Goldberg's constants

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This is a joint work with Walter Bergweiler.

Let F_0 be the class of all holomorphic functions f defined in the rings

$$\mathcal{A}(f) = \{z : \rho(f) < |z| < 1\}, \tag{1}$$

omitting 0 and 1, and such that the curve $\gamma(f) = f(\{z : |z| = (1 + \rho(f))/2\})$ has non-zero distinct indices with respect to 0 and 1.

Let F_1, F_2, F_3, F_4 be subclasses of F_0 consisting of meromorphic, holomorphic, rational and polynomial functions in the unit disc U. We define

$$A_j = \inf\{\rho(f) : f \in F_j\}, \quad 0 \le j \le 4.$$

Goldberg proved that

$$0 < A_0 = A_1 = A_3 < A_2 = A_4 < 0.0319,$$

and extremal functions for A_0 and A_2 exist, but extremal functions for A_1, A_3 and A_4 they do not exist. This is a simple normal families argument; it also shows that extremal functions for A_0 have the boundary of the ring $\{z : A_0 < |z| < 1\}$ as the natural boundary, and extremal functions for A_2 have the unit circle as the natural boundary.

The problem is to find the constants A_0 and A_2 and extremal functions.

Theorem 1.

$$A_0 = J := \exp\left(-\frac{\pi^2}{\log(3 + 2\sqrt{2})}\right) \approx 0.003701599.$$

Proof. The assumption of the theorem implies that $\gamma(f)$ is a non-peripheral curve in $D = \mathbf{C}\backslash\{0,1\}$. Consider the uniformization $D = H/\Gamma(2)$. Here $\Gamma(2)$ is the principal congruence subgroup of level 2 of the modular group,

$$\Gamma(2) = SL_2(\mathbf{Z})/\{\pm I\}.$$

Matrices act on the upper half-plane H as fractional-linear transformations. Choosing a point $z_0 \in D$, we consider the fundamental group $\pi(z_0, D)$. To every element of this group corresponds a fractional-linear transformation $\phi \in \Gamma(2)$. This ϕ is parabolic iff γ is peripheral, and the trace of the matrix of ϕ equals ± 2 in this and only this case. Conditions of the theorem imply that $\gamma(f)$ is not peripheral, so the corresponding element $\phi \in \Gamma(2)$ is hyperbolic. Let $\mathcal{A}' = H/\langle\phi\rangle$. Then the map f lifts to a map $\tilde{f} : \mathcal{A}(f) \to \mathcal{A}'$ and we conclude from the Schwarz lemma that

$$\text{mod } \mathcal{A} \geq \text{mod } \mathcal{A}'.$$

If M is a matrix of ϕ, we must have $|\text{tr } M| \geq 6$ because the traces of the elements of $\Gamma(2)$ have residue $2 (\text{mod } 4)$, and ϕ is hyperbolic. Using the well known formula relating the modulus of a ring and the trace,

$$|\text{tr } \phi| = 2\cosh \frac{\pi^2}{\text{mod } \mathcal{A}},$$

we obtain the inequality $A_0 \geq J$.

Now take any element $\phi_0 \in \Gamma(2)$ with trace ± 6, define the ring $\mathcal{A}' = H/\langle\phi\rangle$ and map it conformally onto the ring of the form (1). The inverse of this conformal map composed with the universal covering $\Lambda : H \to D = H/\Gamma(2)$ gives a function f for which equality holds in Theorem 1.

If the indices of $\gamma(f)$ about 0 and 1 are prescribed, we can improve the lower estimate for $\rho(f)$. The free homotopy class of $\gamma(f)$ can be encoded by a cyclic word

$$A^{m_1}B^{n_1}\ldots, A^{m_k}B^{n_k}, \quad \text{where} \quad m_j \neq 0, n_j \neq 0 \quad \text{for} \quad 1 \leq j \leq k. \qquad (2)$$

where A and B are the free generators of the fundamental group $\pi(z_0, D)$ represented by simple counterclockwise loops around 0 and 1. Let $A_0(m, n)$

be the inf of $\rho(f)$ over all functions $f \in F_0$ such that $\gamma(f)$ has representation (2) with

$$\sum_{j=1}^{k} |m_j| = m, \quad \text{and} \quad \sum_{j=1}^{k} |n_j| = n.$$

Theorem 2.

$$A_0(m,n) \geq \exp\left(-\frac{\pi^2}{\log(N + \sqrt{N^2 - 1})}\right),$$

where $N = m + n$ when $m + n$ is odd, and $N = 2(m+n) - 3$ when $m + n$ is even. The estimate is best possible for every $N \geq 3$.

Evidently $m \geq \text{ind}_0\gamma(f)$ and $n \geq \text{ind}_1\gamma(f)$.

The proof of Theorem 2 consists in estimating from below the absolute value of the trace of matrices of the form (2) in terms of m and n. This is done by induction in the length of the word.

Now we discuss A_2.

If $f \in F_2$ one can improve previous estimates by extending the ring $\mathcal{A}(f)$. For example, we can consider a continuum K of minimal Green capacity[1] which contains $f^{-1}(\{0,1\})$. This gives Theorem 3 below. Let $F_2(m,n) \subset F_2$ be the subclass of holomorphic functions with m zeros and n 1-points in the unit disc.

Theorem 3. *For $f \in F_2(m,n)$, let q be the cardinality of $f^{-1}(\{0,1\})$. Then*

$$\rho(f) \geq \left(1 + \sqrt{16A_0(m,n)^{2q}}\right)^{2/q} A_0(m,n),$$

in particular,

$$A_2 \geq \left(1 + \sqrt{1 - 16A_0^6}\right)^{2/3} \approx 0.00587.$$

The proof uses Dubinin's inequality for capacity.

Theorem 3 is far from being best possible. One can improve it further, by considering the length $\ell(f)$ of the shortest geodesic in the free homotopy

[1]Same as the capacity of the condenser $(\partial U, K)$.

class of $\{z : |z| = (\rho(f)+1)/2\}$ with respect to the hyperbolic metric on $U\backslash f^{-1}(\{0,1\})$. The Schwarz lemma implies that

$$\ell(f) \geq \ell'(f), \tag{3}$$

where the right hand side is the length of the shortest geodesic with respect to the hyperbolic metric in $\mathbf{C}\backslash\{0,1\}$ in the free homotopy class of the curve $\gamma(f)$.

The quantity $\ell(f)$ is hard to estimate from above in terms of $\rho(f)$.

Equality holds in (3) if and only if the map

$$f : U\backslash f^{-1}(\{0,1\}) \to \mathbf{C}\backslash\{0,1\} \tag{4}$$

is a *covering*. Such functions in F_2 are called *locally extremal*. This observation permits us to solve the problem of minimizing $\rho(f)$ in a subclass of F_2.

Let $F_5(m,n) \subset F_2(m,n)$ be the subclass consisting of functions having only one zero and one 1-point in U, of multiplicities m and n, and $F_5 = \cup F_5(m,n)$ over all m,n such that $0 < \min\{m,n\} < \max\{m,n\} < \infty$. We define $A_5(m,n)$ and A_5 as the infima of $\rho(f)$ over the corresponding classes.

Proposition. *Every function $f \in F_5(m,n)$ is subordinate to a locally extremal function $g \in F_5(m,n)$. In particular, we have $\rho(f) \geq \rho(g)$ with equality iff f itself is locally extremal.*

Theorem 4. *$A_5 = \rho(h)$ where h is the unique locally extremal function in $F_5(2,1)$ which has a zero of multiplicity 2 at $-\mu$ and a simple 1-point at $\mu > 0$, and for which the subgroup*

$$\Gamma(f) = h_*(\pi(z_0, U\backslash h^{-1}(\{0,1\}))) \subset \Gamma(2)$$

is conjugate to the subgroup generated by A^2 and B.

Here $\mu \approx 0.0252896$ is an absolute constant. The extremal function h comes from conformal maps of some circular quadrilaterals and can be expressed in terms of solutions of a Lamé equation.

To prove Theorem 4, we first reduce it to the case of a locally extremal function using the Proposition above. The crucial fact is that under the assumptions of this proposition the map f_* between the fundamental groups is injective.

Then we have to compare all locally extremal functions in F_5. This is achieved with the help of Theorem 3 which permits to single out just one conjugacy class of subgroups of $\pi(z_0, \mathbf{C}\backslash\{0,1\})$.

The simplest case to which Theorem 4 does not apply is a function f with one simple zero and two simple 1-points. Such a function does not have to be subordinate to any locally extremal function, because the map f_* does not have to be injective. Nevertheless we conjecture that $\rho(f) \geq \mu$, where μ is the number defined in Theorem 4, but we don't know how to prove this.

This is a special case of the more general conjecture that $A_2 = \mu$.

Suppose that f has a simple zero at 0 and two simple 1-points at $\pm a$, and no other zeros or 1-points. Then inequality (3) with *precise* numerical computation of both sides gives $|a| \geq 0.0145$ which must be much worse than the best possible estimate. The best upper estimate known for the minimal possible $|a|$ is 0.1428, which is consistent with our conjecture that $|a| \geq \mu$.