o(r), \; r \to \infty \).

This lemma is a variant of Lemma 1.4.1 in [3]. We omit the simple proof, which is based on differentiation of the Schwarz-Jensen formula.

Let \( D(R) = \{z : |z| < R\} \), and let \( u \geq 0 \) be the difference of two functions subharmonic on \( D(R) \) and continuous on \( \overline{D}(R) \). Such functions will be called admissible in what follows. The generalized Laplacian \( \Delta u \) is a signed measure with Jordan decomposition \( \mu_u^+ - \mu_u^- \). We use the notation

\[
M(r, u) = \max_\theta u(re^{i\theta}), \quad n(r, u) = \mu_u(D(R)),
\]

\[
N(r, u) = \int_0^r n(t, u) \frac{dt}{t}, \quad 0 \leq r \leq R.
\]

For an admissible function \( u \) we consider the open set \( D = \{z \in D(R) : u(z) > 0\} \). Let \( g(z, \varsigma, D) \) denote the function defined as follows. If \( z \) lies in the same
component of $D$ as $\zeta$, then $g(z, \zeta, D)$ is the (positive) Green’s function of this component with pole at $\zeta$. In all the remaining cases $g(z, \zeta, D) = 0$. Denote by $u(\cdot, D)$ a function harmonic in $D$ with $u(z, D) = u(z)$, $z \in \overline{D(R)} \setminus D$. The Riesz representation

$$u(z) = u(z, D) + \int_D g(z, \zeta, D) d(\mu^- - \mu^+)$$

(1.1)

is valid for an admissible function $u$ in $\overline{D(R)}$.

Denote by $D^*$ the circular symmetrization of the open set $D$, i.e., the open set such that

$$\text{meas}\{D \cap \{z: |z| = r\} \} = \text{meas}\{D^* \cap \{z: |z| = r\} \},$$

where $D^* \cap \{z: |z| = r\}$ is either the whole circle or an arc whose midpoint lies on the positive ray. For any measurable function $\varphi$ on $[-\pi, \pi]$ define the symmetrization $\varphi^*$ as the monotonically decreasing function of $|\theta|$, $\theta \in [-\pi, \pi]$, such that

$$\text{meas}\{\theta: \varphi(\theta) > t\} = \text{meas}\{\theta: \varphi^*(\theta) > t\}$$

for any $t \in \mathbb{R}$. Let $u^*(\cdot, D^*)$ be defined as follows: $u^*(\cdot, D^*)$ is harmonic on $D^*$ and equal to 0 on $D(R) \setminus D^*$, and $u^*(Re^{i\theta}, D^*) = (u(Re^{i\theta}))^*$. The function $u^*(\cdot, D^*)$ is admissible.

**Lemma 2.** 1°. If $u$ is an admissible function on $\overline{D(R)}$, $D = \{z \in D(R): u(z) > 0\}$, then

$$M(r, u(\cdot, D)) \geq M(r, u^*(\cdot, D^*)) = u^*(r, D^*), \quad 0 \leq r \leq R.$$

2°. $M(r, g(\cdot, \zeta, D)) \leq M(r, g(\cdot, |\zeta|, D^*)) = g(r, |\zeta|, D^+), \quad 0 \leq r \leq R$.

Assertion 1° follows from Theorem 7 of Baernstein in [5], and 2° is Theorem 5 in the same paper.

Denote by $c(\mu)$ the circular projection of the measure $\mu$ on the positive ray. Lemma 2 gives us

**Lemma 3.** Suppose that the admissible function $u$ has the form (1.1). Then

$$M(r, u) \leq M(r, u^*) = u^*(r), \quad r \leq R, \quad n(r, u) = n(r, u^*), \quad r < R,$$

(1.2)

where

$$u^*(z) = u^*(z, D^+) + \int_{D^*} g(z, \zeta, D^*) dc(\mu^-).$$

Note that $u^* = 0$ in $D(R) \setminus D^*$, and that $u^*$ is superharmonic in $D^*$ and subharmonic off the positive ray.

**Lemma 4.** Suppose that $u$ is an admissible function,

$$\lim_{|z| \to R} u(z) \leq 1, \quad \mu^-(D) < \infty.$$

(1.3)

Let $v(z) = \min\{u(z), 2\}$. Then $n(R, v) = n(R, u)$. 

(1.4)
PROOF. Let $D_1 = \{z \in D : u(z) > 2\}$. By (1.3) and the fact that $u(z) = 0$ for $z \in \partial D \cap D(R)$, we have that $\overline{D}_1 \subset D$. Let $D_2$ be an open set with smooth boundary $\Gamma$ such that $\overline{D}_1 \subset D_2$ and $\overline{D}_2 \subset D$. Obviously, $u(z) = v(z)$ in a neighborhood of $\Gamma$. By Green’s theorem,

$$
\mu^-(D_2) = \int_{\Gamma} \frac{\partial v}{\partial n} ds = \mu^-(D_2)
$$

in $D(R) \setminus D_1$ we have that $u(z) = v(z)$; therefore

$$
\mu^-(D(R) \setminus D_2) = \mu^-(D(R) \setminus D_2),
$$

which together with (1.5) proves the lemma.

LEMMA 5. Let $(v_k)$ be a sequence of admissible functions with the properties that

$$
n(R, v_k) \leq A \quad (A \text{ does not depend on } k),
$$

$$
v_k(r) \geq \kappa > 0, \quad R/8 \leq r \leq R, \quad r \notin X_k,
$$

$$
\text{meas } X_k \to 0, \quad k \to \infty, \quad \text{and } \kappa \text{ does not depend on } k. \text{ Then } M(r, v_k) \geq \kappa/2, \quad R/4 \leq r \leq R/2, \text{ for sufficiently large } k.
$$

PROOF. We prove the lemma by contradiction. Suppose that (1.6) and (1.7) hold, and that there is a sequence $r_k \in [R/4, R/2]$ such that

$$
M(r_k, v_k) < \kappa/2.
$$

Consider the new sequence of functions

$$
w_k(z) = \frac{2}{\kappa} \left( v_k \left( \frac{r_k}{2} z \right) - \frac{\kappa}{2} \right)^+.
$$

It follows from (1.8), (1.7), and (1.6) that

$$
w_k(2e^{i\theta}) = 0,
$$

$$
w_k(r) \geq 1, \quad 1 \leq r \leq 2, \quad r \notin Y_k, \quad \text{meas } Y_k \to 0, \quad k \to \infty,
$$

$$
n(3, w_k) \leq A.
$$

Without loss of generality it can be assumed that the $w_k$ are harmonic in $G = D(2) \setminus [1, 2]$. Indeed, if we replace $w_k$ in $G$ by the solution to the Dirichlet problem with boundary data $w_k$, then the conditions (1.9)–(1.11) are not violated. We next assume that the $w_k$ are harmonic in $G$. Denote by $\omega(z, \alpha)$ the harmonic measure of an arc $\alpha \subset \partial G$ in $G$. For any continuous function $u$ let

$$
E(r, u) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta.
$$

It follows from (1.10) that

$$
w_k(z) \geq \omega(z, ([1, 2] \setminus Y_k)) = \omega(z, [1, 2]) - \omega(z, Y_k) = \omega_1(z) - \omega_{2, k}(z), \quad |z| < 2.
$$
For any $\varepsilon > 0$ we have that $\omega_{2,k}(z) \to 0$ as $k \to \infty$ uniformly with respect to $z$ for $\varepsilon \leq |\arg| \leq \pi$. Therefore, $E(r, \omega_{2,k}) \to 0$ uniformly for $0 \leq r \leq 2$. Consequently,

$$E(r, w_k) \geq E(r, \omega_1) + o(1), \quad k \to \infty \tag{1.12}$$

uniformly with respect to $r$. For $\omega_1(z)$ there is an explicit expression

$$\omega_1(z) = \frac{\pi}{2} \arcsin \frac{2 - |z|}{|2 - \xi|}, \quad \xi = \frac{4(z - 1)}{4 - z}.$$

From this an uncomplicated direct computation gives us that

$$\lim_{r \to 2 - 0} \frac{rd}{dr} E(r, \omega_1) = -\infty. \tag{1.13}$$

It follows from (1.9) that $E(2, w_k) = 0$. Considering (1.12) and (1.13), we get that

$$\lim_{k \to \infty} \lim_{r \to 2 - 0} \frac{rd}{dr} E(r, w_k) = -\infty.$$

However, by Green's formula, we get from (1.11) that

$$\frac{rd}{dr} E(r, w_k) = n(r, -w_k) - n(r, w_k) \geq -n(r, w_k)$$

$$\geq -n(3, w_k) \geq -A$$

for $r < 2$. This contradiction proves the lemma.

Let $\Gamma_1$ and $\Gamma_2$ be two simple Jordan curves joining the circles of the annulus \(\{z: 1 < |z| < 2\}\). Denote by $S$ one of the curvilinear quadrangles bounded by these curves and arcs of the circles of the annulus. There is a unique conformal and univalent mapping of the domain $S$ onto some rectangle $Q = \{\xi = \xi + i\eta: |\xi| < 2, |\eta| < \delta\}$ with the curves $\Gamma_1$ and $\Gamma_2$ going into the horizontal sides $|\eta| = \pm \delta$, and with the circular arcs going into the vertical sides.

**Lemma 6.** $\delta \leq 2|S| \leq 6\pi$, where $|S|$ is the area of the region $S$.

This is a variant of a known theorem of Grötzsch.

**Proof.** Let $\varphi: Q \to S$ be the mapping function. Then

$$1 \leq \int_{-2}^{2} |\varphi'| d\xi, \quad 1 \leq 4 \int_{-2}^{2} |\varphi'|^2 d\xi, \quad 2\delta \leq 4 \int_{-2}^{2} |\varphi'|^2 d\xi d\eta = 4|S|$$

as required.

Let $w(\xi)$ be a superharmonic function continuous on $\overline{Q}$ such that $0 \leq w(\pm 2 + i\eta) \leq 2$ for $|\eta| \leq \delta$, and $w(\xi \pm i\delta) = 0$ for $|\xi| < 2$ and $\delta < 6\pi$.

**Lemma 7.** Let $M(\xi) = \max_{\eta} w(\xi + i\eta) \geq \kappa > 0$, $|\xi| < 1$. Then $\kappa \leq A(\delta w(\xi) + \delta^2)$, where $A$ is an absolute constant.

**Proof.** We represent $w$ as the sum of a harmonic function $h$ on $Q$ and a Green's potential $p$. If $|\xi| \leq 1$, then it is not hard to get the estimate

$$h(\xi) \leq A_1 \exp(-A_2/\delta) \leq A_3 \delta^2, \quad \Re \xi = \xi, \tag{1.14}$$

where $A_1$, $A_2$, and $A_3$ are absolute constants.
For the potential we have that
\[ p(\zeta) = \int_{Q} g(\zeta, t, Q) \; d\mu_{w} \leq \int_{Q} g(\xi, \text{Re } t, Q) \; d\mu_{w}, \]
where \( g \) is the Green's function. Denote by \( \Pi(\delta) \) the horizontal strip \( \{ \zeta : |\text{Im } \zeta| < \delta \} \) and let \( M_{1}(\xi) = \max_{\eta} p(\xi + i\eta) \). We have that
\[
\int_{-1}^{1} M_{1}(\xi) \; d\xi \leq \int_{-1}^{1} d\xi \int_{Q} g(\xi, \text{Re } t, \Pi(\delta)) \; d\mu_{w} \\
\leq \int_{Q} d\mu_{w} \int_{-\infty}^{\infty} g(\xi, 0, \Pi(\delta)) \; d\xi \\
= \delta \mu_{w}(Q) \int_{-\infty}^{\infty} g(\xi, 0, \Pi(1)) \; d\xi, \tag{1.15}
\]
because \( g(\delta \xi, 0, \Pi(1)) = g(\xi, 0, \Pi(\delta)) \). The last integral in (1.15) obviously converges and is an absolute constant. By (1.14) and (1.15),
\[
\kappa \leq \int_{-1}^{1} M(\xi) \; d\xi \leq A_{3} \delta^{2} + \int_{-1}^{1} M_{1}(\xi) \; d\xi \leq A(\delta \mu_{w}(Q) + \delta^{2}),
\]
which is what we were required to prove.

2. PROOF OF THEOREM. Without loss of generality it can be assumed that \( f(0) = 1 \) and that \( \overline{N}(r, f) \sim T(r, f) \), \( r \to \infty \); consequently,
\[
2T(r, f) \leq T(r, f') \leq 2T(2r, f) = 2T(2r). \]
It is known ([3], p. 64) that for functions of finite lower order the series \( \sum_{a} \beta(a, f) \) converges; therefore, for the proof it can be assumed that the numbers \( \beta(a) = \beta(a, f) \) are sufficiently small. Further, if the lower order \( \lambda \) of \( f \) is equal to 0, then the relation \( \beta(a, f) > 0 \) can hold for at most one value \( a \in \mathbb{C} \) ([3], p. 69). Therefore, it will be assumed that \( \lambda > 0 \).

There exist sequences \( r_{m} \to \infty \) and \( S_{m} \to \infty \) such that
\[
T(Sr_{m}) \leq S^{\lambda + 1} T(r_{m}), \quad 1 \leq S \leq S_{m}. \tag{2.1}
\]
The proof of the theorem is divided into several steps.

1°. By H. Cartan's theorem ([1], Theorem 1.3) and by (2.1),
\[
\int_{0}^{2\pi} n \left( 4r_{m}, \frac{1}{f' - te^{i\varphi}} \right) \; d\varphi \leq \int_{0}^{2\pi} N \left( 12r_{m}, \frac{1}{f' / t - e^{i\varphi}} \right) \; d\varphi + \text{const} \\
\leq \log + \frac{1}{t} + (2 + o(1)) T(24r_{m}) \\
\leq A \left( \log + \frac{1}{t} + T(r_{m}) \right) \quad \forall t > 0.
\]
Here and below, \( A \) denotes various constants depending only on \( \lambda \). Let \( l(t) \) be the total length of the level curves \( |f'(z)| = t \) in the disk \( D(4r_{m}) \), and let \( \gamma_{1} = \exp(-T(r_{m})) \) and \( \gamma_{2} = \gamma_{1} / 2 \). According to the length and area principle ([6], §2.1),
\[
\int_{\gamma_{2}}^{\gamma_{1}} \frac{l^{2}(t) \; dt}{t} \leq Ar_{m}^{2} \left( \log + \frac{1}{\gamma_{2}} + T(r_{m}) \right).
\]
Therefore, there is an \( \alpha_m, \sqrt{T(r_m)} \leq \alpha_m \leq \sqrt{T(r_m)} + \log 2 \), such that
\[
I(e^{-\alpha_m}) \leq Ar_m \sqrt{T(r_m)}.
\] (2.2)

Fix a finite collection of points \( a_1, \ldots, a_q \in \mathbb{C}, \ q \geq 2, \) with \( \min\{|a_i - a_j| : i \neq j\} = c > 0 \) and \( \beta(a_j, f) > 0 \). We consider the set \( G_m = \{z : |z| < 4r_m, \ \log |f'(z)| < -\alpha_m\} \). Let \( G_{jm}, \ 1 \leq j \leq q, \) be the open set formed by the components \( G_m \) containing a point \( z_1 \) at which
\[
|f(z_1) - a_j| < c/4.
\] (2.3)

Then
\[
|f(z) - a_j| < c/2
\] (2.4)
everywhere in \( G_{jm} \). Indeed, \( z \in G_{jm} \). In the same component as \( z \) there is a point \( z_1 \) for which (2.3) holds. By (2.2), there is a curve \( \Gamma \subset G_{jm} \) joining \( z \) and \( z_1 \) with length at most \( Ar_m \sqrt{T(r_m)} \). On this curve, as everywhere in \( G_m \),
\[
|f'(z)| \leq \exp(-\alpha_m) \leq \exp(-\sqrt{T(r_m)}).
\]

Therefore, considering that \( \lambda > 0 \), we get that
\[
|f(z) - f(z_1)| \leq \int_{\Gamma} |f'(z)| |dz| \leq A \exp(-\sqrt{T(r_m)})r_m \sqrt{T(r_m)} = o(1), \ m \to \infty,
\]
and (2.3) implies (2.4). In particular, the \( G_{jm} \) are pairwise disjoint.

2°. By Lemma 1 and (2.1), there is a set \( I_m \subset [r_m/2, 4r_m], \ \text{meas} I_m = o(r_m), \ m \to \infty, \) such that
\[
\log^+ \left| \frac{f'(z)}{f(z) - a_j} \right| = o(T(r_m)), \quad m \to \infty,
\] (2.5)

\( 1 \leq j \leq q, \ |z| \in [r_m/2, 4r_m] \setminus I_m \). We show that for any \( r \in [r_m/2, 4r_m] \setminus I_m \) there is a point \( z \) with \( |z| = r \) such that \( z \in G_{jm} \) and
\[
\log |f'(z)| < -A\beta(a_j)T(r_m).
\] (2.6)

Indeed, since \( \beta(a_j) > 0 \), for any \( r \in [r_m/2, 4r_m] \) there is a point \( z \) with \( |z| = r \) such that
\[
\log |f(z) - a_j| < -\frac{1}{2}\beta(a_j)T(r) \leq -A\beta(a_j)T(r_m).
\] (2.7)

This and (2.5) give us (2.6) for some point \( z \). By the definition of \( G_m \), this point is contained in \( G_m \). Finally, by (2.4), it is contained precisely in \( G_{jm} \).

We remark that \( G_{jm} \) cannot contain any circle \( \{z : |z| = r\} \). This fact (which is important for what follows) is a consequence of (2.4), (2.5) and the fact that \( q > 2 \).

3°. By a theorem of Miles [7], the meromorphic function \( 1/f' \) can be represented as a quotient of two entire functions \( g_1 \) and \( g_2 \) such that \( T(r, g_j) \leq A_1T(A_2r), \ j = 1, 2, \) where \( A_1 \) and \( A_2 \) are absolute constants. Using this theorem and the known estimate of the maximal modulus of an entire function in
terms of the characteristic, we get that \(-\log |f'(z)| = t_1(z) - t_2(z)\), where \(t_1\) and \(t_2\) are subharmonic functions with

\[ t_j(z) \leq AT(r_m), \quad |z| < 12r_m. \quad (2.8) \]

Let

\[ t^*_1 = \max(t_1, t_2 + \alpha_m), \quad t^*_2 = \max(t_1 - T(r_m), t_2 + \alpha_m), \]

\[ y_m(z) = (T(r_m)^{-1}) \times (t^*_1(r_mz) - t^*_2(r_mz)), \quad z \in D(4). \]

Denote by \(D_{jm}, D_m, \) and \(X_m\) the sets such that

\[ r_mD_{jm} = G_{jm}, \quad r_mD_m = G_m, \quad r_mX_m = I_m. \]

It is not hard to see that \(y_m(z) = 0\) if \(z \in D(4) - D_{jm}\), \(y_m(z) = 1\) if \(\log |f'(r_mz)| \leq -T(r_m) - \alpha_m\), and \(y_m(z) = T(r_m)^{-1}(-\log |f'(r_mz)| - \alpha_m)\) otherwise. Therefore

\[ 0 \leq y_m \leq 1, \quad z \in \overline{D(4)}, \quad (2.9) \]

and it follows from (2.6) that

\[ \max_{|z|=r, z \in D_{jm}} y_m(z) \geq A\beta(a_j), \quad r \notin X_m, \quad 1/2 \leq r \leq 4. \quad (2.10) \]

Here and below, it is assumed in analogous inequalities that \(A\beta(a_j) < 1\). By (2.8),

\[ n(4, y_m) = T(r_m)^{-1} n(4r_m, -t^*_2) \leq (T r_m)^{-1} N(12r_m, -t^*_2) \]

\[ \geq T(r_m)^{-1} M(12r_m, t^*_2) \leq A, \quad (2.11) \]

where \(A\) does not depend on \(m\).

Now let

\[ u_{jm} = \begin{cases} y_m(z), & z \in D_{jm}, \\ 0, & z \in D(4) \setminus D_{jm}. \end{cases} \]

It follows from (2.10) that

\[ M(r, u_{jm}) \geq A\beta(a_j), \quad \frac{1}{2} \leq r \leq 4, \quad r \notin X_m, \quad (2.12) \]

and it follows from (2.9) that

\[ 0 \leq u_{jm} \leq 1, \quad z \in \overline{D(4)}. \quad (2.13) \]

Let \(p_{jm} = n(4, u_{jm})\). We get from (2.11) that

\[ \sum_{j=1}^{q} p_{jm} \leq n(4, y_m) \leq A. \quad (2.14) \]

The functions \(u_{jm}\) satisfy the conditions of Lemma 3 (take \(D_{jm}\) as \(D\) and \(R = 4\)). According to this lemma, we get functions \(u^*_{jm}\) and regions \(D^*_{jm}\) satisfying the following conditions:

\[ M(r, u^*_{jm}) \geq A\beta(a_j), \quad 1/2 \leq r \leq 4, \quad r \notin X_m; \quad (2.15) \]

\[ n(4, u^*_{jm}) \leq p_{jm} \leq A; \quad (2.16) \]

\[ \lim_{|z| \to 4} u^*_{jm}(z) = 1; \quad (2.17) \]

\[ u^*_{jm}(z) = 0, \quad z \in D(4) \setminus D^*_{jm}. \quad (2.18) \]
Inequality (2.15) follows from (2.12) and Lemma 3; (2.16) follows from (1.2); (2.17) follows from (2.13); and (2.18) follows from the remark after Lemma 3. By (2.15) and (2.16), the conditions of Lemma 5 hold \((R = 4)\), and this lemma enables us to replace (2.15) by

\[
M(r, u_j^*) \geq A\beta(a_j), \quad 1 \leq r \leq 2. \tag{2.19}
\]

Let us now consider the functions \(v_{jm} = \min(u_j^*, 2)\). We get from (2.19) that

\[
M(r, v_{jm}) \geq A\beta(a_j), \quad 1 \leq r \leq 2. \tag{2.20}
\]

By (2.16)–(2.18), conditions (1.3) and (1.4) of Lemma 4 are satisfied. This lemma gives us that

\[
n(4, v_{jm}) \leq p_{jm}. \tag{2.21}
\]

We now observe that since the regions \(D_{jm}\) are disjoint,

\[
\sum_{j=1}^{q} |D_{jm}^*| = \sum_{j=1}^{q} |D_{jm}| \leq 16\pi. \tag{2.22}
\]

Further, none of the \(D_{jm}\) contains a circle about zero, as follows from the remark at the end of 2°. Therefore, the regions \(D_{jm}^*\) also do not contain such circles. It follows from (2.20) and (2.18) that \([1, 2] \subset D_{jm}^*\); consequently, the sets \(S_{jm} = D_{jm}^* \cap \{z : 1 < |z| < 2\}\) are connected. It is easy to see that the \(S_{jm}\) are simply connected domains.

We map each domain \(S_{jm}\) conformally and univalently onto the rectangle \(Q_{jm} = \{z = \xi + i\eta : \xi < 2; |\eta| < \delta_{jm}\}\) as required in Lemma 6. According to this lemma,

\[
\delta_{jm} \leq 2|S_{jm}| \leq 2|D_{jm}^*|. \tag{2.23}
\]

Let \(\varphi_{jm}: Q_{jm} \to S_{jm}\) be the conformal univalent mapping inverse to the indicated mapping, and consider the composition \(w_{jm}(\xi) = v_{jm}(\varphi_{jm}(\xi))\). By the definition of \(v_{jm}\), it follows that \(0 \leq w_{jm} \leq 2\), and \(w_{jm}(\xi + i\delta_{jm}) = 0\); by (2.21),

\[
\mu_{\bar{w}_{jm}}(Q_{jm}) \leq p_{jm}, \tag{2.24}
\]

and, by (2.20),

\[
\max w_{jm}(\xi + i\eta) \geq A\beta(a_j), \quad |\xi| < 2.
\]

Lemma 7 (with \(\kappa = A\beta(a_j)\)) together with (2.24) and (2.23) gives us that

\[
\beta(a_j) \leq A(\delta_{jm}p_{jm} + \delta_{jm}^2) \leq 4A(|D_{jm}^*|p_{jm} + |D_{jm}^*|^2).
\]

From this, using (2.14), (2.22), and elementary inequalities, we deduce that

\[
\sum_{j=1}^{q} \beta^{1/2}(a_j) \leq A \sum_{j=1}^{q} |D_{jm}| + \sum_{j=1}^{q} p_{jm} \leq A.
\]

The theorem is proved.
BIBLIOGRAPHY


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